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An explicit fusion algebra isomorphism for twisted quantum doubles of finite groups

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Abstract

We exhibit an isomorphism between the fusion algebra of the quantum double of an extraspecial p-group, where p is an odd prime, and the fusion algebra of a twisted quantum double of an elementary abelian group of the same order.

Keywords: fusion algebra, twisted quantum double of finite group

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Abstract

We exhibit an isomorphism between the fusion algebra of the quantum double of an extraspecial p-group, where p is an odd prime, and the fusion algebra of a twisted quantum double of an elementary abelian group of the same order.

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1 Preliminaries

This article extends [3] to include odd primes. By demonstrating that $D(G)$ and $D^\omega(E)$ have isomorphic fusion algebras, we provide another family of examples involving (untwisted) quantum doubles of nonabelian groups and twisted quantum doubles of abelian groups with nonabelian cocycles (in the sense of [8]). The existence of our isomorphism is a special case of a more general theorem in [9]. We give one such isomorphism explicitly in Theorem 4.2 for its potential use in applications (such as [2], e.g.).

For the entirety of this work, p is an odd prime. Fix ϵ to be a primitive p-th root of unity and fix η so that $\eta^p = \epsilon$. A delta with two indices will be the usual Kronecker delta. Also, $\delta_{x \in A} = 1$ if $x \in A$, 0 if not. All tensor products are over $\mathbb{C}$ unless otherwise noted.

2 Quantum Double of a Finite Group

Let G be a finite group with identity element 1_G and let e_g denote the functional on G given by $e_g(h) = \delta_{g,h}$. Then $\mathbb{C}G^* = \text{span}\{e_g \mid g \in G\}$. The quantum double of a finite group, denoted

$D(G) = (\mathbb{C}G^* \otimes \mathbb{C}G, u, \Delta, \epsilon)$ is a bialgebra, where

$$\begin{aligned}
(e_g \otimes x) \cdot (e_h \otimes y) &= \delta_{g,xhx^{-1}} (e_g \otimes xy) \\
u(1) &= \sum_{h \in G} (e_h \otimes 1_G) \\
\Delta (e_g \otimes x) &= \sum_{h \in G} (e_h \otimes x) \otimes (e_{h^{-1}g} \otimes x), \quad \text{and} \\
\epsilon (e_g \otimes x) &= \delta_{g,1_G}
\end{aligned}$$

for all $g, h, x, y \in G$.

Remark 2.1 *$D(G)$ becomes a braided Hopf algebra [6] by adding the antipode, S, and R-matrix given below.*

$$\begin{aligned}
S (e_g \otimes x) &= \left(e_{x^{-1}} \otimes x^{-1} g^{-1} x\right) \\
R &= \sum_{g,h \in G} (e_g \otimes x) \otimes (e_h \otimes g)
\end{aligned}$$

Representations of $D(G)$ are induced from representations of centralizers of G. See [1], [7], [5], or [3] for details. Let K be a conjugacy class of G, $g_K \in K$, and let $C_K = C_G(g_K)$. Assume that $C_K \lhd G$. Let M be an irreducible C_K-module with character ξ. Then $M(K, \xi)$ is an irreducible $D(G)$-module whose character is $\widehat{\xi_K}$, given by

$$\widehat{\xi_K} (e_g \otimes x) = \delta_{g \in K} \delta_{x \in C_K} \xi^{(r)}(x),$$

where $\xi^{(r)}(x) = \xi(rxr^{-1})$ and r satisfies $g = r^{-1} g_K r$.

2.1 Example - G Extraspecial

For a definition of extraspecial groups and a catalog of relevant properties, see [3, §A.1]. Let G be an extraspecial p-group with $|G| = p^{2n+1}$. Choose $z \in Z = Z(G)$ such that $Z = \langle z \rangle$.

Select an element g_K from each conjugacy class K and (right) coset representatives $\{r_{K,i}\}$ of $C_G(g_K) = C_K$ in G so that $r_{K,i}^{-1} g_K r_{K,i} = z^i g_K$. We will omit subscripts when they can be determined from context. Note that $r_i g_K r_i^{-1} = z^{-i} g_K$. Since G is extraspecial, each $C_K \lhd G$.

For each element of Z, we obtain an irreducible representation of $D(G)$ for every irreducible of G. Together, these elements account for $p \cdot p^{2n}$ one-dimensional and $p(p-1)$ p^n-dimensional irreducible modules. The one-dimensional modules will be denoted $M(i, \alpha)$ for $i \in \mathbb{Z}_p$ and $\alpha \in \widehat{G}$, the set of inequivalent one-dimensional characters of G. Here, i stands for the conjugacy class of z^i. The larger modules are $M(i, \Lambda_a)$, where Λ_a is the p^n-dimensional irreducible character of G satisfying $\Lambda_a(z) = p^n \epsilon^a$. Note that $a \in \mathbb{Z}_p^*$.

Let K be a noncentral conjugacy class. Then C_K affords exactly $p(p-1)$ irreducible representations of dimension p^{n-1} (on which Z acts nontrivially) and p^{2n-1} one-dimensional irreducibles (on which Z acts trivially) [3, Lemma A.2].

Let $\widehat{C_K}$ denote the set of all irreducible characters χ of C_K satisfying $\chi(z) = 1$. For $n > 1$, $\widehat{C_K}$ is exactly the set of one-dimensional characters of C_K. Pick $\chi \in \widehat{C_K}$. Then $M(K, \chi)$ denotes the irreducible module of $D(G)$ induced from χ. Note that $\chi^{(r)} = \chi$ for all $r \in G$.

Let ρ_K be an irreducible representation of C_K satisfying $\mathrm{Tr}(\rho_K) \notin \widehat{C_K}$. Then $\rho_K(z) = \epsilon^a \,\mathrm{id}$ for some $a \in \mathbb{Z}_p^*$. The $p-1$ such irreducible representations will be denoted as $\rho_{K,a,\gamma}$, where $\gamma^p = 1$. We know from [3, Lemma A.2] that $g_K^p = z^d$ for some $d \in \mathbb{Z}_p$. For consistency, we must have $\rho_{K,a,\gamma}(g_K) = \gamma \eta^{ad} \,\mathrm{id}$. Since the conjugacy class will be clear from context, we let $\lambda_{a,\gamma}$ denote the character associated to $\rho_{K,a,\gamma}$. The following lemma will help in calculations.

Lemma 2.2 *Let K be a noncentral conjugacy class of G, and let $g_K^p = z^d$. Then* $\lambda_{a,\gamma}^{(r_m)}(z^b g_K^q) = p^{n-1} \epsilon^{a(b-mq)} \gamma^q \eta^{adq}$.

Proof: We have

$$\begin{aligned}
\lambda_{a,\gamma}^{(r_m)}(z^b g_K^q) &= \lambda_{a,\gamma}(z^b r_m g_K^q r_m^{-1}) \\
&= \lambda_{a,\gamma}(z^b (r_m g_K r_m^{-1})^q) \\
&= \lambda_{a,\gamma}(z^b (z^{-m} g_K)^q) \\
&= \lambda_{a,\gamma}(z^{b-mq} g_K^q) \\
&= p^{n-1}\epsilon^{a(b-mq)}(\gamma\eta^{ad})^q,
\end{aligned}$$

where the last equality follows from the scalar actions of z and g_K. □.

We have established the following.

Lemma 2.3 *Let G be an extraspecial p-group with $|G| = p^{2n+1}$. Then there are four types of irreducible $D(G)$-modules.*

- *$M(i, \alpha)$ of dimension 1 for $i \in \mathbb{Z}_p, \alpha \in \widehat{G}$*
- *$M(i, \Lambda_a)$ of dimension p^n for $i \in \mathbb{Z}_p, a \in \mathbb{Z}_p^*$*
- *$M(K, \chi)$ of dimension p for $K \not\subset Z, \chi(z) = 1$*
- *$M(K, \lambda_{a,\gamma})$ of dimension p^n for $K \not\subset Z, a \in \mathbb{Z}_p^*, \gamma^p = 1$*

For $k \in \mathbb{Z}_p$, define $K^k = \{g^k \mid g \in K\}$, which is itself a conjugacy class for extraspecial G.

Theorem 2.4 *The fusion rules of $D(G)$ are given explicitly in the following formulas. Conjugacy classes K and L are noncentral with $C_K \neq C_L$. Let $a_2 \neq -a_1$ and let $k \in \mathbb{Z}_p^*$ with $k \neq -1$.*

1. *$M(i_1, \alpha_1) \otimes M(i_2, \alpha_2) = M(i_1 + i_2, \alpha_1 \otimes \alpha_2)$*

2. $M(i_1,\alpha)\otimes M(i_2,\Lambda_a)=M(i_1+i_2,\Lambda_a)$

3. $M(i_1,\Lambda_a)\otimes M(i_2,\Lambda_{-a})=\bigoplus_{\alpha\in\widehat{G}} M(i_1+i_2,\alpha)$

4. $M(i_1,\Lambda_{a_1})\otimes M(i_2,\Lambda_{a_2})=p^n\cdot M(i_1+i_2,\Lambda_{a_1+a_2})$

5. $M(i,\alpha)\otimes M(K,\chi)=M(K,\operatorname{Res}^G_{C_K}\alpha\otimes\chi)$

6. $M(i,\alpha)\otimes M(K,\lambda_{a,\gamma})=M(K,\lambda_{a,\alpha(g_K)\gamma\epsilon^{ai}})$

7. $M(i,\Lambda_a)\otimes M(K,\chi)=\bigoplus_{\nu^p=1} M(K,\lambda_{a,\nu})$

8. $M(i,\Lambda_a)\otimes M(K,\lambda_{-a,\gamma})=\bigoplus_{\chi\in\widehat{C_K}} M(K,\chi)$

9. $M(i,\Lambda_{a_1})\otimes M(K,\lambda_{a_2,\gamma})=\bigoplus_{\nu^p=1} p^{n-1}\cdot M(K,\lambda_{a_1+a_2,\nu})$

10. $M(K,\chi)\otimes M(K^{-1},\psi)=\bigoplus_{i=1}^p\bigoplus_{\operatorname{Res}^G_{C_K}\alpha=\chi\otimes\psi} M(i,\alpha)$

11. $M(K,\chi)\otimes M(K^{-1},\lambda_{a,\gamma})=\bigoplus_{i=1}^p M(i,\Lambda_a)$

12. $M(K,\lambda_{a,\gamma_1})\otimes M(K^{-1},\lambda_{-a,\gamma_2})=\bigoplus_{i=1}^p\bigoplus_{\alpha(g_K)=\gamma_1\gamma_2^{-1}\epsilon^{-ai}} M(i,\alpha)$

13. $M(K,\lambda_{a_1,\gamma_1})\otimes M(K^{-1},\lambda_{a_2,\gamma_2})=\bigoplus_{i=1}^p p^{n-1}\cdot M(i,\Lambda_{a_1+a_2})$

14. $M(K,\chi)\otimes M(K^k,\psi)=p\cdot M(K^{k+1},\chi\otimes\psi)$

15. $M(K,\chi)\otimes M(K^k,\lambda_{a,\gamma})=\bigoplus_{\nu^p=1} M(K^{k+1},\lambda_{a,\nu})$

16. $M(K,\lambda_{a,\gamma_1})\otimes M(K^k,\lambda_{-a,\gamma_2})=\bigoplus_{\chi\in\widehat{C_K}} M(K^{k+1},\chi)$

17. $M(K,\lambda_{a,\gamma_1})\otimes M(K^k,\lambda_{ka,\gamma_2})=p^n\cdot M(K^{k+1},\lambda_{(k+1)a,\gamma_1^{k+1}\gamma_2^{1+k^{-1}}})$

18. $M(K,\lambda_{a_1,\gamma_1})\otimes M(K^k,\lambda_{a_2,\gamma_2})=\bigoplus_{\nu^p=1} p^{n-1}\cdot M(K^{k+1},\lambda_{a_1+a_2,\nu}),$ *where* $a_2\neq ka_1$

19. $M(K,\chi)\otimes M(L,\psi)=\bigoplus_{\mathrm{Res}_Q\,\rho=\mathrm{Res}_Q\,\chi\otimes\mathrm{Res}_Q\,\psi} M(KL,\rho),$ *where* $Q=C_K\cap C_L$

20. $M(K,\chi)\otimes M(L,\lambda_{a,\gamma})=\bigoplus_{\nu^p=1} M(KL,\lambda_{a,\nu})$

21. $M(K,\lambda_{a,\gamma_1})\otimes M(L,\lambda_{-a,\gamma_2})=\bigoplus_{\chi\in\widehat{C_{KL}}} M(KL,\chi)$

22. $M(K,\lambda_{a_1,\gamma_1})\otimes M(L,\lambda_{a_2,\gamma_2})=\bigoplus_{\nu^p=1} p^{n-1}\cdot M(KL,\lambda_{a_1+a_2,\nu})$

Proof:

1-4. These cases follow directly from the decomposition of irreducible representations of G.

5. $\Delta(e_{g_K}\otimes x)=\sum_{g\in G}(e_g\otimes x)\otimes(e_{g^{-1}g_K}\otimes x)$. The only nonzero contribution to the trace occurs when $x\in C_K$ and $g=z^i$. So, $\mathrm{Tr}\,(e_{g_K}\otimes x)=\alpha(x)\chi(x)\delta_{x\in C_K}$.

6. Let $g_K^p=z^d$. $\Delta(e_{g_K}\otimes g_K)=\sum_{g\in G}(e_g\otimes g_K)\otimes(e_{g^{-1}g_K}\otimes g_K)$. The only nonzero contribution to the trace occurs when $g=z^i$. So,

$$\begin{aligned}\mathrm{Tr}\,(e_{g_K}\otimes g_K) &= \alpha(g_K)\lambda_{a,\gamma}^{(r-i)}(g_K)\\ &= p^{n-1}\alpha(g_K)\epsilon^{ai}\gamma\eta^{ad}\\ &= \lambda_{a,\alpha(g_K)\gamma\epsilon^{ai}}(g_K).\end{aligned}$$

7. $\mathrm{Tr}\,(e_{g_K}\otimes z)=p^n\epsilon^a$ and $\mathrm{Tr}\,(e_{g_K}\otimes g_K)=0$. Hence $M(K,\lambda_{a,\nu})$ must occur exactly once in the decomposition for each ν. Cases 8 and 9 are similar.

10. Pick $i\in\mathbb{Z}_p$. $\Delta(e_{z^i}\otimes x)=\sum_{g\in G}(e_g\otimes x)\otimes\left(e_{g^{-1}z^i}\otimes x\right)$. We get a nonzero contribution to the trace for every $g\in K$. So $\mathrm{Tr}\,(e_{z^i}\otimes x)=p\chi(x)\psi(x)$, which means that the tensor product module decomposes into precisely those p $M(i,\alpha)$ for which the restriction of α to $C_K(=C_{K^{-1}})$ is $\chi\otimes\psi$. Cases 11, 12, and 13 are obtained by considering $\mathrm{Tr}\,(e_{z^i}\otimes z)$ and $\mathrm{Tr}\,(e_{z^i}\otimes g_K)$.

14. $\Delta\left(e_{g_{K^{k+1}}} \otimes x\right) = \sum_{g \in G} (e_g \otimes x) \otimes \left(e_{g^{-1} g_{K^{k+1}}} \otimes x\right)$. We get a nonzero contribution to the trace for every $g \in K$. Thus $\mathrm{Tr}\left(e_{g_{K^{k+1}}} \otimes x\right) = p\chi(x)\psi(x)$ for all $x \in C_K \,(= C_{K^k})$. Cases 15 and 16 are similar.

17-18. Consider the conjugacy classes K^i, where $i \in \mathbb{Z}_p$. Note that each g_{K^i} belongs to the (abelian) supgroup generated by z and g_K. Choose $d \in \mathbb{Z}_p$ so that $g_K^p = z^d$. Then $g_{K^i}^p = z^{di}$ for all i.

Fix k and let $J = K^{k+1}$. There exist $j, b, c \in \mathbb{Z}_p$ satisfying $g_K g_{K^k} = z^j g_J$, $g_J = z^b g_K^{k+1}$, and $g_J = z^c g_{K^k}^{1+k^{-1}}$. So,

$$\begin{aligned} z^{j(k+1)} g_J^{k+1} &= g_K^{k+1} g_{K^k}^{k+1} \\ &= (z^{-b} g_J)(z^{-ck} g_J^k) \\ &= z^{-b-ck} g_J^{k+1}, \end{aligned}$$

which implies that $j(k+1) + b + ck \equiv 0 \mod p$.

Since $\Delta(e_{g_J} \otimes g_J) = \sum_{g \in G} (e_g \otimes g_J) \otimes (e_{g^{-1} g_J} \otimes g_J)$, we get a nonzero contribution to the trace for every $g \in K$. So, $\Delta\left(e_{g_J} \otimes g_J\right) = \sum_{m=1}^{p}\left(e_{z^m g_K} \otimes g_J\right) \otimes \left(e_{z^{-m} g_K^{-1} g_J} \otimes g_J\right)$.

$$\begin{aligned}
\operatorname{Tr}\left(e_{g_J} \otimes g_J\right) &= \sum_{m=1}^{p} \lambda_{a_1,\gamma_1}^{(r_{K,m})}(g_J)\lambda_{a_2,\gamma_2}^{(r_{K^k,-j-m})}(g_J) \\
&= \sum_{m=1}^{p} \lambda_{a_1,\gamma_1}^{(r_{K,m})}(z^b g_K^{k+1})\lambda_{a_2,\gamma_2}^{(r_{K^k,-j-m})}(z^c g_{K^k}^{1+k^{-1}}) \\
&= p^{2n-2}\sum_{m=1}^{p} \epsilon^{a_1(b-m(k+1))}\gamma_1^{k+1}\eta^{a_1 d(k+1)}\epsilon^{a_2(c+(j+m)(1+k^{-1}))}\gamma_2^{1+k^{-1}}\eta^{a_2 k d(1+k^{-1})} \\
&= p^{2n-2}\sum_{m=1}^{p} \epsilon^{k^{-1}[m(a_2-ka_1)(k+1)+ka_1 b+a_2(j(k+1)+ck)]}\gamma_1^{k+1}\gamma_2^{1+k^{-1}}\eta^{(a_1+a_2)(k+1)d} \\
&= p^{2n-2}\sum_{m=1}^{p} \epsilon^{k^{-1}[m(a_2-ka_1)(k+1)+b(ka_1-a_2)]}\gamma_1^{k+1}\gamma_2^{1+k^{-1}}\eta^{(a_1+a_2)(k+1)d} \\
&= p^{2n-2}\sum_{m=1}^{p} \epsilon^{k^{-1}(a_2-ka_1)(m(k+1)-b)}\gamma_1^{k+1}\gamma_2^{1+k^{-1}}\eta^{(a_1+a_2)(k+1)d}
\end{aligned}$$

Now, if $a_2 = ka_1$, then the epsilon factor drops out, leaving $p^{2n-1}\gamma_1^{k+1}\gamma_2^{1+k^{-1}}\eta^{(k+1)a_1(k+1)d}$, which implies case 17. On the other hand, if $a_2 \neq ka_1$, then (since $k+1 \neq 0$) $\operatorname{Tr}\left(e_{g_J} \otimes g_J\right) = 0$, which implies case 18.

19. Let $x \in Q = C_K \cap C_L$. Then $\Delta\left(e_{g_{KL}} \otimes x\right) = \sum_{g \in G}\left(e_g \otimes x\right) \otimes \left(e_{g^{-1}g_{KL}} \otimes x\right)$. We get a nonzero contribution to the trace for each $g \in K$. So $\operatorname{Tr}\left(e_{g_{KL}} \otimes x\right) = p\chi(x)\psi(x)$. There are exactly p representations ρ of C_{KL} for which $\operatorname{Res}_Q \rho = \operatorname{Res}_Q \chi \otimes \operatorname{Res}_Q \psi$.

20-22. These cases follow from the fact that $\operatorname{Tr}\left(e_{g_{KL}} \otimes g_K\right) = \operatorname{Tr}\left(e_{g_{KL}} \otimes g_L\right) = 0$ [3, Lemma A.4].

□

3 Twisted Quantum Double of a Finite Group

Let G be a finite group and let $\omega \in Z^3(G, \mathbb{C}^*)$. Without loss of generality, we choose ω to be normalized. The *twisted quantum double of* G, $D^\omega(G) = (\mathbb{C}G^* \otimes \mathbb{C}G, u, \Delta, \epsilon, \Phi)$ is a quasi-bialgebra with structure maps given below. The maps u and ϵ are the same as in the untwisted case.

$$(e_g \otimes x) \cdot (e_h \otimes y) = \delta_{g,xhx^{-1}} \theta_g(x,y) \, (e_g \otimes xy)$$

$$\Delta \, (e_g \otimes x) = \sum_{h \in G} \gamma_x(h, h^{-1}g) \, (e_h \otimes x) \otimes (e_{h^{-1}g} \otimes x)$$

$$\Phi = \sum_{g,h,k \in G} \omega(g,h,k)^{-1} \, (e_g \otimes 1) \otimes (e_h \otimes 1) \otimes (e_k \otimes 1) \, ,$$

where

$$\theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)}{\omega(x,x^{-1}gx,y)}$$

and

$$\gamma_x(g,h) = \frac{\omega(g,h,x)\omega(x,x^{-1}gx,x^{-1}hx)}{\omega(g,x,x^{-1}hx)}$$

for all $g, h, x, y \in G$. Notice that if $\omega \equiv 1$, then we recover the definition of $D(G)$. It is well known that the fusion algebra of $D^\omega(G)$ depends only on G and the cohomology class $[\omega] \in H^3(G, \mathbb{C}^*)$ [6], [1].

Remark 3.1 *$D^\omega(G)$ is actually a braided quasi-Hopf algebra. See [6, §XV.5] for the additional structures.*

3.1 Example - E Elementary Abelian

Let E be an elementary abelian p-group with $|E| = p^{2n+1}$ and let $\theta \in Z^2(E, \mathbb{C}^*)$. Projective θ-representations of E are in one-to-one correspondence with (linear) representations of X, a central extension of E by $\mathbb{C}^*$ with associated 2-cocycle θ.

Let $\theta \notin B^2(E, \mathbb{C}^*)$ and let X be defined by the short exact sequence

$$1 \to \mathbb{C}^* \to X \xrightarrow{\pi} E \to 1.$$

By [3, Theorem A.5], there exists a subgroup $F = \pi(Z(X)) \lneq E$ and an extraspecial p-group G such that

$$X \cong F \oplus (\mathbb{C}^* * G),$$

where the central product identifies the subgroup $\{\nu \in \mathbb{C}^* \mid \nu^p = 1\}$ with the center of G via the isomorphism $\epsilon \mapsto z$ for fixed $z \in Z(G)$. Note that $|F| \cdot |G| = p|E|$.

Let ψ be an irreducible θ-representation of E and let Ψ be the associated linear representation of X. Since $\theta \notin B^2(E, \mathbb{C}^*)$, $\dim \psi = \dim \Psi > 1$. By Clifford's Theorem, $\operatorname{Res}_F^X \Psi = (\dim \Psi)\beta$ for some one-dimensional irreducible representation $\beta \in \widehat{F}$. Thus $\operatorname{Res}_G^X \Psi$ must be irreducible. If $|G| = p^{2k+1}$, then $\dim \operatorname{Res}_G^X \Psi = p^k = \dim \psi$. We have just shown the following.

Lemma 3.2 *Let $\theta \notin B^2(E, \mathbb{C}^*)$. Then all irreducible θ-representations have the same dimension. Indeed,*

$$\dim \psi = \sqrt{\frac{|E|}{|F|}}.$$

We now identify F with its isomorphic copy (as a direct summand) in X.

Lemma 3.3 *The map $\mathcal{R} : \psi \mapsto \beta$ is a natural bijection between (inequivalent) irreducible θ-representations of E and (inequivalent) irreducible linear representations of F.*

Proof: This was proved in [3] for $p = 2$, but much of that proof holds here. In particular, the two sets in question still have the same cardinality. We will show that $\mathcal{R}$ is surjective.

Choose $\beta \in \widehat{F}$ and recall $\Lambda_1 : G \to \operatorname{End} V$ is the p^k-dimensional irreducible representation of G satisfying $\Lambda_1(z) = \epsilon\,\mathrm{id}$. Define $\Psi_{\beta,\Lambda_1} : X \to \operatorname{End} V$ via $\Psi_{\beta,\Lambda_1}(f\nu g) = \nu\beta(f)\Lambda_1(g)$ for all $f \in F$, $\nu \in \mathbb{C}^*$, $g \in G$. The choice of Λ_1 guarantees that Ψ_{β,Λ_1} respects the central product identification. Furthermore,

$$
\begin{aligned}
\Psi_{\beta,\Lambda_1}(f_1\nu_1 g_1)\Psi_{\beta,\Lambda_1}(f_2\nu_2 g_2) &= \nu_1\beta(f_1)\Lambda_1(g_1)\nu_2\beta(f_2)\Lambda_1(g_2) \\
&= \nu_1\nu_2\beta(f_1 f_2)\Lambda_1(g_1 g_2) \\
&= \Psi_{\beta,\Lambda_1}(f_1 f_2 \nu_1\nu_2 g_1 g_2).
\end{aligned}
$$

Hence Ψ_{β,Λ_1} is a linear representation of X. Moreover Ψ_{β,Λ_1} is irreducible because Λ_1 is. So, there exists ψ, an irreducible θ-representation of E corresponding to Ψ_{β,Λ_1}. Thus, $\mathcal{R}(\psi) = \operatorname{Res}_F^X \Psi_{\beta,\Lambda_1} = \beta$. $\square$

Now consider E as a $(2n+1)$-dimensional vector space over $\mathbb{Z}_p$ and let H be a $2n$-dimensional subspace of E. Then H admits a nondegenerate symplectic form $\langle\,,\rangle$. Select such a form and choose a symplectic basis of H, $\{b_1, b_2; \ldots; b_{2n-1}, b_{2n}\}$. Pick $b_0 \in E \setminus H$. For $r \in E$, we write $r = \sum_{i=0}^{2n} r_i b_i$. Let

$$\omega(r, s, t) = \epsilon^{r_0(s_1 t_2 + s_3 t_4 + \ldots + s_{2n-1} t_{2n})} \tag{1}$$

for all $r, s, t \in E$. Since ω is trilinear, it is a 3-cocycle on E. Note that ω is normalized.

Lemma 3.4 *Pick $r \in E$. Let X be a central extension of E $(= \pi(X))$ by $\mathbb{C}^*$ with 2-cocycle θ_r and let $F_r = \pi(Z(X))$.*

1. *If $r = 0$, then $F_r = E$.*

2. *If $r \in H \setminus \{0\}$, then $F_r = r^{\perp}$ $(\subseteq H)$.*

3. If $r \notin H$*, then* $F_r = \mathrm{span}\{r\}$.

Proof: Note that $s \in F_r$ if and only if $\theta_r(s,t) = \theta_r(t,s)$ for all $t \in E$. Direct calculation gives

$$\frac{\theta_r(s,t)}{\theta_r(t,s)} = \epsilon^{r_0\langle s,t\rangle + s_0\langle t,r\rangle + t_0\langle r,s\rangle}.$$

Here, $\langle a,b\rangle = \langle P_H(a), P_H(b)\rangle$, where P_H is orthogonal projection onto H.

1. Assume $r = 0$. Then $r_0\langle s,t\rangle + s_0\langle t,r\rangle + t_0\langle r,s\rangle \equiv 0$ for all $s,t \in E$. So $F_r = E$.

2. Assume $r \in H$, $r \neq 0$. Let $s \in F_r$ and pick $t \in H, t \notin r^\perp$. Then $s_0\langle t,r\rangle = 0$, implying $s \in H$. So we have $t_0\langle r,s\rangle = 0$ for all $t \in E$. Thus $s \in r^\perp$. Conversely, pick $s \in r^\perp$. Then $s_0\langle t,r\rangle + t_0\langle r,s\rangle \equiv 0$ for all $t \in E$. So $F_r = r^\perp$.

3. Assume $r \notin H$. Then $r_0 \neq 0$. Let $s \in F_r$ and pick $t \in H$. Then

$$\begin{aligned} 0 &= r_0\langle s,t\rangle + s_0\langle t,r\rangle \\ &= \langle r_0 s, t\rangle - \langle s_0 r, t\rangle \\ &= \langle r_0 s - s_0 r, t\rangle . \end{aligned}$$

   Since t was arbitrary, $r_0 P_H(s) = s_0 P_H(r)$, implying $s = \frac{s_0}{r_0} r$. Thus $s \in \mathrm{span}\{r\}$. Conversely, let $s = ar$ for some $a \in \mathbb{Z}_p$. Then we have $r_0\langle ar,t\rangle + ar_0\langle t,r\rangle + t_0\langle r,ar\rangle = ar_0\langle r,t\rangle - ar_0\langle r,t\rangle + at_0\langle r,r\rangle \equiv 0$ for all $t \in E$. Thus $F_r = \mathrm{span}\{r\}$.

□

Lemma 3.5 *Let* $r \in E, k \in \mathbb{N}$. *If* ρ *is a* θ_r*-representation of* E*, then*

$$\rho(kr) = \rho(r)^k \omega(r,r,r)^{-\frac{k(k-1)}{2}}.$$

Moreover, $\rho(r) = \mu\,\mathrm{id}$*, where* $\mu^p = 1$.

Proof: It is clear that the result holds if $k = 1$. By induction,

$$\begin{aligned}
\rho(kr + r) &= \rho(kr)\rho(r)\theta_r^{-1}(kr, r) \\
&= \rho(r)^k \omega(r, r, r)^{-\frac{k(k-1)}{2}} \rho(r)\omega(r, kr, r)^{-1} \\
&= \rho(r)^{k+1} \omega(r, r, r)^{-\frac{k(k-1)}{2}} \omega(r, r, r)^{-k} \\
&= \rho(r)^{k+1} \omega(r, r, r)^{-\frac{(k+1)k}{2}}.
\end{aligned}$$

Also, $\mathrm{id} = \rho(pr) = \rho(r)^p \omega(r, r, r)^{-\frac{p(p-1)}{2}} = \rho(r)^p$, because p is odd. Since $r \in F_r$, $\rho(r)$ must be a nonzero scalar. $\square$

Irreducible modules of $D^\omega(E)$ are induced from irreducible θ_r-representations of E as r ranges over the conjugacy classes (elements) of E.

If $r = 0$, then we obtain p^{2n+1} inequivalent irreducible one-dimensional $D^\omega(E)$-modules. Let $N(0, \beta)$ denote the module arising from the representation β on E.

If $r \in H$, $r \neq 0$, then we obtain p^{2n-1} inequivalent irreducible p-dimensional $D^\omega(E)$-modules. Each such projective representation corresponds to a one-dimensional representation ψ of $F_r \leq H$ via the bijection of Lemma 3.3. Denote this $D^\omega(E)$-module as $N(r, \psi)$.

If $r \notin H$ then we have p inequivalent irreducible $D^\omega(E)$-modules of dimension p^n. Denote these modules by $N(r, \mu)$ for $\mu^p = 1$, where $\lambda(r) = \mu\,\mathrm{id}$. We have established the following.

Lemma 3.6 *There are three types of irreducible representations of $D^\omega(E)$.*

- *$N(0, \beta)$ of dimension 1 for $\beta \in \widehat{E}$*
- *$N(h, \psi)$ of dimension p for $h \in H$, $h \neq 0$, and $\psi \in \widehat{h^\perp}$*
- *$N(t, \mu)$ of dimension p^n for $t \notin H$ and $\mu^p = 1$* $\square$

Theorem 3.7 *The fusion rules of $D^\omega(E)$ are given explicitly in the following formulas. Below, h_1 and h_2 are linearly independent, as are t_1 and t_2. Let $k \in \mathbb{Z}_p^*$ with $k \neq -1$.*

1. $N(0,\beta_1) \otimes N(0,\beta_2) = N(0,\beta_1 \otimes \beta_2)$
2. $N(0,\beta) \otimes N(h,\psi) = N(h, \mathrm{Res}^E_{h^\perp}\, \beta \otimes \psi)$
3. $N(0,\beta) \otimes N(t,\mu) = N(t,\beta(t)\mu)$
4. $N(h,\psi) \otimes N(-h,\chi) = \bigoplus_{\mathrm{Res}^E_{h^\perp}\, \beta=\psi\otimes\chi} N(0,\beta)$
5. $N(h,\psi) \otimes N(kh,\chi) = p \cdot N((k+1)h, \psi \otimes \chi)$
6. $N(h_1,\psi) \otimes N(h_2,\chi) = \bigoplus_{\mathrm{Res}_P\, \zeta=\mathrm{Res}_P\, \psi\otimes\mathrm{Res}_P\, \chi} N(h_1+h_2,\zeta)$, *where* $P = h_1^\perp \cap h_2^\perp$
7. $N(h,\psi) \otimes N(t,\mu) = \bigoplus_{\nu^p=1} N(t+h,\nu)$
8. $N(t,\mu_1) \otimes N(-t,\mu_2) = \bigoplus_{\beta(t)=\mu_1\mu_2^{-1}} N(0,\beta)$
9. $N(t,\mu_1) \otimes N(kt,\mu_2) = p^n \cdot N((k+1)t, \mu_1^{k+1}\mu_2^{1+k^{-1}})$
10. $N(t_1,\mu_1) \otimes N(t_2,\mu_2) = \bigoplus_{\psi\in\widehat{(t_1+t_2)^\perp}} N(t_1+t_2,\psi)$, *if* $t_1+t_2 \in H$
11. $N(t_1,\mu_1) \otimes N(t_2,\mu_2) = \bigoplus_{\nu^p=1} p^{n-1} \cdot N(t_1+t_2,\nu)$, *if* $t_1+t_2 \notin H$

Proof:

1. $\Delta(e_0 \otimes x) = \sum_{g\in E} \gamma_x(g,-g)\,(e_g \otimes x) \otimes (e_{-g} \otimes x)$. The only term with nonzero trace occurs when $g=0$. So $\mathrm{Tr}\,(e_0 \otimes x) = \beta_1(x)\beta_2(x)$. Cases 2 and 3 are similar.

4. $\Delta(e_0 \otimes x) = \sum_{g\in E} \gamma_x(g,-g)\,(e_g \otimes x) \otimes (e_{-g} \otimes x)$. The only term with nonzero trace occurs when $g=h$. So $\mathrm{Tr}\,(e_0 \otimes x) = p^2 \delta_{x\in\widehat{h^\perp}} \gamma_x(h,-h)\psi(x)\chi(x)$. Note that $\gamma_x(h,-h) = 1$

for $x, h \in H$. Therefore, the tensor product module must decompose into precisely those $N(0,\beta)$ where the restriction of β to $\widehat{h^\perp}$ is $\psi \otimes \chi$. Since $|\widehat{h^\perp}| = |h^\perp| = p^{2n-1}$, there are exactly p^2 distinct such $\beta \in \widehat{E}$. Cases 5 and 6 are similar.

7. $\Delta\left(e_{h+t} \otimes (h+t)\right) = \sum_{g\in E} \gamma_{h+t}(g, -g+h+t)\left(e_g \otimes (h+t)\right) \otimes \left(e_{-g+h+t} \otimes (h+t)\right)$. Again, the only term with nonzero trace occurs when $g = h$. So $\mathrm{Tr}\left(e_{h+t} \otimes (h+t)\right) = p^{n+1}\gamma_{h+t}(h,t)\psi(h+t)\mu = 0$ because $h+t \notin H$. Since $h+t$ acts as a scalar p-th root of unity in each $N(h+t,\nu)$, we must have each one appearing once to guarantee a trace of zero. Cases 10 and 11 are similar because the linear independence of t_1 and t_2 guarantees that $F_{t_1} \cap F_{t_2} = 1$.

8. $\Delta\left(e_0 \otimes x\right) = \sum_{g\in E} \gamma_x(g,-g)\left(e_g \otimes x\right) \otimes \left(e_{-g} \otimes x\right)$. The only term with nonzero trace occurs when $g = t$. Note that if t acts as the scalar μ, then $-t$ acts as $\mu^{-1}\omega(t,t,t)^{-1}$ by Lemma 3.5. Hence, if $x = t$, then the trace is

$$p^{2n}\mu_1\mu_2^{-1}\omega(-t,-t,-t)^{-1}\gamma_t(t,-t) = p^{2n}\mu_1\mu_2^{-1}\omega(t,t,t)\omega(t,t,t)^{-1} = p^{2n}\mu_1\mu_2^{-1}.$$

Moreover, $\mathrm{Tr}\left(e_0 \otimes x\right) = 0$ if $x \notin \mathrm{span}\{t\}$. So we must have every $N(0,\beta)$ appearing in which $\beta(t) = \mu_1\mu_2^{-1}$. There are p^{2n} distinct such $\beta \in \widehat{E}$.

9. $\Delta\big(e_{(k+1)t} \otimes (k+1)t\big) = \sum_{g\in E} \gamma_{(k+1)t}(g, -g+(k+1)t)\left(e_g \otimes (k+1)t\right) \otimes \big(e_{-g+(k+1)t} \otimes (k+1)t\big)$. The only term with nonzero trace occurs when $g = t$. Using Lemma 3.5, we can determine how $(k+1)t$ acts on each tensor factor.

$$\begin{aligned}
\mathrm{Tr}\big(e_{(k+1)t} \otimes (k+1)t\big) &= p^{2n}\mu_1^{k+1}\omega(t,t,t)^{-\frac{(k+1)k}{2}}\mu_2^{1+k^{-1}}\omega(kt,kt,kt)^{-\frac{(1+k^{-1})(k^{-1})}{2}}\gamma_{(k+1)t}(t,kt) \\
&= p^{2n}\mu_1^{k+1}\mu_2^{1+k^{-1}}\omega(t,t,t)^{-\frac{(k+1)k}{2}-\frac{(1+k^{-1})(k^{-1})k^3}{2}}\omega(t,t,t)^{k(k+1)} \\
&= p^{2n}\mu_1^{k+1}\mu_2^{1+k^{-1}}\omega(t,t,t)^{-\frac{(k+1)k}{2}-\frac{(k+1)k}{2}+(k+1)k} \\
&= p^{2n}\mu_1^{k+1}\mu_2^{1+k^{-1}}.
\end{aligned}$$

Therefore, the tensor product module must decompose into p^n copies of $N((k+1)t, \mu_1^{k+1}\mu_2^{1+k^{-1}})$.

$\square$

4 An Explicit Fusion Algebra Isomorphism

For the remainder of this work, G will denote an extraspecial p-group with $|G| = p^{2n+1}$ and E an elementary abelian group with $|E| = |G|$. Pick $H \leq E$ of index p and fix $t \in E \setminus H$. Both G/Z and H admit nondegenerate symplectic forms as $\mathbb{Z}_p$-spaces. Choose the form on G/Z given in [3, (A.1)], and identify it with a form on H by requiring the linear isomorphism $\phi : G/Z \to H$ to be an isometry. Elements of G/Z will be denoted either by Z or by the corresponding noncentral conjugacy class K. Since G is extraspecial, one-dimensional representations of G (and thus of G/Z) are in one-to-one correspondence with one-dimensional representations of H via ϕ. Similarly, elements of $\widehat{C_K}$ can be paired with one-dimensional representations of the subgroup $\phi(C_K/Z) = \phi(K^\perp) = \phi(K)^\perp$ of H.

Let $i \in \mathbb{Z}_p$. If $\alpha \in \widehat{G}$, then define $\overline{\alpha}_i$ to be the representation of E arising from the representation $\alpha \circ \phi^{-1}$ of H with t acting as ϵ^i. That is, $\overline{\alpha}_i(bt + h) = \alpha(\phi^{-1}(h))\epsilon^{bi}$ for $b \in \mathbb{Z}_p$.

Pick $\omega \in Z^3(E, \mathbb{C}^*)$ as in (1). Then $D(G)$ and $D^\omega(E)$ have the same number of one-dimensional, p-dimensional, and p^n-dimensional modules, respectively. Let F be the map from the irreducibles of $D(G)$ to the irreducibles of $D^\omega(E)$ such that

$$
\begin{aligned}
FM(i,\alpha) &= N(0,\overline{\alpha}_i) &(2)\\
FM(i,\Lambda_a) &= N(at,\epsilon^{ai}) &(3)\\
FM(K,\chi) &= N(\phi(K),\chi\circ\phi^{-1}|_{\phi(K)^\perp}) &(4)\\
FM(K,\lambda_{a,\gamma}) &= N(at+\phi(K),\gamma). &(5)
\end{aligned}
$$

Lemma 4.1 *F is a bijection.*

Proof: Recall the three types of $D^\omega(E)$-modules given in Lemma 3.6. Pick $\beta\in\widehat{E}$. Using (2), we have that $FM(\beta(t),\mathrm{Res}^E_H\,\beta\circ\phi)=N(0,\overline{\mathrm{Res}^E_H\,\beta\circ\phi}_{\beta(t)})$. But

$$\overline{\mathrm{Res}^E_H\,\beta\circ\phi}_{\beta(t)}(bt+h)=\mathrm{Res}^E_H\,\beta\circ\phi(\phi^{-1}(h))\beta(t)^b=\beta(h)\beta(t)^b=\beta(bt+h)$$

for all $b\in\mathbb{Z}_p$, $h\in H$. So $N(0,\beta)$ is in the image of F.

Now pick $h\in H$ and $\psi\in\widehat{h^\perp}$. Let K denote the conjugacy class of G that satisfies $\phi(K)=h$. Then we have

$$FM(K,\psi\circ\phi|_{C_K/Z})=N(\phi(K),\psi\circ\phi|_{C_K/Z}\circ\phi^{-1}|_{\phi(K)^\perp})=N(h,\psi).$$

From (3) and (5), modules of type $N(at+h,\mu)$ are also in the image of F for all $a\in\mathbb{Z}_p^*, h\in H$. Hence F is a bijection. $\square$

Theorem 4.2 *F extends to an isomorphism of fusion algebras.*

Proof: We can extend F additively on the irreducible elements in order to obtain a bijection from the fusion algebra of $D(G)$ to that of $D^\omega(E)$. We now show that F preserves the tensor product multiplication, using the fusion rule numbering from Theorem 2.4. Most of the proof involves straightforward checking and so will be omitted. A few of the less clear rules are shown below.

1.

$$FM(i_1,\alpha_1)\otimes FM(i_2,\alpha_2) = N(0,\overline{\alpha_1}_{i_1})\otimes N(0,\overline{\alpha_2}_{i_2}) = N(0,\overline{\alpha_1}_{i_1}\otimes\overline{\alpha_2}_{i_2}).$$

$$FM(i_1+i_2,\alpha_1\otimes\alpha_2) = N(0,(\overline{\alpha_1\otimes\alpha_2})_{i_1+i_2}).$$

Note that

$$\begin{aligned}(\overline{\alpha_1}_{i_1}\otimes\overline{\alpha_2}_{i_2})(bt+h) &= \alpha_1\circ\phi^{-1}(h)\epsilon^{bi_1}\alpha_2\circ\phi^{-1}(h)\epsilon^{bi_2}\\ &= (\alpha_1\otimes\alpha_2)\circ\phi^{-1}(h)\epsilon^{bi_1+bi_2}\\ &= (\overline{\alpha_1\otimes\alpha_2})_{i_1+i_2}(bt+h).\end{aligned}$$

2.

$$\begin{aligned}FM(i_1,\alpha)\otimes FM(i_2,\Lambda_a) &= N(0,\overline{\alpha}_{i_1})\otimes N(at,\epsilon^{ai_2})\\ &= N(at,\overline{\alpha}_{i_1}(at)\epsilon^{ai_2})\\ &= N(at,\epsilon^{ai_1}\epsilon^{ai_2}).\end{aligned}$$

$$FM(i_1+i_2,\Lambda_a) = N(at,\epsilon^{a(i_1+i_2)}).$$

3.

$$\begin{aligned}FM(i_1,\Lambda_a)\otimes FM(i_2,\Lambda_{-a}) &= N(at,\epsilon^{ai_1})\otimes N(-at,\epsilon^{-ai_2})\\ &= \bigoplus_{\beta(at)=\epsilon^{ai_1}(\epsilon^{-ai_2})^{-1}} N(0,\beta)\\ &= \bigoplus_{\beta(t)=\epsilon^{i_1+i_2}} N(0,\beta).\end{aligned}$$

$$\bigoplus_{\alpha\in\widehat{G}} FM(i_1+i_2,\alpha) = \bigoplus_{\alpha\in\widehat{G}} N(0,\overline{\alpha}_{i_1+i_2}).$$

4.

$$\begin{aligned} FM(i_1, \Lambda_{a_1}) \otimes FM(i_2, \Lambda_{a_2}) &= N(a_1 t, \epsilon^{a_1 i_1}) \otimes N(a_2 t, \epsilon^{a_2 i_2}) \\ &= p^n \cdot N(a_1 t + a_2 t, \epsilon^{a_1 i_1 (k+1)} \epsilon^{a_2 i_2 (1+k^{-1})}) \text{ where } k = a_2 a_1^{-1} \\ &= p^n \cdot N((a_1 + a_2)t, \epsilon^{a_2 i_1 + a_1 i_1 + a_2 i_2 + a_1 i_2}) \\ &= p^n \cdot N((a_1 + a_2)t, \epsilon^{(a_1+a_2)(i_1+i_2)}). \end{aligned}$$

$$p^n \cdot FM(i_1 + i_2, \Lambda_{a_1+a_2}) = p^n \cdot N((a_1 + a_2)t, \epsilon^{(a_1+a_2)(i_1+i_2)}).$$

6.

$$\begin{aligned} FM(i, \alpha) \otimes FM(K, \lambda_{a,\gamma}) &= N(0, \overline{\alpha}_i) \otimes N(at + \phi(K), \gamma) \\ &= N(at + \phi(K), \overline{\alpha}_i(at + \phi(K))\gamma) \\ &= N(at + \phi(K), \alpha(g_K)\epsilon^{ai}\gamma). \end{aligned}$$

$$FM(K, \lambda_{a,\alpha(g_K)\gamma\epsilon^{ai}}) = N(at + \phi(K), \alpha(g_K)\gamma\epsilon^{ai}).$$

10.

$$\begin{aligned} FM(K, \chi) \otimes FM(K^{-1}, \psi) &= N(\phi(K), \chi \circ \phi^{-1}|_{\phi(K)^\perp}) \otimes N(-\phi(K), \psi \circ \phi^{-1}|_{\phi(K)^\perp}) \\ &= \bigoplus_{\mathrm{Res}^E_{\phi(K)^\perp} \beta = (\chi\circ\phi^{-1})\otimes(\psi\circ\phi^{-1})} N(0, \beta) \\ &= \bigoplus_{i=1}^{p} \bigoplus_{\substack{\beta(t)=\epsilon^i \\ \mathrm{Res}^E_{\phi(K)^\perp} \beta = (\chi\otimes\psi)\circ\phi^{-1}}} N(0, \beta). \end{aligned}$$

$$\bigoplus_{i=1}^{p} \bigoplus_{\mathrm{Res}^G_{C_K} \alpha = \chi\otimes\psi} FM(i, \alpha) = \bigoplus_{i=1}^{p} \bigoplus_{\mathrm{Res}^G_{C_K} \alpha = \chi\otimes\psi} N(0, \overline{\alpha}_i).$$

12.

$$
\begin{aligned}
FM(K,\lambda_{a,\gamma_1})\otimes FM(K^{-1},\lambda_{-a,\gamma_2}) &= N(at+\phi(K),\gamma_1)\otimes N(-at-\phi(K),\gamma_2)\\
&= \bigoplus_{\beta(at+\phi(K))=\gamma_1\gamma_2^{-1}} N(0,\beta)\\
&= \bigoplus_{i=1}^{p} \bigoplus_{\substack{\beta(t)=\epsilon^i\\ \beta(\phi(K))=\epsilon^{-ai}\gamma_1\gamma_2^{-1}}} N(0,\beta).
\end{aligned}
$$

$$
\bigoplus_{i=1}^{p}\bigoplus_{\alpha(g_K)=\gamma_1\gamma_2^{-1}\epsilon^{-ai}} FM(i,\alpha)=\bigoplus_{i=1}^{p}\bigoplus_{\alpha(g_K)=\gamma_1\gamma_2^{-1}\epsilon^{-ai}} N(0,\overline{\alpha}_i).
$$

17.

$$
\begin{aligned}
FM(K,\lambda_{a,\gamma_1})\otimes FM(K^{k},\lambda_{ka,\gamma_2}) &= N(at+\phi(K),\gamma_1)\otimes N(kat+k\phi(K),\gamma_2)\\
&= p^n\cdot N((k+1)(at+\phi(K)),\gamma_1^{k+1}\gamma_2^{1+k^{-1}}).
\end{aligned}
$$

$$
p^n\cdot FM(K^{k+1},\lambda_{(k+1)a,\gamma_1^{k+1}\gamma_2^{1+k^{-1}}})=p^n\cdot N((k+1)at+(k+1)\phi(K),\gamma_1^{k+1}\gamma_2^{1+k^{-1}}).
$$

18.

$$
\begin{aligned}
FM(K,\lambda_{a_1,\gamma_1})\otimes FM(K^{k},\lambda_{a_2,\gamma_2}) &= N(a_1t+\phi(K),\gamma_1)\otimes N(a_2t+k\phi(K),\gamma_2)\\
&= \bigoplus_{\nu^p=1} p^{n-1}\cdot N((a_1+a_2)t+(k+1)\phi(K),\nu).
\end{aligned}
$$

$$
\bigoplus_{\nu^p=1} p^{n-1}\cdot FM(K^{k+1},\lambda_{a_1+a_2,\nu})=\bigoplus_{\nu^p=1} p^{n-1}\cdot N((a_1+a_2)t+(k+1)\phi(K),\nu).
$$

Therefore, F is a homomorphism. Together with Lemma 4.1, this implies that F is an isomorphism of fusion algebras. □

5 Appendix - Fusion Rules for Extraspecial Groups

Let G be an extraspecial p-group. Fix $z\in G$ so that $\langle z\rangle = Z(G)(\cong \mathbb{Z}_p)$. Recall that $|G|=p^{2n+1}$ for some positive integer n and that $G/G'\cong \mathbb{Z}_p^{2n}$. Hence G admits p^{2n} inequivalent one-dimensional irreducible representations. Let $\widehat{G}$ denote the set of these irreducibles. Then $\widehat{G} \cong \mathbb{Z}_p^{2n}$, and for all $\alpha \in \widehat{G}$, $\alpha(z) = \mathrm{id}$. The remaining $p-1$ irreducible representations have dimension p^n and can be distinguished by their action on z. Let $\rho_i(z) = \epsilon^i$. Further explanations can be found in [4] and [3, Appendix A].

The fusion rules of G are given by the following and hold for all $\alpha, \beta \in \widehat{G}$, and $i, j \in \mathbb{Z}_p$.

- $\alpha \otimes \beta = \alpha\beta$
- $\alpha \otimes \rho_i = \rho_i$
- $\rho_i \otimes \rho_j = \begin{cases} p \cdot \rho_{i+j} & i \neq -j \\ \bigoplus_{\alpha \in \widehat{G}} \alpha & i = -j \end{cases}$

Note that these rules hold even if $p = 2$; the rule with multiplicity p would not occur.

References

[1] R. Dijkgraaf, V. Pasquier, and P. Roche, Quasi-Hopf Algebras, Group Cohomology and Orbifold Models. Nucl. Phys. B (Proc. Suppl.) 18B, 1990, 60–72.

[2] C. Dong and G. Mason, Vertex Operator Algebras and Moonshine: A Survey. Advanced Studies in Pure Mathematics 24, 1996, 101–136.

[3] C. Goff, A family of isomorphic fusion algebras of twisted quantum doubles of finite groups. J. Alg. 259, 2003, 494–511.

[4] D. Gorenstein, Finite Groups. Chelsea Publishing Company, New York, 1980.

[5] M. D. Gould, Quantum double finite group algebras and their representations. Bull. Austral. Math. Soc. 48 (2), 1993, 275–301.

[6] C. Kassel, Quantum Groups. Springer-Verlag, New York, 1995.

[7] G. Mason, The Quantum Double of a Finite Group and its Role in Conformal Field Theory. London Math. Society Lecture Note Series 212, v.2, 1995, 405–417.

[8] G. Mason and S.-H. Ng, Group cohomology and gauge equivalence of some twisted quantum doubles. Trans. Am. Math. Soc. 353(9), 2001, 3465–3509.

[9] S. Natale, On group theoretical Hopf algebras and exact factorizations of finite groups. J. Alg. 270, 2003, 199–211.