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A Family of Isomorphic Fusion Algebras of Twisted Quantum Doubles of Finite Groups

by

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Abstract

Let $D^\omega(G)$ be the twisted quantum double of a finite group, G, where $\omega \in Z^3(G, \mathbb{C}^*)$. For each $n \in \mathbb{N}$, there exists an ω such that $D(G)$ and $D^\omega(E)$ have isomorphic fusion algebras, where G is an extraspecial 2-group with 2^{2n+1} elements, and E is an elementary abelian group with $|E| = |G|$.

1 Introduction

We aim to study the rules which determine how the tensor product of two irreducible algebra modules decomposes into a direct sum of irreducibles. Those algebras whose modules admit a tensor product are called *quasi-bialgebras*. We will denote a quasi-bialgebra A over a field k as $(A, \Delta, \epsilon, \Phi)$, where A is a semisimple unital k-algebra, $\Phi \in A \otimes A \otimes A$ is invertible, and $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ are algebra homomorphisms. Moreover, $(\mathrm{id} \otimes \Delta)(\Delta) = \Phi(\Delta \otimes \mathrm{id})(\Delta)\Phi^{-1}$, $(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1$, and Φ satisfies the pentagon condition

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = (1 \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(\Phi \otimes 1).$$

For simplicity, we set the field of scalars to be $\mathbb{C}$.

Definition 1.1 *Let $(A, \Delta, \epsilon, \Phi)$ be a quasi-bialgebra where A is a semisimple associative algebra (over $\mathbb{C}$). The* fusion algebra *of A is an associative $\mathbb{Z}$-algebra with basis elements being the inequivalent irreducible A-modules (by abuse of notation) and with multiplication ($\otimes$) on basis elements given below and extended bilinearly.*

$$V \otimes W = \sum_U N_{V,W}^U U, \tag{1}$$

where U ranges over all inequivalent irreducible A-modules and $N_{V,W}^U$ is the multiplicity of U in the decomposition of $V \otimes W$.

We compare different quasi-bialgebras by comparing their corresponding fusion algebras. An isomorphism of fusion algebras arises whenever two quasi-bialgebras have tensor equivalent module categories. However, the converse is not necessarily true: while the dihedral group of order eight and the quaternion group of order eight have isomorphic fusion algebras, their module categories are not tensor equivalent [16].

Quantum groups, which arise in physics, constitute an important class of quasi-bialgebras. One class of quantum groups, called the *quantum double of a finite group*, $D(G)$, is constructed from a finite group and the dual of its group algebra [1], [14]. A generalization can be made to a *twisted* quantum double of G, denoted $D^{\omega}(G)$, where one includes certain constants arising from ω, a 3-cocycle of G with coefficients in the trivial G-module, $\mathbb{C}^*$. While such an algebra is in fact quasi-Hopf [12], we are only concerned with the quasi-bialgebra structures. Hence we will not consider antipodes or R-matrices in this work. Reasons for studying the fusion algebra of $D^{\omega}(G)$ can be found in the theory of vertex operator algebras (VOAs) and the conjectured monoidal equivalence between the module categories of a fixed point VOA V^G and $D^{\omega}(G)$ [5]. Applications of twisted quantum doubles to conformal field theory and VOAs can be found in various works, including [2], [3], [4], and [13].

In Theorem 4.2, we prove the existence of a family of fusion algebra isomorphisms. In particular, we demonstrate the existence of $\omega \in Z^3(E, \mathbb{C}^*)$ such that the fusion algebras of $D(G)$ and $D^{\omega}(E)$ are isomorphic, where G is an extraspecial 2-group, and E is elementary abelian with $|E| = |G|$.

2 Quantum Double of a Finite Group

Let e_g denote the functional on the group G given by

$$e_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \neq h. \end{cases}$$

Then $\mathbb{C}G^* = \text{span}\{e_g \mid g \in G\}$. If a delta has two subscripts, then it is the usual Kronecker delta. Also, $\delta_{x\in A}$ is equal to 1 if $x \in A$ and zero if not. All tensor products are over $\mathbb{C}$ unless otherwise noted.

2.1 Definition and Representation Theory

Let G be a finite group with identity element 1. The *quantum double of a finite group*, denoted $D(G)$, has $\mathbb{C}G \otimes \mathbb{C}G^*$ as its underlying vector space and the following linear operations which make it into a semisimple unital bialgebra.

$$\begin{aligned}
m((x \otimes e_g), (y \otimes e_h)) &= \delta_{g,xhx^{-1}} (xy \otimes e_g) \\
u(1) &= \sum_{h\in G} (1 \otimes e_h) \\
\Delta (x \otimes e_g) &= \sum_{\substack{k,l\in G \\ g=kl}} (x \otimes e_k) \otimes (x \otimes e_l) \\
\epsilon (x \otimes e_g) &= \delta_{g,1}
\end{aligned}$$

Representations of $D(G)$ are induced from representations of centralizers of G [1], [14]. An irreducible right $D(G)$-module is isomorphic to $M \otimes_{C_G(g)} D(g)$ for some $g \in G$, where M is an irreducible right module for $C_G(g)$ and $D(g) = \text{span}\{(x \otimes e_g) \mid x \in G\}$. $D(g)$ is a left $C_G(g)$-module by letting g' act as left multiplication by $(g' \otimes e_g)$. Choose an element g_K from each conjugacy class K of G and denote $C_G(g_K)$ as C_K. By choosing one element from each class of G, we obtain a complete set of inequivalent irreducible modules [1]. We also fix one representative $r_{K,i}$ from each right coset of C_K in G, but we drop the subscripts when when the conjugacy class and coset can be determined from context.

Assume that each $C_K \lhd G$. If M is an irreducible C_K-module with character ξ, then we denote $M \otimes_{C_K} D(g_K)$ as $M(K,\xi)$. The irreducible character of $M(K,\xi)$ is $\widehat{\xi}_K (x \otimes e_g) = \delta_{g\in K}\delta_{x\in C_K}\xi^{(r)}(x)$, where $\xi^{(r)}(x) = \xi(rxr^{-1})$ and r satisfies $g = r^{-1}g_K r$. Note that $\xi^{(r)}$ is another character of C_K, possibly different from ξ.

2.2 Example - G Extraspecial 2-group

For a definition of extraspecial groups and a catalog of properties we will need, see Section 5.1. Let G be an extraspecial 2-group for the remainder of this section with $|G| = 2^{2n+1}$. Choose an element g_K from each conjugacy class K and coset representatives $r_{K,i}$ as before. Since G is extraspecial, each C_K is normal in G.

Let the center of G be denoted $Z = \{1_G, z\}$. Since each of these elements has G as its centralizer, we obtain an irreducible representation of $D(G)$ for every irreducible of G. Together, these elements account for $2 \cdot 2^{2n}$ one-dimensional and two 2^n-dimensional irreducible modules. Denote these one-dimensional modules as $M(1, \alpha)$ and $M(-1, \alpha)$ for $\alpha \in \widehat{G}$, the set of inequivalent one-dimensional characters of G. Here, 1 denotes the class of the identity and -1 the nonidentity central conjugacy class. The larger modules are $M(1, \Lambda)$ and $M(-1, \Lambda)$, where Λ is the unique 2^n-dimensional irreducible character of G.

Let K be a noncentral conjugacy class. Then Lemma 5.2 implies that each C_K affords exactly two irreducibles of dimension 2^{n-1} (on which z acts as the scalar -1) and 2^{2n-1} one-dimensional irreducibles (on which z acts as the identity).

Let χ be an irreducible character of C_K. If $\chi(z) = 1$, then let $M(K, \chi)$ denote the irreducible module of $D(G)$ induced from χ. If $\chi(z) \neq 1$, then z acts as the scalar -1. The two such irreducible representations will be denoted as ρ_ϵ where $\epsilon = \pm 1$. The value of ϵ depends on the scalar by which g_K acts. In particular, if $g_K^2 = 1_G$, then $\rho_\epsilon(g_K) = \epsilon$. However, if $g_K^2 = z$, then $\rho_\epsilon(g_K) = \epsilon i$ where i is a square root of -1. Let λ_ϵ be the character associated to ρ_ϵ. We have established the following.

Lemma 2.1 *Let G be an extraspecial 2-group with $|G| = 2^{2n+1}$ and let K be a noncentral conjugacy class of G. Then there are four types of irreducible modules of $D(G)$.*

- *$M(\epsilon, \alpha)$ of dimension 1 for $\alpha \in \widehat{G}$, $\epsilon = \pm 1$*

- *$M(\epsilon, \Lambda)$ of dimension 2^n with $\epsilon = \pm 1$*

- *$M(K, \chi)$ of dimension 2 for χ an irreducible character of C_K with $\chi(z) = 1$*

- *$M(K, \lambda_\epsilon)$ of dimension 2^n with $\epsilon = \pm 1$*

□

The reader may verify the completeness of this list.

Pick a $C_K \neq G$ and choose coset representatives 1_G and $r \notin C_K$ for C_K in G. Recall that $\xi^{(r)}(x) = \xi(rxr^{-1})$ for ξ a character of C_K. Since G is extraspecial, $rxr^{-1} = x$ or xz, depending on whether or not r and x commute. If ξ comes from a C_K-representation where z acts as the identity, then $\xi^{(r)} = \xi$. If instead z acts as -1, then $\xi^{(r)} \neq \xi$ because, for example, $\xi^{(r)}(g_K) = \xi(g_K z) = -\xi(g_K) \neq \xi(g_K)$ because $g_K \in Z(C_K)$ and must therefore act as a nonzero scalar. We have thus shown that $\chi^{(r)} = \chi$ for χ an irreducible character of C_K with $\chi(z) = 1$ and that $\lambda_\epsilon^{(r)} = \lambda_{-\epsilon}$.

2.3 Fusion Rules for $D(G)$

Theorem 2.2 *Let K and L be distinct noncentral conjugacy classes of G. The fusion rules of $D(G)$ are given explicitly in the following formulas. Here, $\widehat{C_K}$ denotes the one-dimensional characters of C_K in which z acts as the identity and $Q = C_K \cap C_L$.*

1. $M(\epsilon_1, \alpha) \otimes M(\epsilon_2, \beta) = M(\epsilon_1\epsilon_2, \alpha \otimes \beta)$

2. $M(\epsilon_1, \alpha) \otimes M(\epsilon_2, \Lambda) = M(\epsilon_1\epsilon_2, \Lambda)$

3. $M(\epsilon, \alpha) \otimes M(K, \chi) = M(K, \mathrm{Res}^G_{C_K} \alpha \otimes \chi)$

4. $M(\epsilon_1, \alpha) \otimes M(K, \lambda_{\epsilon_2}) = M(K, \lambda_{\epsilon_1\epsilon_2\alpha(g_K)})$

5. $M(K,\chi)\otimes M(K,\psi) = \bigoplus_{\mathrm{Res}^G_{C_K}\alpha=\chi\otimes\psi} M(1,\alpha)\oplus M(-1,\alpha)$

6. $M(K,\chi)\otimes M(L,\psi) = \bigoplus_{\mathrm{Res}_Q\,\xi=\mathrm{Res}_Q\,\chi\otimes\mathrm{Res}_Q\,\psi} M(KL,\xi)$

7. $M(K,\chi)\otimes M(\epsilon,\Lambda) = M(K,\lambda_1)\oplus M(K,\lambda_{-1})$

8. $M(K,\chi)\otimes M(K,\lambda_\epsilon) = M(1,\Lambda)\oplus M(-1,\Lambda)$

9. $M(K,\chi)\otimes M(L,\lambda_\epsilon) = M(KL,\lambda_1)\oplus M(KL,\lambda_{-1})$

10. $M(\epsilon_1,\Lambda)\otimes M(\epsilon_2,\Lambda) = \bigoplus_{\alpha\in\widehat{G}} M(\epsilon_1\epsilon_2,\alpha)$

11. $M(\epsilon_1,\Lambda)\otimes M(K,\lambda_{\epsilon_2}) = \bigoplus_{\chi\in\widehat{C_K}} M(K,\chi)$

12. $M(K,\lambda_{\epsilon_1})\otimes M(K,\lambda_{\epsilon_2}) = \bigoplus_{\alpha(g_K)=\epsilon_1\epsilon_2} M(1,\alpha) \bigoplus_{\beta(g_K)=-\epsilon_1\epsilon_2} M(-1,\beta)$

13. $M(K,\lambda_{\epsilon_1})\otimes M(L,\lambda_{\epsilon_2}) = \bigoplus_{\chi\in\widehat{C_{KL}}} M(KL,\chi)$

Proof: We will prove only a few cases to give the reader an idea of the straightforward proof. Full details require properties outlined in Section 5.1 and can also be found in [7].

Case 1: Consider the trace of the element $(x\otimes e_g)$ on the left hand side of the equation. Via the coproduct, Δ, this element acts with trace zero unless $g=\epsilon_1\epsilon_2$. In this case, the trace becomes $\alpha(x)\beta(x)=(\alpha\otimes\beta)(x)$. Cases 2 and 10 are similar.

Case 4: Consider the trace of the element $(g_K\otimes e_{g_K})$ on the left hand side. The only term in $\Delta\,(g_K\otimes e_{g_K})$ that has nonzero action is $(g_K\otimes e_{\epsilon_1})\otimes(g_K\otimes e_{\epsilon_1 g_K})$. Thus the trace is $\alpha(g_K)\lambda_{\epsilon_2}(g_K)$ if $\epsilon_1=1$ and $\alpha(g_K)\lambda_{-\epsilon_2}(g_K)$ if ϵ_1 is -1.

Case 5: Consider the trace of $(x\otimes e_{1_G})$ on the left hand side. There are two terms that have nonzero action. If g_K has order 2, then the terms are $(x\otimes e_{g_K})\otimes(x\otimes e_{g_K})$ and $(x\otimes e_{g_K z})\otimes(x\otimes e_{g_K z})$. Hence $(x\otimes e_{1_G})$ acts as $2\chi(x)\psi(x)\delta_{x\in C_K}$. This character thus decomposes into two representations, each of which agree with $\chi \otimes \psi$ on C_K, but disagree on $G \setminus C_K$. If instead g_K has order 4, then the terms with nonzero action are $(x \otimes e_{g_K z}) \otimes (x \otimes e_{g_K})$ and $(x \otimes e_{g_K}) \otimes (x \otimes e_{g_K z})$. However, the trace is still $2\chi(x)\psi(x)\delta_{x \in C_K}$. The argument for $(x \otimes e_z)$ is similar.

Case 11: Consider the trace of $(x \otimes e_{g_K})$ on the left hand side. The only term with nonzero action is $(x \otimes e_{\epsilon_1}) \otimes (x \otimes e_{\epsilon_1 g_K})$, which yields $\Lambda(x)\lambda_{\epsilon_1\epsilon_2}(x)\delta_{x\in C_K} = 2^{2n-1}\delta_{x\in Z}$. Hence, it decomposes as a sum over all of the 2^{2n-1} irreducible representations of C_K in which z acts as the identity.

Case 12: Consider the trace of $(g_K \otimes e_{1_G})$ on the left hand side. If g_K has order two, then there are two terms in $\Delta(g_K \otimes e_{1_G})$ with nonzero action, namely $(g_K \otimes e_{g_K}) \otimes (g_K \otimes e_{g_K})$ and $(g_K \otimes e_{g_K z}) \otimes (g_K \otimes e_{g_K z})$. Thus $(g_K \otimes e_{1_G})$ acts with trace $(\epsilon_1 2^{n-1})(\epsilon_2 2^{n-1}) + (-\epsilon_1 2^{n-1})(-\epsilon_2 2^{n-1}) = 2^{2n-1}\epsilon_1\epsilon_2$. Lemma 5.4 implies that $(x \otimes e_{1_G})$ acts with trace zero for all x not in the subgroup generated by g_K and z. So this character decomposes into a sum over all one-dimensional characters in which g_K acts as $\epsilon_1\epsilon_2$. This comprises half of the 2^{2n} one-dimensional characters of G. If g_K instead has order four, then the terms of $\Delta(g_K \otimes e_{1_G})$ acting with nonzero trace are $(g_K \otimes e_{g_K z}) \otimes (g_K \otimes e_{g_K})$ and $(g_K \otimes e_{g_K}) \otimes (g_K \otimes e_{g_K z})$. In this case, the trace is $(-i\epsilon_1 2^{n-1})(i\epsilon_2 2^{n-1}) + (i\epsilon_1 2^{n-1})(-i\epsilon_2 2^{n-1}) = 2^{2n-1}\epsilon_1\epsilon_2$, as before. The trace of $(g_K \otimes e_z)$ is handled similarly.

□

3 Twisted Quantum Double of a Finite Group

3.1 Definition and Representation Theory

Let G be a finite group. The *twisted quantum double of G*, denoted $D^\omega(G)$, has the same underlying vector space as the ordinary quantum double, but some of the operations involve a cocycle $\omega \in Z^3(G, \mathbb{C}^*)$.

The maps u and ϵ are unchanged.

$$m((x \otimes e_g), (y \otimes e_h)) = \delta_{g,xhx^{-1}} \theta_g(x,y)\,(xy \otimes e_g)$$

$$\Delta(x \otimes e_g) = \sum_{h \in G} \gamma_x(h, h^{-1}g)\,(x \otimes e_h) \otimes \left(x \otimes e_{h^{-1}g}\right)$$

$$\Phi = \sum_{g,h,k \in G} \omega(g,h,k)^{-1}\,(1 \otimes e_g) \otimes (1 \otimes e_h) \otimes (1 \otimes e_k)\,,$$

where

$$\theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)}{\omega(x,x^{-1}gx,y)}$$

and

$$\gamma_x(g,h) = \frac{\omega(g,h,x)\omega(x,x^{-1}gx,x^{-1}hx)}{\omega(g,x,x^{-1}hx)}.$$

These equations make $D^\omega(G)$ into a semisimple unital quasi-bialgebra. Note that $D^1(G) = D(G)$. If we replace ω by a cohomologous 3-cocycle $\omega(\delta\mu)$, then $D^{\omega(\delta\mu)}(G)$ is equal to the twist of $D^\omega(G)$ by a gauge transformation [1], and so $D^\omega(G)$ and $D^{\omega(\delta\mu)}(G)$ have tensor-equivalent module categories [12, Section XV.3]. Thus, while various structures of $D^\omega(G)$ depend on the specific cocycle ω, the fusion algebra depends only on the cohomology class $[\omega] \in H^3(G,\mathbb{C}^*)$. Without loss of generality, then, we choose ω to be normalized. Note also that, on the subgroup $C_G(g)$, $\theta_g = \gamma_g$ and $\theta_g \in Z^2(C_G(g),\mathbb{C}^*)$.

Representations of $D^\omega(G)$ are induced from projective representations of $C_G(g)$ with cocycle θ_g for some $g \in G$ [1]. If G is abelian, then an irreducible θ_g-representation of G is also an irreducible representation of $D^\omega(G)$. In this case, we let $M(g,\lambda)$ denote the irreducible $D^\omega(G)$-module which equals the irreducible θ_g-representation λ of G. The following specialization will be helpful in establishing Theorem 3.4.

Lemma 3.1 *Let $r, s \in E$, a finite elementary abelian group (written additively). Let $M(r,\lambda)$ and $M(s,\mu)$ be irreducible right $D^\omega(E)$-modules. Then*

$$\mathrm{Tr}_{M(r,\lambda)\otimes M(s,\mu)}(g \otimes e_t) = \delta_{t,r+s}\delta_{g\in F_r \cap F_s}\gamma_g(r,s)\,\mathrm{Tr}_{M(r,\lambda)}(g)\cdot \mathrm{Tr}_{M(s,\mu)}(g),$$

where $F_x = \pi(Z(X_x))$ and X_x is the central extension of E with associated 2-cocycle θ_x, as in Theorem 5.5.

Proof: Let $m_1 \in M(r,\lambda), m_2 \in M(s,\mu)$.

$$\begin{aligned}(m_1 \otimes m_2) \circ (g \otimes e_t) &= \sum_{\substack{k,l \in G \\ t=k+l}} (m_1 \otimes m_2)\left((g \otimes e_k) \otimes (g \otimes e_l)\right) \gamma_g(k,l) \\ &= \sum_{\substack{k,l \in G \\ t=k+l}} \delta_{k,r}(g \cdot m_1) \otimes \delta_{l,s}(g \cdot m_2) \gamma_g(k,l) \\ &= \delta_{t,r+s}\gamma_g(r,s)(g \cdot m_1) \otimes (g \cdot m_2),\end{aligned}$$

which is zero if $t \neq r+s$. But Corollary 5.6 implies that it is also zero when g is not in F_r or F_s. □

3.2 Example - E Elementary Abelian 2-group

For an explicit determination of the projective representations of an elementary abelian 2-group, and for an explanation of some of the notation used in the sequel, consult Section 5.2. Let E denote an elementary abelian group of order 2^{2n+1} for the remainder of this section. Then E is isomorphic to a vector space over $\mathbb{Z}_2$. Let H be a subspace of E of index 2. Then H admits a nondegenerate symplectic form $\langle\, ,\rangle$. Select such a form and choose a symplectic basis of H, $\{b_1, b_2, \ldots, b_{2n-1}, b_{2n}\}$. Pick $b_0 \in E \setminus H$ and extend the form to E by placing b_0 in the radical. To avoid confusion, we will use h^+ to denote the radical of the extended form on E. That is, $h^\perp = h^+ \cap H$. Let $r,s,t \in E$ and write $r = \sum_{i=0}^{2n} r_i b_i$, where each $r_i \in \mathbb{Z}_2$. Define s_i and t_i similarly. Define a 3-cocycle ω by

$$\omega(r,s,t) = (-1)^{r_0(s_1t_2 + s_3t_4 + \ldots + s_{2n-1}t_{2n})}. \tag{2}$$

Notice that ω is normalized and that if $r \in H$, then $\omega(r,s,t) \equiv 1$. Since ω is trilinear, it is a 3-cocycle on E.

Lemma 3.2 *Let ω be given by (2) and let $r \in E$. Let X be a central extension of E $(= \pi(X))$ by $\mathbb{C}^*$ with cocycle θ_r. Let $F_r = \pi(Z(X))$.*

1. *If $r = 0$, then $F_r = E$.*

2. *If $r \in H \setminus \{0\}$, then $F_r = r^\perp = r^+ \cap H$.*

3. *If $r \notin H$, then $F_r = \langle r \rangle$.*

Proof: Let $s \in E$ with $s \neq 0$. If $s \in F_r$, then (s, μ) is in the center of X for all $\mu \in \mathbb{C}^*$. Hence the ratio $\theta_r(s,t)/\theta_r(t,s) = 1$ for arbitrary $t \in E$. This means

$$\begin{aligned} \frac{\theta_r(s,t)}{\theta_r(t,s)} &= \frac{\omega(r,s,t)\omega(s,t,r)}{\omega(s,r,t)} \cdot \frac{\omega(t,r,s)}{\omega(r,t,s)\omega(t,s,r)} \\ &= (-1)^{r_0\langle s,t\rangle + s_0\langle r,t\rangle + t_0\langle r,s\rangle}. \end{aligned}$$

Therefore

$$r_0 \langle s,t\rangle + s_0 \langle r,t\rangle + t_0 \langle r,s\rangle = 0 \qquad (3)$$

for all $t \in E$. Conversely, if (3) holds for all $t \in E$, then $s \in F_r$.

1. $r = 0$. Since ω is normalized, $\theta_r \equiv 1$. Thus X is abelian and $F_r = E$.

2. $r_0 = 0 \neq r$. If $t \in H$ and $t \notin r^+$, then (3) implies that $s \in H$ (i.e. $s_0 = 0$). If $t \notin H$, then (3) implies that $s \in r^+$. Hence $F_r = r^+ \cap H = r^\perp$.

3. $r_0 = 1$; i.e., $r = b_0 + h$ for some $h \in H$. If $h = 0$, then r is in the radical of the form. Hence (3) holds if and only if $s = b_0 = r$. Symmetrically, if $s = b_0$, then (3) implies $r = b_0$. On the other hand, if $h \neq 0$, then $s \neq b_0$. Thus we can find $t \in H$ such that $t \notin s^+$, in which case (3) becomes $1 + s_0 \langle r,t\rangle = 0$, implying $s_0 = 1$. So (3) now simplifies to $\langle r+s,t\rangle = 0$ for all $t \in H$. Since the form is nondegenerate on H, $s = r$. Therefore $F_r = \langle r \rangle$ in either case. □

By Corollary 5.7, the size of F_r determines the size and number of projective representations of E with cocycle θ_r. If $r = 0$, then we obtain 2^{2n+1} inequivalent irreducible $D^\omega(E)$-modules, each of dimension one. Denote the module arising from the representation β on E as $N(0,\beta)$. If $r \in H$, $r \neq 0$, then we obtain 2^{2n-1} inequivalent irreducible two-dimensional $D^\omega(E)$-modules. Each such λ corresponds to a one-dimensional representation of $F_r \leq H$ via the bijection of Corollary 5.7. If $\mathcal{R}(\lambda) = \psi$, then denote the $D^\omega(E)$-module as $N(r,\psi)$. Fix $t \notin H$. If $r \notin H$ then $r = t + h'$ for some $h' \in H$. We have two inequivalent irreducible $D^\omega(E)$-modules of dimension 2^n. Denote these modules by $N(r,\epsilon)$ for $\epsilon = \pm 1$, where the value of ϵ is determined by the scalar action of r. If λ is a projective representation of E, then

$$\mathrm{id} = \lambda(r^2) = \lambda(r)\lambda(r)\theta_r(r,r)^{-1}$$

which implies that $\lambda(r)^2 = \theta_r(r,r)$. So r acts as the complex scalar $\epsilon\sqrt{\theta_r(r,r)}$. We have established the following.

Lemma 3.3 *There are three types of irreducible representations of $D^\omega(E)$. $\widehat{A}$ denotes a set of inequivalent one-dimensional characters of the subgroup A.*

- $N(0,\beta)$ *of dimension* 1 *where* $\beta \in \widehat{E}$
- $N(h,\psi)$ *of dimension* 2 *where* $h \in H$, $h \neq 0$, *and* $\psi \in \widehat{h^\perp}$
- $N(t+h',\epsilon)$ *of dimension* 2^n *where* $h' \in H$ *and* $\epsilon = \pm 1$ □

Again, the reader may verify the completeness of this list.

3.3 Fusion Rules for $D^\omega(E)$

Theorem 3.4 *Pick ω as in (2). The fusion rules of $D^\omega(E)$ are given explicitly in the following formulas, which hold for all $h, h' \in H$ with $h \neq 0$.*

1. $N(0,\beta) \otimes N(0,\gamma) = N(0,\beta \otimes \gamma)$
2. $N(0,\beta) \otimes N(h,\psi) = N(h, \mathrm{Res}^E_{h^\perp}\, \beta \otimes \psi)$

3. $N(0,\beta) \otimes N(t+h',\epsilon) = N(t+h', \epsilon \cdot \beta(t+h'))$

4. $N(h,\psi) \otimes N(h,\chi) = \bigoplus_{\mathrm{Res}^E_{h^\perp} \beta = \psi \otimes \chi} N(0,\beta)$

5. $N(h_1,\psi) \otimes N(h_2,\chi) = \bigoplus_{\mathrm{Res}_P \xi = \mathrm{Res}_P \psi \otimes \mathrm{Res}_P \chi} N(h_1+h_2,\xi)$

   where $h_1, h_2 \in H \setminus \{0\}$, $h_1 \neq h_2$, *and* $P = h_1^\perp \cap h_2^\perp$

6. $N(h,\psi) \otimes N(t+h',\epsilon) = N(t+h+h',1) \oplus N(t+h+h',-1)$

7. $N(t+h',\epsilon_1) \otimes N(t+h',\epsilon_2) = \bigoplus_{\beta(t+h') = \epsilon_1\epsilon_2} N(0,\beta)$

8. $N(t+h_1,\epsilon_1) \otimes N(t+h_2,\epsilon_2) = \bigoplus_{\psi \in \widehat{(h_1+h_2)}^\perp} N(h_1+h_2,\psi)$, *where* $h_1, h_2 \in H$, $h_1 \neq h_2$

Proof: As in Theorem 2.2, we will only establish a handful of the cases in order to describe the manner of proof. Complete details can be found in [7].

Case 5: Let $P = h_1^\perp \cap h_2^\perp$. Since any elements orthogonal to h_1 and h_2 are also orthogonal to their sum, $P \leq (h_1+h_2)^\perp$. Lemma 3.1 implies

$$\mathrm{Tr}\,(x \otimes e_{h_1+h_2}) = \theta_x(h_1,h_2)\,\mathrm{Tr}_{N(h_1,\psi)}(x)\,\mathrm{Tr}_{N(h_2,\chi)}(x)\delta_{x\in P}.$$

If $x \in P \subset H$, then x acts as the scalar $\psi(x)\chi(x)$ on the tensor product module, making the trace equal to $4\psi(x)\chi(x)$. So x must act as the scalar $\psi(x)\chi(x)$ on each of the modules appearing in the decomposition. Since P has index two in $(h_1+h_2)^\perp$, there are two elements $\xi_1, \xi_2 \in (h_1+h_2)^\perp$ such that $\mathrm{Res}_P\,\xi_1 = \mathrm{Res}_P\,\xi_2 = \mathrm{Res}_P\,\psi \otimes \mathrm{Res}_P\,\chi$. We claim that each of these appears in the decomposition once.

Consider the trace of $(x \otimes e_{h_1+h_2})$ on $N(h_1+h_2,\xi_i) \oplus N(h_1+h_2,\xi_j)$ where i and j are possibly equal elements of $\{1,2\}$. We have

$$\mathrm{Tr}\,(x \otimes e_{h_1+h_2}) = 2\delta_{x\in(h_1+h_2)^\perp}(\xi_i(x) + \xi_j(x)),$$

If $x \in P$, then this simplifies to $2(2\psi(x)\chi(x)) = 4\psi(x)\chi(x)$, as desired, independent of the choice of i and j. But if $x \in (h_1 + h_2)^\perp \setminus P$ and $i = j$, the trace becomes $4\xi_i(x)$, which is a nonzero constant for either choice of i. Thus $i \neq j$.

Case 6: Lemma 3.1 implies that $\operatorname{Tr}(x \otimes e_{t+h+h'}) = \theta_x(h, t+h') \operatorname{Tr}_{N(h,\psi)}(x) \operatorname{Tr}_{N(t+h',\epsilon)}(x)\delta_{x\in h^\perp \cap \langle t+h'\rangle}$. But $h^\perp$ and $\langle t + h'\rangle$ have trivial intersection, implying that every nonidentity element acts with trace zero. This occurs only if both $N(t + h + h', 1)$ and $N(t + h + h', -1)$ occur in the sum.

Case 7: $\operatorname{Tr}(x \otimes e_0) = \theta_x(t + h', t + h') \operatorname{Tr}_{N(t+h',\epsilon_1)}(x) \operatorname{Tr}_{N(t+h',\epsilon_2)}(x)\delta_{x\in\langle t+h'\rangle}$. If $x = t + h'$, then the trace is

$$2^{2n} \left(\epsilon_1 \sqrt{\theta_{t+h'}(t + h', t + h')}\right) \left(\epsilon_2 \sqrt{\theta_{t+h'}(t + h', t + h')}\right) \theta_{t+h'}(t + h', t + h') = 2^{2n}\epsilon_1\epsilon_2.$$

Taking the inner product with an arbitrary character $\beta \in \widehat{E}$ gives

$$\frac{1}{|E|} \left(2^{2n}\beta(1) + 2^{2n}\epsilon_1\epsilon_2\beta(t + h')\right) = \frac{1}{2} \left(1 + \epsilon_1\epsilon_2\beta(t + h')\right)$$

Thus, β appears in the decomposition if and only if $\beta(t + h') = \epsilon_1\epsilon_2$.

Case 8: $\operatorname{Tr}(x \otimes e_{h_1+h_2}) = \theta_x(t+h_1, t+h_2) \operatorname{Tr}_{N(t+h_1,\epsilon_1)}(x) \operatorname{Tr}_{N(t+h_2,\epsilon_2)}(x)\delta_{x\in\langle t+h_1\rangle\cap\langle t+h_2\rangle}$. As in Case 6, this is zero for all $x \neq 0$ because the intersection of $\langle t + h_1\rangle$ and $\langle t + h_2\rangle$ is trivial. Thus, it is exactly the character of the regular representation of the group $(h_1 + h_2)^\perp$. □

4 An Isomorphism of Fusion Algebras

Let G be an extraspecial 2-group with $|G| = 2^{2n+1}$ and $Z = \{1_G, z\}$. Let E be an elementary abelian 2-group with $|E| = |G|$. We now link these two groups. Pick $H \leq E$ of index 2 and choose $t \in E \setminus H$. Both G/Z and H admit nondegenerate symplectic forms as $\mathbb{Z}_2$-spaces. Choose the form on G/Z given by (8) and identify it with a form on H by choosing $\phi : G/Z \to H$ to be a linear isomorphism and an isometry. Thus $\phi(Z) = 0$. Other elements of G/Z will be denoted simply by the corresponding conjugacy class K. Since G is extraspecial, one-dimensional representations of G (and thus of G/Z) are in one-to-one correspondence with one-dimensional representations of H via the isomorphism ϕ. Similarly, one-dimensional representations of a centralizer C_K of G in which z acts as the identity can be paired with one-dimensional representations of the subgroup $\phi(C_K/Z) = \phi(K)^\perp$ of H.

Let $\epsilon = \pm 1$. If α is a one-dimensional representation of G, then define $\overline{\alpha}_\epsilon$ to be the representation of E arising from the representation $\alpha \circ \phi^{-1}$ of H with t acting as ϵ. That is, $\overline{\alpha}_\epsilon(at + h) = \alpha(\phi^{-1}(h))\epsilon^a$ for $a \in \mathbb{Z}_2$.

Pick $\omega \in Z^3(E, \mathbb{C}^*)$ as in equation (2). Then $D(G)$ and $D^\omega(E)$ have the same number of one-dimensional, two-dimensional, and 2^n-dimensional modules, respectively. Let $\mathcal{F}$ be the map from the fusion algebra of $D(G)$ to the fusion algebra of $D^\omega(E)$ such that

$$\mathcal{F}M(\epsilon, \alpha) = N(0, \overline{\alpha}_\epsilon) \tag{4}$$

$$\mathcal{F}M(\epsilon, \Lambda) = N(t, \epsilon) \tag{5}$$

$$\mathcal{F}M(K, \chi) = N(\phi(K), \chi \circ \phi^{-1}|_{\phi(K)^\perp}) \tag{6}$$

$$\mathcal{F}M(K, \lambda_\epsilon) = N(t + \phi(K), \epsilon). \tag{7}$$

Lemma 4.1 *$\mathcal{F}$ is a bijection on the irreducible modules.*

Proof: We will show that $\mathcal{F}$ maps onto irreducible $D^\omega(E)$-modules of the three types given in Lemma 3.3. Using (4), we have that $\mathcal{F}M(\beta(t), \mathrm{Res}^E_H \beta \circ \phi) = N(0, \overline{\mathrm{Res}^E_H \beta \circ \phi}_{\beta(t)})$. But by definition of $\overline{\alpha}_\epsilon$, we have

$$\overline{\mathrm{Res}^E_H \beta \circ \phi}_{\beta(t)}(at + h) = \mathrm{Res}^E_H \beta \circ \phi(\phi^{-1}(h))\beta(t)^a = \beta(h)\beta(t)^a = \beta(at + h)$$

for all $a \in \mathbb{Z}_2$, $h \in H$. So $N(0, \beta)$ is in the image of $\mathcal{F}$. Now let $\phi(K) = h$. Then we have

$$\begin{aligned} \mathcal{F}M(K, \psi \circ \phi|_{C_K/Z}) &= N(\phi(K), \psi \circ \phi|_{C_K/Z} \circ \phi^{-1}|_{\phi(K)^\perp}) \\ &= N(h, \psi). \end{aligned}$$

From (5) and (7), modules of type $N(t+h,\epsilon)$ are also in the image of $\mathcal{F}$. $\square$

Theorem 4.2 *$\mathcal{F}$ extends to an isomorphism of $\mathbb{Z}$-algebras.*

Proof: We can extend $\mathcal{F}$ $\mathbb{Z}$-additively on these irreducible elements in order to obtain a bijection from the fusion algebra of $D(G)$ to that of $D^\omega(E)$. We now show that $\mathcal{F}$ preserves the tensor product multiplication, using the fusion rule numbering from Theorem 2.2.

1. $\mathcal{F}[M(\epsilon_1,\alpha)\otimes M(\epsilon_2,\beta)] = \mathcal{F}M(\epsilon_1\epsilon_2,\alpha\otimes\beta) = N(0,\overline{\alpha\otimes\beta}_{\epsilon_1\epsilon_2})$.

   $\mathcal{F}M(\epsilon_1,\alpha)\otimes\mathcal{F}M(\epsilon_2,\beta) = N(0,\overline{\alpha}_{\epsilon_1})\otimes N(0,\overline{\beta}_{\epsilon_2}) = N(0,\overline{\alpha}_{\epsilon_1}\otimes\overline{\beta}_{\epsilon_2})$.

   To see these are equal, let $a\in\mathbb{Z}_2$. Then $(\overline{\alpha}_{\epsilon_1}\otimes\overline{\beta}_{\epsilon_2})(at+h) = \overline{\alpha}_{\epsilon_1}(at+h)\cdot\overline{\beta}_{\epsilon_2}(at+h) = \alpha(\phi^{-1}(h))\epsilon_1^a\cdot\beta(\phi^{-1}(h))\epsilon_2^a = \alpha\otimes\beta(\phi^{-1}(h))(\epsilon_1\epsilon_2)^a = \overline{\alpha\otimes\beta}_{\epsilon_1\epsilon_2}(at+h)$.

2. $\mathcal{F}[M(\epsilon_1,\alpha)\otimes M(\epsilon_2,\Lambda)] = \mathcal{F}M(\epsilon_1\epsilon_2,\Lambda) = N(t,\epsilon_1\epsilon_2)$.

   $\mathcal{F}M(\epsilon_1,\alpha)\otimes\mathcal{F}M(\epsilon_2,\Lambda) = N(0,\overline{\alpha}_{\epsilon_1})\otimes N(t,\epsilon_2) = N(t,\epsilon_2\overline{\alpha}_{\epsilon_1}(t)) = N(t,\epsilon_2\epsilon_1)$.

3. $\mathcal{F}[M(\epsilon,\alpha)\otimes M(K,\chi)] = \mathcal{F}M(K,\mathrm{Res}^G_{C_K}\alpha\otimes\chi) = N(\phi(K),(\mathrm{Res}^G_{C_K}\alpha\otimes\chi)\circ\phi^{-1}|_{\phi(K)^\perp})$.

$$\begin{aligned}\mathcal{F}M(\epsilon,\alpha)\otimes\mathcal{F}M(K,\chi) &= N(0,\overline{\alpha}_\epsilon)\otimes N(\phi(K),\chi\circ\phi^{-1}|_{\phi(K)^\perp})\\ &= N(\phi(K),\mathrm{Res}^E_{\phi(K)^\perp}\overline{\alpha}_\epsilon\otimes(\chi\circ\phi^{-1}|_{\phi(K)^\perp}))\\ &= N(\phi(K),(\alpha\circ\phi^{-1}|_{\phi(K)^\perp})\otimes(\chi\circ\phi^{-1}|_{\phi(K)^\perp}))\\ &= N(\phi(K),(\mathrm{Res}^G_{C_K}\alpha\otimes\chi)\circ\phi^{-1}|_{\phi(K)^\perp}).\end{aligned}$$

4. $\mathcal{F}[M(\epsilon_1,\alpha)\otimes M(K,\lambda_{\epsilon_2})] = \mathcal{F}M(K,\lambda_{\epsilon_1\epsilon_2\alpha(g_K)}) = N(t+\phi(K),\epsilon_1\epsilon_2\alpha(g_K))$.

   $\mathcal{F}M(\epsilon_1,\alpha)\otimes\mathcal{F}M(K,\lambda_{\epsilon_2}) = N(0,\overline{\alpha}_{\epsilon_1})\otimes N(t+\phi(K),\epsilon_2) = N(t+\phi(K),\epsilon_2\epsilon_1\alpha(g_K))$.

5.

$$\begin{aligned}
\mathcal{F}[M(K,\chi)\otimes M(K,\psi)] &= \bigoplus_{\mathrm{Res}^G_{C_K}\alpha=\chi\otimes\psi} \mathcal{F}M(1,\alpha)\oplus\mathcal{F}M(-1,\alpha) \\
&= \bigoplus_{\mathrm{Res}^G_{C_K}\alpha=\chi\otimes\psi} N(0,\overline{\alpha}_1)\oplus N(0,\overline{\alpha}_{-1}) \\
&= \bigoplus_{\mathrm{Res}^H_{\phi(K)^\perp}(\alpha\circ\phi^{-1})=(\chi\otimes\psi)\circ\phi^{-1}} N(0,\overline{\alpha}_1)\oplus N(0,\overline{\alpha}_{-1}) \\
&= \bigoplus_{\mathrm{Res}^E_{\phi(K)^\perp}\beta=(\chi\otimes\psi)\circ\phi^{-1}} \left(\bigoplus_{\beta(t)=1} N(0,\beta) \bigoplus_{\beta(t)=-1} N(0,\beta)\right) \\
&= \bigoplus_{\mathrm{Res}^E_{\phi(K)^\perp}\beta=(\chi\otimes\psi)\circ\phi^{-1}} N(0,\beta),
\end{aligned}$$

where the penultimate equality holds because $\overline{\alpha}_\epsilon = \alpha\circ\phi^{-1} = \beta$ on any subgroup of H.

$\mathcal{F}M(K,\chi)\otimes\mathcal{F}M(K,\psi) = N(\phi(K),\chi\circ\phi^{-1})\otimes N(\phi(K),\psi\circ\phi^{-1}) =$

$$\bigoplus_{\mathrm{Res}^E_{\phi(K)^\perp}\beta=(\chi\circ\phi^{-1})\otimes(\psi\circ\phi^{-1})} N(0,\beta).$$

6.

$$\begin{aligned}
\mathcal{F}[M(K,\chi)\otimes M(L,\psi)] &= \bigoplus_{\mathrm{Res}_Q\xi=\mathrm{Res}_Q\chi\otimes\mathrm{Res}_Q\psi} \mathcal{F}M(KL,\xi) \\
&= \bigoplus_{\mathrm{Res}_Q\xi=\mathrm{Res}_Q\chi\otimes\mathrm{Res}_Q\psi} N(\phi(KL),\xi\circ\phi^{-1}) \\
&= \bigoplus_{\mathrm{Res}_P(\xi\circ\phi^{-1})=\mathrm{Res}_P(\chi\circ\phi^{-1})\otimes\mathrm{Res}_P(\psi\circ\phi^{-1})} N(\phi(K)+\phi(L),\xi\circ\phi^{-1})
\end{aligned}$$

because $P=\phi(Q)$.

$$\begin{aligned}
\mathcal{F}M(K,\chi)\otimes\mathcal{F}M(L,\psi) &= N(\phi(K),\chi\circ\phi^{-1})\otimes N(\phi(L),\psi\circ\phi^{-1}) \\
&= \bigoplus_{\mathrm{Res}_P\sigma=\mathrm{Res}_P(\chi\circ\phi^{-1})\otimes\mathrm{Res}_P(\psi\circ\phi^{-1})} N(\phi(K)+\phi(L),\sigma).
\end{aligned}$$

7. $\mathcal{F}[M(K,\chi)\otimes M(\epsilon,\Lambda)] = \mathcal{F}M(K,\lambda_1)\oplus\mathcal{F}M(K,\lambda_{-1}) = N(t+\phi(K),1)\oplus N(t+\phi(K),-1)$.

$\mathcal{F}M(K,\chi)\otimes\mathcal{F}M(\epsilon,\Lambda) = N(\phi(K),\chi\circ\phi^{-1})\otimes N(t,\epsilon) = N(t+\phi(K),1)\oplus N(t+\phi(K),-1)$.

8. $\mathcal{F}[M(K,\chi)\otimes M(K,\lambda_\epsilon)] = \mathcal{F}M(1,\Lambda)\oplus\mathcal{F}M(-1,\Lambda) = N(t,1)\oplus N(t,-1)$.

$\mathcal{F}M(K,\chi)\otimes\mathcal{F}M(K,\lambda_\epsilon) = N(\phi(K),\chi\circ\phi^{-1})\otimes N(t+\phi(K),\epsilon) = N(t,1)\oplus N(t,-1)$.

9. $\mathcal{F}[M(K,\chi)\otimes M(L,\lambda_\epsilon)] = \mathcal{F}M(KL,\lambda_1)\oplus\mathcal{F}M(KL,\lambda_{-1}) = N(t+\phi(KL),1)\oplus N(t+\phi(KL),-1)$.

$\mathcal{F}M(K,\chi)\otimes\mathcal{F}M(L,\lambda_\epsilon) = N(\phi(K),\chi\circ\phi^{-1})\otimes N(t+\phi(L),\epsilon) = N(t+\phi(KL),1)\oplus N(t+\phi(KL),-1)$.

10. $\mathcal{F}[M(\epsilon_1,\Lambda)\otimes M(\epsilon_2,\Lambda)] = \bigoplus_{\alpha\in\widehat{G}}\mathcal{F}M(\epsilon_1\epsilon_2,\alpha) = \bigoplus_{\alpha\in\widehat{G}} N(0,\overline{\alpha}_{\epsilon_1\epsilon_2}) = \bigoplus_{\beta(t)=\epsilon_1\epsilon_2} N(0,\beta)$.

$\mathcal{F}M(\epsilon_1,\Lambda)\otimes\mathcal{F}M(\epsilon_2,\Lambda) = N(t,\epsilon_1)\otimes N(t,\epsilon_2) = \bigoplus_{\beta(t)=\epsilon_1\epsilon_2} N(0,\beta)$.

11. $\mathcal{F}[M(\epsilon_1,\Lambda)\otimes M(K,\lambda_{\epsilon_2})] = \bigoplus_{\chi\in\widehat{C_K}}\mathcal{F}M(K,\chi) = \bigoplus_{\chi\circ\phi^{-1}\in\widehat{\phi(K)}^\perp} N(\phi(K),\chi\circ\phi^{-1})$.

$\mathcal{F}M(\epsilon_1,\Lambda)\otimes\mathcal{F}M(K,\lambda_{\epsilon_2}) = N(t,\epsilon_1)\otimes N(t+\phi(K),\epsilon_2) = \bigoplus_{\psi\in\widehat{\phi(K)}^\perp} N(\phi(K),\psi)$.

12. $$\mathcal{F}[M(K,\lambda_{\epsilon_1})\otimes M(K,\lambda_{\epsilon_2})] = \bigoplus_{\alpha(g_K)=\epsilon_1\epsilon_2}\mathcal{F}M(1,\alpha)\bigoplus_{\beta(g_K)=-\epsilon_1\epsilon_2}\mathcal{F}M(-1,\beta) =$$

$$\bigoplus_{\alpha(g_K)=\epsilon_1\epsilon_2} N(0,\overline{\alpha}_1)\bigoplus_{\beta(g_K)=-\epsilon_1\epsilon_2} N(0,\overline{\beta}_{-1}).$$

$$\mathcal{F}M(K,\lambda_{\epsilon_1})\otimes\mathcal{F}M(K,\lambda_{\epsilon_2}) = N(t+\phi(K),\epsilon_1)\otimes N(t+\phi(K),\epsilon_2) = \bigoplus_{\gamma(t+\phi(K))=\epsilon_1\epsilon_2} N(0,\gamma) =$$

$$\bigoplus_{\substack{\gamma(t)=1\\ \gamma(\phi(K))=\epsilon_1\epsilon_2}} N(0,\gamma)\bigoplus_{\substack{\gamma(t)=-1\\ \gamma(\phi(K))=-\epsilon_1\epsilon_2}} N(0,\gamma).$$

13. $\mathcal{F}[M(K,\lambda_{\epsilon_1})\otimes M(L,\lambda_{\epsilon_2})] = \bigoplus_{\chi\in\widehat{C_{KL}}}\mathcal{F}M(KL,\chi) = \bigoplus_{\chi\circ\phi^{-1}\in\widehat{\phi(KL)}^\perp} N(\phi(KL),\chi\circ\phi^{-1})$.

$\mathcal{F}M(K,\lambda_{\epsilon_1})\otimes\mathcal{F}M(L,\lambda_{\epsilon_2}) = N(t+\phi(K),\epsilon_1)\otimes N(t+\phi(L),\epsilon_2) = \bigoplus_{\psi\in\widehat{\phi(KL)}^\perp} N(\phi(KL),\psi)$.

Together with Lemma 4.1, we have shown that $\mathcal{F}$ is a $\mathbb{Z}$-algebra isomorphism. □

5 Appendix

5.1 Extraspecial p-groups

Let p be a prime number, and let G be a finite p-group. Denote the center of G as Z, the commutator subgroup as G', and the Frattini subgroup as $\Xi(G)$. Then G is *extraspecial* if $Z = G' = \Xi(G) \cong \mathbb{Z}_p$. As a consequence, $G/Z \cong (\mathbb{Z}_p)^n$ [8].

Let G be extraspecial and consider G/Z to be a vector space over the field $\mathbb{Z}_p$. Let $Z = \langle z \rangle$ and define a function $\langle \, , \rangle : G \times G \to \mathbb{Z}_p$ as follows:

$$\langle x, y \rangle = \alpha, \tag{8}$$

where $xyx^{-1}y^{-1} = z^{\alpha}$. This function passes to the quotient group G/Z, where it becomes a symplectic, nondegenerate, bilinear form [9]. If $p = 2$, then the form is also symmetric. It is immediate that G/Z is even-dimensional. If $x \in Z$, then $(xZ)^{\perp} = Z^{\perp} = G/Z$. Otherwise, if $x \notin Z$, then x is in some noncentral conjugacy class K, and $xZ = K$ because G is extraspecial. Thus we can define $K^{\perp}$ to be $(xZ)^{\perp}$ if $x \in K$.

Lemma 5.1 *Let G be an extraspecial p-group of order p^{2n+1}. Let $x \in K$ and $y \in L$, where K and L are noncentral conjugacy classes of G.*

1. *$|C_G(x)| = p^{2n}$. In particular, $C_G(x)$ is a normal subgroup of G.*

2. *If $K \neq L$, then $|C_G(x) : C_G(x) \cap C_G(y)| = p$.*

Proof:

1. By definition, $C_G(x)/Z = K^{\perp}$, a subspace of G/Z of dimension $2n - 1$.

2. Part 1 implies that $|G : C_G(x)| = p$. If $C_G(x) = C_G(y)$, then $K^{\perp} = L^{\perp}$. But the nondegeneracy of the form implies that $K = L$, contrary to hypothesis. Thus $C_G(x) \neq C_G(y)$. Therefore $G =$

$C_G(x)C_G(y)$, implying

$$G/C_G(x) \cong C_G(x)/C_G(x) \cap C_G(y). \quad \square$$

Lemma 5.2 *Let G_n denote an extraspecial group of order p^{2n+1} for each $n \in \mathbb{N}$ and let $G_0 = \mathbb{Z}_p$. Let $x \in G_n \setminus Z(G_n)$. Then $C_{G_n}(x) \cong \mathbb{Z}_p \times G_{n-1}$ if $x^p = 1$ and $C_{G_n}(x) \cong \mathbb{Z}_{p^2} * G_{n-1}$ if $x^p \neq 1$, where $*$ denotes the central product. In each case, the first factor is generated by x.*

Proof: The result follows inductively from the fact that any extraspecial p-group can be expressed as a central product of extraspecial p-groups of order p^3 [8, Theorem 5.5.2]. $\square$

Let G be extraspecial with $|G| = p^{2n+1}$. Then G has p^{2n} inequivalent one-dimensional irreducible representations and $p-1$ inequivalent p^n-dimensional irreducible representations [8, Theorem 5.5.5].

Lemma 5.3 *Let G be extraspecial with $|G| = p^{2n+1}$ and let χ be an irreducible character of G of dimension greater than one. Then $\chi(g) = 0$ if and only if $g \notin Z$.*

Proof: This result follows from the character table of G. $\square$

Choose a generator z of the center Z of extraspecial G. Let $x \in G \setminus Z$. Then Lemma 5.2 implies that $C_G(x)$ affords exactly $p(p-1)$ inequivalent irreducible representations of dimension p^{n-1} in which z acts as a scalar different from one. Pick an irreducible representation ρ of $C_G(x)$ such that $\rho(z) \neq \mathrm{id}$.

Lemma 5.4 *Let $x \in G \setminus Z$. Then $\operatorname{Tr} \rho(y) = 0$ for all y not contained in the subgroup generated by x and z.*

Proof: If $|G| = p^3$, then the statement is vacuously true. Let $|G| = p^{2n+1}$ with $n > 1$ and choose y not in the subgroup generated by x and z. Then Lemma 5.2 implies that y can be expressed as $x^a g$ for some $a \in \mathbb{Z}_p$ and some $g \in G_{n-1} \setminus Z$. Hence

$$\rho(y) = \rho(x^a)\rho(g) = \zeta^a \rho(g),$$

where ζ is a scalar because $x \in Z(C_G(x))$. Lemma 5.3 now implies that $\operatorname{Tr} \rho(y) = 0$. $\square$

5.2 Group Cohomology and Projective Representations

Let G be a finite group, V a complex vector space, and $\theta \in Z^2(G, \mathbb{C}^*)$ where $\mathbb{C}^*$ is a trivial G-module. Then a projective representation of G on V with 2-cocycle θ (a θ-representation) is equivalent to a linear representation of the central extension X of G by $\mathbb{C}^*$ with associated 2-cocycle θ [10, Exercise 6.10]. Recall that $X = G \times \mathbb{C}^*$ as a set, but has a group operation given by

$$(g_1, a_1)(g_2, a_2) = (g_1 g_2, \theta(g_1, g_2) a_1 a_2)$$

for all $g_i \in G$ and $a_i \in \mathbb{C}^*$. If (ψ, θ) is a projective representation of G, then one defines a linear representation Ψ of X via $\Psi(g, a) = a\psi(g)$ for all $g \in G, a \in \mathbb{C}^*$. Conversely, given a linear representation Ψ of X (with $\Psi(1, a) = a \in \mathbb{C}^*$), one constructs a projective representation (ψ, θ) of G via $\psi(g) = \Psi(g, 1)$. Choosing a cohomologous cocycle θ' instead of θ naturally leads to an isomorphic central extension. Hence, equivalence classes of central extensions by $\mathbb{C}^*$ are in bijective correspondence with $H^2(G, \mathbb{C}^*)$ [11, Theorem 2.1.2].

Theorem 5.5 *Let E be an elementary p-group and let θ be an element of $Z^2(E, \mathbb{C}^*)$ that is not a coboundary. Let X be a central extension of E by $\mathbb{C}^*$ with associated 2-cocycle θ. Then there exists a subgroup $F \lneq E$ and an extraspecial p-group G such that*

$$X \cong F \times (\mathbb{C}^* * G).$$

In the central product, the subgroup $\{\nu \in \mathbb{C}^* \mid \nu^p = 1\}$ is identified with the center of G.

Proof: We begin with the exact sequence defining X,

$$1 \to \mathbb{C}^* \to X \xrightarrow{\pi} E \to 1.$$

Let $F = \pi(Z(X))$. Since θ is not a coboundary, X is not abelian. Thus $F = Z(X)/\mathbb{C}^* \lneq X/\mathbb{C}^* \cong E$. Note that $Z(X) \cong F \times \mathbb{C}^*$ by the injectivity of $\mathbb{C}^*$ [10, Section 3.11]. Since E is elementary abelian, we can choose a complementary subgroup B of E such that $E \cong F \times B$. Let $Y = \pi^{-1}(B) \leq X$. Then

$$X = \pi^{-1}(F \times B) = \pi^{-1}(F)\pi^{-1}(B) = Z(X)Y.$$

Hence $Z(Y) \leq Z(X)$. If $y \in Z(Y)$, then $\pi(y) \in F \cap B = 1$, implying $Z(Y) = \mathbb{C}^*$. We have just shown that $X \cong Z(X) *_{\mathbb{C}} Y$ where $*_{\mathbb{C}}$ means that the central product identifies the $\mathbb{C}^*$ contained in each of these subgroups.

Without loss of generality, assume that θ only takes values that are p-th roots of unity [6]. Therefore, we have that $G = \{(b, \nu) \mid b \in B, \nu^p = 1\}$ is a subgroup of Y. Let $C = \{(1, \nu) \mid \nu^p = 1\} \leq Z(Y) = \mathbb{C}^*$. Direct computation shows that G is an extraspecial p-group with center C. Identifying C with the subgroup $\{\nu \in \mathbb{C}^* \mid \nu^p = 1\}$ of $\mathbb{C}^*$, we obtain $Y \cong \mathbb{C}^* * G$. Hence,

$$X \cong Z(X) *_{\mathbb{C}} (\mathbb{C}^* * G) \cong (F \times \mathbb{C}^*) *_{\mathbb{C}} (\mathbb{C}^* * G) \cong F \times (\mathbb{C}^* * G). \quad \square$$

As an immediate consequence,

$$|F| \cdot |G| = p|E|. \tag{9}$$

Corollary 5.6 *Let E, θ, X, F, and G be as in Theorem 5.5. Let (ψ, θ) be an irreducible projective representation of E. Then $x \in F$ if and only if $\operatorname{Tr}\psi(x) \neq 0$.*

Proof: First, note that $\dim \psi > 1$. Let $\mu \in \mathbb{C}^*$. Assume $x \in F$. Then $(x, \mu) \in Z(X)$. Hence (x, μ) acts as a nonzero scalar in all its irreducible representations, including Ψ. So $\operatorname{Tr}\psi(x) \neq 0$. On the other hand, if $x \notin F$, then $(x, \mu) \notin Z(X)$. Theorem 5.5 implies the existence of f in the isomorphic copy of F contained in X, $\nu \in \mathbb{C}^*$, and $g \in G \setminus Z(G)$ such that $f\nu g = (x, \mu)$. We have

$$\Psi(f\nu g) = \Psi(f\nu)\Psi(g).$$

Since $f\nu \in Z(X)$, $\Psi(f\nu)$ is a scalar matrix $\kappa\,\mathrm{id}$, making the trace of $\Psi(f\nu g)$ equal to $\kappa \operatorname{Tr} \Psi(g)$. Lemma 5.3 now implies that $\operatorname{Tr} \psi(x) = 0$. □

Let $p = 2$ and let $|G| = 2^{2k+1}$. Choose an irreducible projective representation (ψ, θ) of E with $\dim \psi = \dim \Psi$ greater than one. Then $\operatorname{Res}^X_G \Psi$ is the *unique* irreducible representation of G of dimension 2^k. Hence (ψ, θ) is 2^k-dimensional.

Corollary 5.7 *Let E, θ, X, F, and G be as in Theorem 5.5 with $|E| = 2^{2n+1}$. Identify F with its isomorphic copy in X and let $|G| = 2^{2k+1}$ for some k between 1 and n. Then there is a natural bijection between inequivalent irreducible projective θ-representations of E and inequivalent irreducible linear representations of F.*

Proof: Let (ψ, θ) be an irreducible projective representation of E on a complex vector space V. Construct the associated irreducible linear representation Ψ of X. By Clifford's Theorem [10, Section 5.2], we have that $\operatorname{Res}^X_F \Psi$ decomposes into conjugate irreducible representations, each occurring with the same multiplicity. Since F is abelian, $\operatorname{Res}^X_F \Psi$ is a direct sum of copies of a unique one-dimensional irreducible representation $\beta \in \widehat{F}$. We show that $\mathcal{R} : (\psi, \theta) \mapsto \beta$ is the desired natural bijection.

Let (ψ', θ) be another projective representation of E which is projectively equivalent to (ψ, θ). Then there exists an invertible linear transformation f of V such that $\psi' = f\psi f^{-1}$. In other words,

$$\operatorname{Res}^X_F \Psi' = f(\operatorname{Res}^X_F \Psi)f^{-1} = (\dim \Psi) f\beta f^{-1} = (\dim \Psi)\beta.$$

Therefore, $\mathcal{R}$ respects projective equivalence.

Let (ψ_0, θ) be an irreducible projective representation of E which is not equivalent to (ψ, θ). Then Ψ_0 and Ψ are also inequivalent. However, since Ψ_0 and Ψ agree in their restrictions to G, they must differ in their restrictions to F. Thus $\mathcal{R}$ is injective.

Let m be the number of inequivalent projective θ-representations of E. Since each has dimension 2^k, we have $|E| = 2^{2n+1} = m(2^k)^2 = 2^{2k}m$. Hence $m = 2^{2n+1-2k} = |\widehat{F}| = |F|$. $\square$

This allows us to determine the dimension of projective θ-representations from the size of F via

$$\dim \psi = 2^k = \sqrt{\frac{|E|}{|F|}}\ .$$

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