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On the Gauge Equivalence of Twisted Quantum Doubles of Elementary Abelian and Extra-Special 2-Groups

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Abstract

We establish braided tensor equivalences among module categories over the twisted quantum double of a finite group defined by an extension of a group $\overline{G}$ by an abelian group, with 3-cocycle inflated from a 3-cocycle on $\overline{G}$. We also prove that the canonical ribbon structure of the module category of any twisted quantum double of a finite group is preserved by braided tensor equivalences. We give two main applications: first, if G is an extra-special 2-group of width at least 2, we show that the quantum double of G twisted by a 3-cocycle ω is gauge equivalent to a twisted quantum double of an elementary abelian 2-group if, and only if, ω^2 is trivial; second, we discuss the gauge equivalence classes of twisted quantum doubles of groups of order 8, and classify the braided tensor equivalence classes of these quasi-triangular quasi-bialgebras. It turns out that there are exactly 20 such equivalence classes.

1 Introduction

Given a finite group G and a (normalized) 3-cocycle $\omega \in Z^3(G, \mathbb{C}^*)$, there is associated the *twisted quantum double* $D^\omega(G)$. This is a certain braided quasi-Hopf algebra introduced by Dijkgraaf-Pasquier-Roche in the context of orbifold conformal field theory [DPR]. For a holomorphic vertex operator algebra (VOA) V admitting a faithful action of G as automorphisms, one expects that the orbifold VOA V^G has

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a module category V^G-**Mod** which (among other things) carries the structure of a braided tensor category, and that there is an equivalence of braided tensor categories V^G-**Mod** $\cong D^\omega(G)$-**Mod** for some choice of ω. The conjectured equivalence of categories is deep, and motivates the results of the present paper. These, however, are concerned purely with twisted quantum doubles and their module categories. No familiarity with VOA theory is required to understand our main results, and we use the language of VOAs purely for motivation. For further background on the connection between VOAs and quasi-Hopf algebras, see [DPR], [M1], [MN1] and [DM].

There is an interesting phenomenon, akin to mirror symmetry, that arises as follows: we are given two pairs $(V,G),(W,H)$, where V,W are holomorphic VOAs with finite, faithful automorphism groups G,H respectively, together with an isomorphism of VOAs, $V^G \cong W^H$. If the conjectured equivalence of orbifold categories is true, there must also be a tensor equivalence

$$D^\omega(G)\text{-}\mathbf{Mod} \cong D^\eta(H)\text{-}\mathbf{Mod} \tag{1}$$

for some choice of 3-cocycles ω,η on G and H respectively. Conversely, deciding when equivalences such as (1) can hold gives information about the VOAs and the cocycles that they determine. This is an interesting problem in its own right, and is the one we consider here.

The case in which the two twisted doubles in question are *commutative* was treated at length in [MN1]. Here we are concerned with a particular case of the more difficult situation in which the twisted doubles are not commutative. It arises in orbifold theory when one applies the $\mathbb{Z}_2$-orbifold construction to a holomorphic lattice theory V_L (L is a self-dual even lattice) [FLM], [DGM]. One takes G to be the *elementary abelian* 2*-group* $L/2L \oplus \mathbb{Z}_2$, and it turns out that H is an *extra-special* 2-group. We study this situation in the present paper. Changing notation, one of our main results (Theorems 4.6 and 4.7) is as follows:

> Let Q be an extra-special group of order 2^{2l+1} *not* isomorphic to the dihedral group of order 8, and let η be *any* 3-cocycle such that $[\eta] \in H^3(Q,\mathbb{C}^*)$ has order at most 2. Then there are $\mu \in Z^3(E,\mathbb{C}^*)$, $E = \mathbb{Z}_2^{2l+1}$, and a braided tensor equivalence $D^\mu(E)$-**Mod** $\cong D^\eta(Q)$-**Mod**. Such a tensor equivalence does *not* exist if $[\eta]$ has order *greater* than 2. (2)

As is well-known (cf. [EG], [NS2]), such an equivalence of braided tensor categories corresponds to a *gauge equivalence* of the twisted doubles as quasi-triangular quasi-bialgebras. The proof of (2) relies on Theorem 2.1 together with the gauge-invariance of *Frobenius-Schur exponents* (cf. [NS3]) of semisimple quasi-Hopf algebras. In Theorem 2.1 we establish the existence of gauge equivalences of quasi-triangular quasi-bialgebras among twisted quantum doubles $D^\omega(G)$ where G is a group defined as an extension of $\overline{G}$ by an abelian group and the 3-cocycle $\omega \in Z^3(G,\mathbb{C}^*)$ is inflated from a 3-cocycle on $\overline{G}$. Several results of Schauenburg ([S1], [S2]) play a significant rôle in the proof.

As long as $l \geq 2$, the group of elements of $H^3(Q, \mathbb{C}^*)$ of order at most 2 has index 2 in the full degree 3 cohomology ([HK]). The proof of the last assertion of (2) involves some computations involving the cohomology of Q. We use these to calculate (Theorem 4.7) the Frobenius-Schur exponents of $D^\omega(G)$ where G is either extra-special group or elementary abelian, and ω is any 3-cocycle. Frobenius-Schur indicators, their higher analogs, and Frobenius-Schur exponents have been investigated in [MN2], [NS1], [NS2], [NS3] in the general context of semisimple quasi-Hopf algebras and pivotal fusion categories. They provide valuable gauge invariants which are reasonably accessible in the case of twisted quantum doubles.

The case $l = 1$ is exceptional in several ways, and we consider the problem of enumerating the gauge equivalence classes defined by quasi-Hopf algebras obtained by twisting the quantum doubles of $\mathbb{Z}_2^3, Q_8$ and D_8. There are 88 such quasi-Hopf algebras which are noncommutative, corresponding to the 64 nonabelian cohomology classes for $\mathbb{Z}_2^3$, together with 8 classes for Q_8 and 16 for D_8. Some of the subtlety of this problem, which remains open, can be illustrated by observing that *all* 88 twisted doubles have the *same* fusion algebra, moreover some of them are isomorphic as *bialgebras* yet are not gauge equivalent. We will see that there are at least 8, and no more than 20, gauge equivalence classes. This makes use of (2) to identify equivalence classes of quasi-bialgebras, together with Frobenius-Schur indicators and their higher analogs (loc. cit) to distinguish between equivalence classes. Now there is a *canonical braiding* of these quasi-Hopf algebras, and using the *invariance* of the canonical ribbon structure under braided tensor equivalences, which we establish separately, we show that the 20 gauge equivalence classes constitute a complete list of gauge equivalence classes of the quasi-triangular quasi-bialgebras under consideration.

The paper is organized as follows: following some background, in Section 2 we give the proof of Theorem 2.1. There are connections here to some results of Natale [N1]. In Section 3 we discuss a variant of Theorem 2.1 involving bialgebra isomorphisms. In Section 4 we give the proofs of Theorems 4.6 and 4.7. The question of the gauge equivalence classes for twisted doubles of groups of order 8 is presented in Section 5. In Section 6 we show that the twist associated with a pivotal braided tensor category is preserved by tensor equivalences which preserve both pivotal structure and braiding. In particular, the *canonical ribbon structure* of the module category of a semisimple quasi-triangular quasi-Hopf algebra is preserved by braided tensor equivalences. In an Appendix we give a complete set of Frobenius-Schur indicators, their higher analogs, and the scalars of the ribbon structures for the particular quasi-Hopf algebras arising from twisted doubles of groups of order 8. This data is interesting in its own right, and may be useful in the future.

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We use the following conventions: all linear spaces are defined over the complex numbers $\mathbb{C}$ and have finite dimension; all groups are finite; all cocycles for a group G

are defined with respect to a *trivial* G-module and are *normalized*; a *tensor category* is a $\mathbb{C}$-linear monoidal category; a *tensor functor* is a $\mathbb{C}$-linear monoidal functor. We generally do not differentiate between a cocycle and the cohomology class it defines, and often use the isomorphism $H^{n+1}(G,\mathbb{Z}) \cong H^n(G,\mathbb{C}^*)$ without explicit reference to it. Unexplained notation is as in [K].

2 A Family of Gauge Equivalences

We first review some facts about twisted quantum doubles ([DPR], [K]). Fix a group G and 3-cocycle $\omega \in Z^3(G,\mathbb{C}^*)$. $D^\omega(G)$ is a crossed product with underlying linear space $\mathbb{C}G^* \otimes \mathbb{C}G$, where the dual group algebra $\mathbb{C}G^*$ has basis $e(g)$ dual to the group elements $g \in G$. Multiplication, comultiplication, associator, counit, antipode, R-matrix, α and β are given by the following formulas:

$$
\begin{aligned}
&e(g) \otimes x.e(h) \otimes y = \theta_g(x,y)\delta_g^{xhx^{-1}} e(g) \otimes xy,\\
&\Delta(e(g) \otimes x) = \sum_{\substack{h,k\\ hk=g}} \gamma_x(h,k) e(h) \otimes x \otimes e(k) \otimes x,\\
&\Phi = \sum_{g,h,k} \omega(g,h,k)^{-1} e(g) \otimes 1 \otimes e(h) \otimes 1 \otimes e(k) \otimes 1,\\
&\epsilon(e(g) \otimes x) = \delta_g^1,\\
&S(e(g) \otimes x) = \theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1} e(x^{-1}g^{-1}x) \otimes x^{-1},\\
&R = \sum_{g,h} e(g) \otimes 1 \otimes e(h) \otimes g,\\
&\alpha = 1,\ \beta = \sum_g \omega(g,g^{-1},g) e(g) \otimes 1.
\end{aligned}
$$

Here, θ_g, γ_g are 2-cochains on G derived from ω via

$$\theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)}{\omega(x,x^{-1}gx,y)}, \tag{3}$$

$$\gamma_g(x,y) = \frac{\omega(x,y,g)\omega(g,g^{-1}xg,g^{-1}yg)}{\omega(x,g,g^{-1}yg)}. \tag{4}$$

For any quasi-Hopf algebra $H = (H,\Delta,\epsilon,\Phi,\alpha,\beta,S)$ and any unit $u \in H$, we can *twist* H by u to obtain a second quasi-Hopf algebra $H_u = (H,\Delta,\epsilon,\Phi,u\alpha,\beta u^{-1},S_u)$, where $S_u(h) = uS(h)u^{-1}$ for $h \in H$. See [D] for details. We obtain another quasi-Hopf algebra $H_F = (H,\Delta_F,\epsilon,\Phi_F,\alpha_F,\beta_F,S)$ via a *gauge transformation* determined by suitable unit $F \in H \otimes H$ ([K]).

The notion dual to quasi-bialgebra is coquasi-bialgebra (cf. [Ma]). A coquasi-bialgebra $H = (H,\phi)$ is a bialgebra H (although the multiplication is not necessarily associative), together with a convolution-invertible map $\phi : H^{\otimes 3} \to \mathbb{C}$, called *coassociator*, which satisfies the dual conditions of associators. The category ${}^H\mathcal{M}$ of left comodules of a coquasi-bialgebra H is a tensor category with associativity isomorphism determined by the coassociator ϕ. Suppose that K is a Hopf algebra, $\iota : K \to H$ a bialgebra map such that $\phi \circ (\iota \otimes \mathrm{id}_H \otimes \mathrm{id}_H) = \varepsilon_K \otimes \varepsilon \otimes \varepsilon$ (ε is the counit of H), and that there exists a convolution invertible left K-module map $\pi : H \to K$ satisfying $\pi(1) = 1$, $\varepsilon\pi = \varepsilon$ and

$$\phi(g \otimes x \otimes h_{(1)})\pi(h_{(2)}) = \phi(g \otimes x \otimes \pi(h)_{(1)})\pi(h)_{(2)}, \quad \text{for } g, h \in H,\, x \in K.$$

Then by [S2, Corollary 3.4.4], ${}^H_K\mathcal{M}_K$ is a tensor category and there exists a coquasi-bialgebra B such that ${}^B\mathcal{M}$ is equivalent as tensor category to ${}^H_K\mathcal{M}_K$. The structure maps of B are given in [S2, 3.4.2 and 3.4.5]. This result of Schauenburg plays an important rôle in the proof of Theorem 2.1.

For a group G, and 3-cocycle $\omega \in Z^3(G, \mathbb{C}^*)$, the bialgebra $\mathbb{C}G$ together with the *coassociator*

$$\phi = \sum_{x,y,z} \omega^{-1}(x, y, z) e(x) \otimes e(y) \otimes e(z) \in (\mathbb{C}G^{\otimes 3})^* \tag{5}$$

is a *coquasi-bialgebra*. We simply write $\mathbb{C}^\omega G$ for this coquasi-bialgebra. Note that ${}^{\mathbb{C}^\omega G}\mathcal{M}$ is tensor equivalent to the category of modules over the quasi-bialgebra $\mathbb{C}G^*$ equipped with the associator ϕ, and every $\mathbb{C}^\omega G$-comodule is a G-graded vector space. The associativity isomorphism $\Phi : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is given by

$$\Phi(u \otimes v \otimes w) = \omega^{-1}(g, g', g'') u \otimes v \otimes w$$

for homogeneous elements $u \in U$, $v \in V$, $w \in W$ of degree g, g', g'' respectively. Moreover, the center of ${}^{\mathbb{C}^\omega G}\mathcal{M}$ is equivalent to $D^\omega(G)$-**Mod** as braided tensor categories.

Let $\overline{G}$ be a group, A a right $\overline{G}$-module with $\overline{G}$-action $\triangleleft$, and E the semidirect product $A \rtimes \overline{G}$ with underlying set $\overline{G} \times A$ and multiplication

$$(x, a)(y, b) = (xy, (a \triangleleft y)b)$$

for $x, y \in \overline{G}, a, b \in A$. Then E fits into a *split* exact sequence of groups:

$$1 \longrightarrow A \longrightarrow E \longrightarrow \overline{G} \longrightarrow 1\,. \tag{6}$$

The character group $\hat{A} = \mathrm{Hom}(A, \mathbb{C}^*)$ of A admits a left $\overline{G}$-module structure $\triangleright$ given by

$$(h \triangleright \chi)(a) = \chi(a \triangleleft h)\,.$$

Let G be an extension of $\overline{G}$ by $\hat{A}$ associated with the exact sequence of groups:

$$1 \longrightarrow \hat{A} \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1\,. \tag{7}$$

Display (7) determines a 2-cocycle $\epsilon \in Z^2(\overline{G}, \hat{A})$, i.e. a function $\epsilon : \overline{G} \times \overline{G} \to \hat{A}$ satisfying:

$$\epsilon(x, 1_{\overline{G}}) = \epsilon(1_{\overline{G}}, y) = 1_{\hat{A}}\,, \tag{8}$$
$$(x \triangleright \epsilon(y, z))\epsilon(x, yz) = \epsilon(xy, z)\epsilon(x, y) \tag{9}$$

for all $x, y, z \in \overline{G}$. Then G is isomorphic to a group $\hat{A} \rtimes_\epsilon \overline{G}$ which has $\hat{A} \times \overline{G}$ as underlying set and multiplication

$$(\mu, x)(\nu, y) = (\mu(x \triangleright \nu)\epsilon(x, y), xy)$$

for $x, y \in \overline{G}, \mu, \nu \in \hat{A}$. Moreover, the exact sequence of groups

$$1 \longrightarrow \hat{A} \xrightarrow{\iota} \hat{A} \rtimes_\epsilon \overline{G} \xrightarrow{p} \overline{G} \longrightarrow 1 \tag{10}$$

is equivalent to (7), where $\iota : \hat{A} \to \hat{A} \rtimes_\epsilon \overline{G}$ and $p : \hat{A} \rtimes_\epsilon \overline{G} \to \overline{G}$ are given by

$$\iota(\mu) = (\mu, 1), \quad \text{and} \quad p(\mu, x) = x$$

for $\mu \in \hat{A}, x \in \overline{G}$.

We are going to compare twisted quantum doubles of $G = \hat{A} \rtimes_\epsilon \overline{G}$ and $E = A \rtimes \overline{G}$. Suppose that $\zeta \in Z^3(\overline{G}, \mathbb{C}^*)$. We may *inflate* ζ along the projections $E \to \overline{G}, G \to \overline{G}$ in (6) and (10) to obtain 3-cocycles

$$\begin{aligned} \zeta_E &= \mathrm{Infl}_{\overline{G}}^{E} \zeta \in Z^3(E, \mathbb{C}^*), \\ \zeta_G &= \mathrm{Infl}_{\overline{G}}^{G} \zeta \in Z^3(G, \mathbb{C}^*). \end{aligned}$$

One can also check directly that the 3-cochain ω on E defined by

$$\omega((h_1, a_1), (h_2, a_2), (h_3, a_3)) = \epsilon(h_2, h_3)(a_1), \ h_i \in \overline{G}, a_i \in A, \tag{11}$$

actually lies in $Z^3(E, \mathbb{C}^*)$. Moreover, for any $\sigma \in \mathrm{Aut}_{\overline{G}}(A)$,

$$\sigma\omega((h_1, a_1), (h_2, a_2), (h_3, a_3)) = \epsilon(h_2, h_3)(\sigma(a_1))$$

also defines a normalized 3-cocycle on E, and we let $\Omega = \{\sigma\omega \mid \sigma \in \mathrm{Aut}_{\overline{G}}(A)\}$.

We can now state our first main result. Notation and assumptions are as above.

Theorem 2.1. *Let $\zeta \in Z^3(\overline{G}, \mathbb{C}^*)$ and $\omega' \in \Omega$. Then the braided tensor categories $D^{\zeta_G}(G)$-**Mod** and $D^{\omega'\zeta_E}(E)$-**Mod** are equivalent.*

Proof. Bearing in mind the definition $G = \hat{A} \rtimes_\epsilon \overline{G}$, it follows from the definition of inflated cocycle that

$$\zeta_G((\mu, x), (\mu', x'), (\mu'', x'')) = \zeta(x, x', x'')$$

for $(\mu, x), (\mu', x'), (\mu'', x'') \in G$.

Consider the coquasi-bialgebra $H = \mathbb{C}^{\zeta_G} G$ with coassociator ϕ as in (5), and the Hopf algebra $K = \mathbb{C}\hat{A}$. Let $\sigma \in \mathrm{Aut}_{\overline{G}}(A)$. We define $\mathbb{C}$-linear maps $\iota_\sigma : K \to H$ by $\iota_\sigma(\mu) = (\mu \circ \sigma^{-1}, 1)$ for $\mu \in \hat{A}$, and $\pi_\sigma : H \to K$ by $\pi_\sigma(\mu, x) = \mu \circ \sigma$ for $(\mu, x) \in G$. It is easy to see that ι_σ is a bialgebra map which satisfies

$$\phi \circ (\iota_\sigma \otimes \mathrm{id}_H \otimes \mathrm{id}_H) = \varepsilon_K \otimes \varepsilon \otimes \varepsilon\,.$$

Furthermore, π_σ is a K-module coalgebra map and is convolution invertible with inverse $\bar{\pi}_\sigma : H \to K$ given by $\bar{\pi}_\sigma(\mu, x) = \mu^{-1} \circ \sigma$. Clearly, $\pi_\sigma(1_H) = 1_K$ and $\varepsilon_K \circ \pi_\sigma = \varepsilon_H$. By the definitions of ϕ and ζ_G, for $g, h \in G$ and $\mu \in \hat{A}$, we have

$$\phi(g \otimes (\mu, 1) \otimes h)\pi_\sigma(h) = \pi_\sigma(h) = \phi(g \otimes (\mu, 1) \otimes \pi_\sigma(h))\pi_\sigma(h)\,.$$

It follows from [S2, Corollary 3.4.4] that there exists a coquasi-bialgebra B such that

$$ {}^H_K\mathcal{M}_K \cong {}^B\mathcal{M}$$

as tensor categories. By [S1], we also have the equivalence

$$\mathcal{Z}({}^H_K\mathcal{M}_K) \cong \mathcal{Z}({}^H\mathcal{M})$$

of braided tensor categories, where $\mathcal{Z}(\mathcal{C})$ denotes the center of the monoidal category $\mathcal{C}$. Since $D^{\zeta_G}(G)$-**Mod** and $\mathcal{Z}({}^H\mathcal{M})$ are equivalent braided tensor categories (cf. [Ma1]), we have

$$D^{\zeta_G}(G)\text{-}\mathbf{Mod} \cong \mathcal{Z}({}^B\mathcal{M})$$

as braided tensor categories.

We proceed to show that $B \cong \mathbb{C}^{\omega'}E$ using [S2, 3.4.2 and 3.4.5], where $\omega' = \sigma\tau\omega \cdot \zeta_E$ and $\tau : A \to A$, $\tau : a \mapsto a^{-1}$. We will use the notation defined in [S2] for the remaining discussion. By [S2, Lemma 3.4.2], $B = \mathbb{C}\overline{G} \otimes K^*$ as vector space, and we write $x \rtimes a$ for $x \otimes a$ whenever $x \in \overline{G}$ and $a \in A$. Here, we have used the canonical identification of the Hopf algebra K^* with $\mathbb{C}A$. Note that $\mathbb{C}\overline{G} \cong H/K^+H$ as coalgebras. Since $\iota_\sigma(\hat{A})$ is a normal subgroup of G, $K^+H = HK^+$. Thus, the right K-action on H/K^+H is trivial, and so the associated left K^*-comodule structure on $\mathbb{C}\overline{G}$ is also trivial. Hence, $\tilde{\rho}(q) = 1_A \otimes q$ for $q \in \mathbb{C}\overline{G}$. For any $x \in \overline{G}$, $\mu, \nu \in \hat{A}$,

$$\phi((1, x) \otimes (\mu, 1) \otimes (\nu, 1)) = 1\,.$$

So the $\tilde{\alpha} : \mathbb{C}\overline{G} \to K^* \otimes K^*$ defined in [S2, Lemma 3.4.2] for our context is given by $\tilde{\alpha} : x \mapsto 1 \otimes 1$. Thus, the comultiplication $\tilde{\Delta}$ and counit $\tilde{\varepsilon}$ of B are given by

$$\tilde{\Delta}(x \rtimes a) = x \rtimes a \otimes x \rtimes a \quad \text{and} \quad \tilde{\varepsilon}(x \rtimes a) = 1$$

for $x \in \overline{G}$ and $a \in A$. With the *cleaving* map π_σ, the associated map $j_\sigma : \mathbb{C}\overline{G} \to H$ is given by $j_\sigma(x) = (1, x)$ for $x \in \overline{G}$, and so

$$x \rightharpoonup \mu = \pi_\sigma(j_\sigma(x)(\mu \circ \sigma, 1)) = (x \triangleright (\mu \circ \sigma)) \circ \sigma^{-1} = x \triangleright \mu$$

for $\mu \in \hat{A}$. Thus for $a \in A$, $x \in \overline{G}$ and $\mu \in \hat{A}$, we have

$$\mu(a \overset{\sim}{\leftharpoonup} x) = (x \rightharpoonup \mu)(a) = (x \triangleright \mu)(a) = \mu(a \triangleleft x)$$

and hence

$$a \overset{\sim}{\leftharpoonup} x = a \triangleleft x\,.$$

Recall that

$$e(a) = \frac{1}{|A|}\sum_{\chi\in\hat{A}} \frac{1}{\chi(a)}\chi\,, \quad a \in A,$$

form a dual basis of A. It follows from [S2, Theorem 3.4.5] that the multiplication on B is given by

$$\begin{aligned}
(x \rtimes a)\cdot(y \rtimes b) &= \sum_{c\in A} \overline{j_\sigma(x)j_\sigma(y)} \rtimes c(a \stackrel{\sim}{-} y)b\phi(j_\sigma(x)\otimes j_\sigma(y)\otimes \iota_\sigma(e(c)))\pi_\sigma(j_\sigma(y))(a) \\
&= \frac{1}{|A|}\sum_{c\in A} xy \rtimes c(a \triangleleft y)b \sum_{\chi\in\hat{A}} \frac{1}{\chi(c)}\phi((1,x)\otimes(1,y)\otimes(\chi\circ\sigma^{-1},1)) \\
&= \frac{1}{|A|}\sum_{c\in A} xy \rtimes c(a \triangleleft y)b|A|\delta_c^1 \\
&= xy \rtimes (a \triangleleft y)b,
\end{aligned}$$

and the coassociator is

$$\begin{aligned}
\tilde{\phi}(x \rtimes a \otimes y \rtimes b \otimes z \rtimes c) &= \phi(j_\sigma(x)\otimes j_\sigma(y)\otimes j_\sigma(z))\pi_\sigma(j_\sigma(y)j_\sigma(z))(a)\pi_\sigma(j_\sigma(z))(b)\varepsilon(c) \\
&= \zeta_G((1,x),(1,y),(1,z))^{-1}\pi_\sigma(\epsilon(y,z),yz)(a) \\
&= \zeta(x,y,z)^{-1}\epsilon(y,z)(\sigma a) \\
&= \zeta(x,y,z)^{-1}\epsilon(y,z)(\sigma\tau a)^{-1} \\
&= \zeta_E((x,a),(y,b),(z,c))^{-1}\sigma\tau\omega((x,a),(y,b),(z,c))^{-1} \\
&= \omega'((x,a),(y,b),(z,c))^{-1}\,.
\end{aligned}$$

Thus, the map $x \rtimes a \mapsto (x,a)$ defines an isomorphism of coquasi-bialgebras from B to $\mathbb{C}^{\omega'}E$.

Now we have equivalences

$$D^{\zeta_G}(G)\text{-}\mathbf{Mod} \cong \mathcal{Z}({}^{B}\mathcal{M}) \cong \mathcal{Z}({}^{\mathbb{C}^{\omega'}E}\mathcal{M}) \cong D^{\sigma\tau\omega\zeta_E}(E)\text{-}\mathbf{Mod}$$

of braided tensor categories. Since σ is arbitrary, $\sigma\tau\omega$ can be any element of Ω. This completes the proof of the Theorem. □

Remark 2.2. *If A is a trivial H-module, one can verify directly via rather extensive computation that*

$$F = \sum_{\substack{p,q\in H\\ \mu,\nu\in\widehat{A}}} e(p,\mu)\otimes(1,\nu^{-1})\otimes e(q,\nu)\otimes(1,1), \tag{12}$$

is a gauge transformation in $D^{\zeta_G}(G)\otimes D^{\zeta_G}(G)$, and that $\varphi\colon D^{\omega\zeta_E}(E)\to D^{\zeta_G}(G)_{F,u}$, given by

$$\varphi : e(h,a)\otimes(k,b) \mapsto \frac{\epsilon(h,k)(b)}{\epsilon(k,k^{-1}hk)(b)}\cdot\frac{1}{|A|}\sum_{\chi,\psi\in\widehat{A}}\frac{\chi(b)}{\psi(a)}e(h,\chi)\otimes(k,\psi) \tag{13}$$

with unit

$$u=\sum_{\substack{p\in H\\ \mu\in\widehat{A}}} e(p,\mu)\otimes(1,\mu), \tag{14}$$

is an isomorphism of quasi-triangular quasi-bialgebras. We also note that the special case of Theorem 2.1 in which ζ is trivial can be deduced without difficulty from some results of Natale [N1].

3 An Isomorphism of Bialgebras

We next consider a variant of Theorem 2.1, which however is more limited in scope. We take G to be a group with $N \trianglelefteq G$ a normal subgroup of *index* 2. Denote the projection onto the quotient as

$$G \longrightarrow G/N, \quad x \mapsto \overline{x}.$$

Let $\eta \in Z^3(G/N,\mathbb{C}^*)$ be the 3-cocycle which represents the nontrivial cohomology class of $G/N=\mathbb{Z}_2$, so that

$$\eta(\overline{x},\overline{y},\overline{z})=\begin{cases} -1 & \text{if } \overline{x},\overline{y},\overline{z}\neq 1,\\ 1 & \text{otherwise.}\end{cases}$$

η is certainly an abelian 3-cocycle, that is the associated 2-cocycles $\hat{\theta}_{\overline{g}}$ are *coboundaries*. Thus

$$\hat{\theta}_{\overline{g}}(\overline{x},\overline{y})=f_{\overline{g}}(\overline{x})f_{\overline{g}}(\overline{y})/f_{\overline{g}}(\overline{xy})=\begin{cases} -1 & \text{if } \overline{g},\overline{x},\overline{y}\neq 1,\\ 1 & \text{otherwise,}\end{cases}$$

where $f_{\overline{g}}$ satisfies

$$f_{\overline{g}}(\overline{x})=\begin{cases} i & \text{if } \overline{g},\overline{x}\neq 1,\\ 1 & \text{otherwise.}\end{cases}$$

Theorem 3.1. *Let the notation be as above, with $\eta'=\mathrm{Infl}_{G/N}^{G}\,\eta$ the inflation of η to G. For any $\omega\in Z^3(G,\mathbb{C}^*)$, the map $\phi: D^{\omega}(G)\to D^{\omega\eta'}(G)$ given by*

$$\phi: e(g)\otimes x\mapsto f_{\overline{g}}(\overline{x})^{-1}e(g)\otimes x$$

is an isomorphism of bialgebras.

Proof. Let θ_g,θ'_g and γ_g,γ'_g denote the 2-cochains (which are in fact 2-cocycles) associated with ω and $\omega\eta'$ respectively (cf. (3), (4)). Clearly

$$\theta'_g(x,y)=\theta_g(x,y)\hat{\theta}_{\overline{g}}(\overline{x},\overline{y}).$$

Similarly,

$$\gamma'_g(x,y)=\gamma_g(x,y)\hat{\theta}_{\overline{g}}(\overline{x},\overline{y})^{-1}.$$

Then for any $g, x, h, y \in G$, we have

$$
\begin{aligned}
\phi((e(g) \otimes x) \cdot (e(h) \otimes y)) &= \theta_g(x,y)\delta^h_{g^x}\phi(e(g) \otimes xy) \\
&= \theta_g(x,y) f_{\overline{g}}(\overline{xy})^{-1}\delta^h_{g^x} e(g) \otimes xy \\
&= \theta_g(x,y)\hat{\theta}_{\overline{g}}(\overline{x},\overline{y}) f_{\overline{g}}(\overline{x})^{-1} f_{\overline{g}}(\overline{y})^{-1}\delta^h_{g^x} e(g) \otimes xy \\
&= \theta'_g(x,y) f_{\overline{g}}(\overline{x})^{-1} f_{\overline{h}}(\overline{y})^{-1}\delta^h_{g^x} e(g) \otimes xy \\
&= \phi(e(g) \otimes x) \cdot \phi(e(h) \otimes y).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(\phi \otimes \phi)\Delta(e(g) \otimes x) &= \sum_{ab=g} \gamma_x(a,b)\phi(e(a) \otimes x) \otimes \phi(e(b) \otimes x) \\
&= \sum_{ab=g} \gamma_x(a,b) f_{\overline{a}}(\overline{x})^{-1} f_{\overline{b}}(\overline{x})^{-1} e(a) \otimes x \otimes e(b) \otimes x \\
&= \sum_{ab=g} \gamma_x(a,b) f_{\overline{x}}(\overline{a})^{-1} f_{\overline{x}}(\overline{b})^{-1} e(a) \otimes x \otimes e(b) \otimes x \\
&= \sum_{ab=g} \gamma_x(a,b) f_{\overline{x}}(\overline{g})^{-1} \hat{\theta}_{\overline{x}}(\overline{a},\overline{b})^{-1} e(a) \otimes x \otimes e(b) \otimes x \\
&= \sum_{ab=g} \gamma'_x(a,b) f_{\overline{g}}(\overline{x})^{-1} e(a) \otimes x \otimes e(b) \otimes x \\
&= \Delta(\phi(e(g) \otimes x)). \quad \square
\end{aligned}
$$

Despite its similarity to Theorem 2.1, we will later show that Theorem 3.1 cannot be improved in the sense that the two twisted doubles that occur are generally *not* gauge equivalent.

Corollary 3.2. *Let notation be as in Theorem 3.1. Then $D^\omega(G)$ and $D^{\omega\eta'}(G)$ have isomorphic fusion algebras.*

4 Extra-Special 2-Groups

Let $l \geq 1$ be an integer and $V \cong \mathbb{Z}_2^{2l}$. Throughout this Section, Q denotes an *extra-special* 2-group, defined via a central extension of groups

$$1 \longrightarrow Z \longrightarrow Q \xrightarrow{\pi} V \longrightarrow 1 \tag{15}$$

together with the additional constraint that $\mathbb{Z}_2 \cong Z = Z(Q)$. Then Z coincides with the commutator subgroup Q' and $|Q| = 2^{2l+1}$. Note that Q has exponent exactly 4. l is called the *width* of Q, and for each value of l, Q is isomorphic to one of just two groups, denoted by Q_+, Q_-. If $l = 1$ then $Q_+ \cong D_8$ (dihedral group) and $Q_- \cong Q_8$ (quaternion group). In general Q_+ is isomorphic to the central product of l copies of D_8 and Q_- to the central product of $l-1$ copies of D_8 and one copy of Q_8. The squaring map $x \mapsto x^2, x \in Q$, induces a nondegenerate quadratic form $q : V \to \mathbb{F}_2$, and Q has type $+$ or $-$ according as the Witt index w of q is l or $l-1$ respectively.

We need to develop some facts concerning $H^3(Q, \mathbb{C}^*)$. For background on the cohomology of extra-special groups one may consult [Q], [HK] and [BC]. We use the following notation: for an abelian group B, $\Omega B = \{b \in B | b^2 = 1\}$; ΩB is an elementary abelian 2-group.

Lemma 4.1. *The following hold:*

$$a) H^3(Q, \mathbb{C}^*) = \mathbb{Z}_2^N \oplus \mathbb{Z}_4 \ (Q \ncong Q_8),\ N = \binom{2l}{1} + \binom{2l}{2} + \binom{2l}{3} - 1;$$
$$b)\ H^3(D_8, \mathbb{C}^*) = \mathbb{Z}_2^2 \oplus \mathbb{Z}_4;$$
$$c)\ H^3(Q_8, \mathbb{C}^*) = \mathbb{Z}_8;$$

Proof. Part a) is proved in [HK], and b) is a special case. It is well-known ([CE], Theorem 11.6) that Q_8 has periodic cohomology. In particular $H^4(Q_8, \mathbb{Z})$ is cyclic of order 8, and c) holds. □

Lemma 4.2. *Suppose that $Q \ncong D_8$. Then*

$$\Omega H^3(Q, \mathbb{C}^*) = \mathrm{Infl}_V^Q H^3(V, \mathbb{C}^*).$$

Proof. Let β and β' be the connecting maps (Bocksteins) associated with the short exact sequences of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{F}_2 \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{exp} \mathbb{S}^1 \longrightarrow 1.$$

From the associated long exact sequences in cohomology we obtain a commuting diagram

$$\begin{array}{ccccc} H^3(Q, \mathbb{F}_2) & \xrightarrow{\beta_Q} & \Omega H^4(Q, \mathbb{Z}) & \xleftarrow{\beta'} & \Omega H^3(Q, \mathbb{C}^*) \\ \uparrow & & \uparrow & & \uparrow \\ H^3(V, \mathbb{F}_2) & \xrightarrow{\beta_V} & H^4(V, \mathbb{Z}) & \xleftarrow{\beta'} & H^3(V, \mathbb{C}^*) \end{array} \qquad (16)$$

where vertical maps are inflations, β' is an isomorphism, and β a surjection. It therefore suffices to show that the leftmost vertical map is surjective. In [Q], Quillen describes the cohomology ring $H^*(Q, \mathbb{F}_2)$ as a tensor product

$$H^*(Q, \mathbb{F}_2) = S(V)/J \otimes \mathbb{F}_2[\zeta]$$

where the left factor $S(V)/J$ (a certain quotient of the cohomology $H^*(V, \mathbb{F}_2) = S(V)$) coincides with $\mathrm{Infl}_V^Q H^*(V, \mathbb{F}_2)$, and ζ is a cohomology class of degree 2^{2l-w}. From this we see that if the leftmost vertical map in (16) is *not* surjective then $\mathbb{F}_2[\zeta]$ has a nonzero element of degree 2. Then $2l - w \leq 1$, and this can only happen if $l = w = 1$, i.e. $Q \cong D_8$. This completes the proof of the Lemma. □

There is a distinguished element $\nu \in H^3(V, \mathbb{F}_2)$ that we will need. Namely, pick a basis of V and let $\lambda_1, ..., \lambda_{2l} \in \mathrm{Hom}(V, \mathbb{F}_2)$ be the dual basis. Let the quadratic form q corresponding to Q be $q = \sum_{i,j} a_{ij}x_i x_j$, and set

$$\nu = \sum_{i,j} a_{ij}\lambda_i^2\lambda_j \in S^3(V) = H^3(V, \mathbb{F}_2). \tag{17}$$

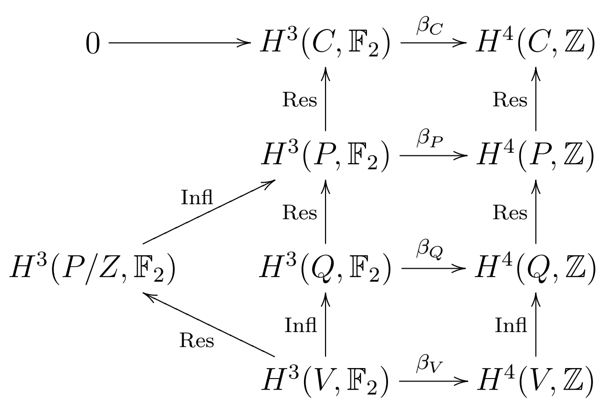
Lemma 4.3. *Suppose that $Q \ncong D_8$. Then $H^4(Q, \mathbb{Z}) \cap$ rad $H(Q, \mathbb{Z})$ is cyclic, and the unique element of order 2 that it contains is* $\mathrm{Infl}_V^Q \beta_V(\nu) = \beta_Q(\mathrm{Infl}_V^Q \nu)$.

Proof. From Lemma 4.2 we know that any element $\zeta \in \Omega H^4(Q, \mathbb{Z}) \cap \mathrm{rad}\ H(Q, \mathbb{Z})$ satisfies

$$\zeta = \beta_Q(\mathrm{Infl}_V^Q \delta) = \mathrm{Infl}_V^Q \beta_V(\delta),$$
$$\mathrm{Res}_C^Q \zeta = 0, \forall\ C \subseteq Q \text{ of order } 2,$$

for some $\delta \in H^3(V, \mathbb{F}_2)$.

We claim that $\mathrm{Res}_D^V(\delta) = 0$ for every subgroup $D \subseteq V$ of order 2 generated by a *singular vector* x (i.e. $q(x) = 0$). First note that with the notation of (15), the singularity of x means that $P = \pi^{-1}D$ is the direct product of Z and C for some subgroup $C \subseteq Q$ of order 2. Consider the commutative diagram



Since $\mathrm{Res}_C^Q \circ \beta_Q \circ \mathrm{Infl}_V^Q(\delta) = \mathrm{Res}_C^Q \zeta = 0$, we find $\mathrm{Res}_C^Q \circ \mathrm{Infl}_V^Q(\delta) = 0$ by the injectivity of β_C and the commutative diagram. Notice that the composition $\mathrm{Res}_C^P \circ \mathrm{Infl}_D^P$ is an isomorphism. It follows from the commutative diagram that $\mathrm{Res}_D^V(\delta) = 0$.

The classes δ with this property are spanned by the elements $\lambda_i^2\lambda_j + \lambda_i\lambda_j^2$ together with ν. However, the Bockstein annihilates all of these elements with the exception of ν. This proves that $\Omega H^4(Q, \mathbb{Z}) \cap \mathrm{rad}\ H(Q, \mathbb{Z})$ is spanned by $\beta_Q(\mathrm{Infl}_V^Q \nu)$, and in particular $H^4(Q, \mathbb{Z}) \cap \mathrm{rad}\ H(Q, \mathbb{Z})$ is cyclic. □

Lemma 4.4. *Let $\zeta \in H^3(D_8, \mathbb{C}^*)$ have order 4, and set $I = \mathrm{Infl}_V^{D_8} H^3(V, \mathbb{C}^*)$. Then*

$$H^3(D_8, \mathbb{C}^*) = \langle\zeta\rangle \times I.$$

Proof. In the case $Q = D_8$, the proof of Lemma 4.2 shows that $|I| = 4$, so after Lemma 4.1b) it suffices to show that I contains no nonidentity squares. But it is easily checked (cf. [M2]) that the three nonzero elements of I are inflated from elements $\delta \in H^3(V, \mathbb{C}^*)$ with the property that δ restricts nontrivially to exactly two subgroups of V of order 2. Then $\mathrm{Infl}_V^{D_8} \delta$ must restrict nontrivially to some order 2 subgroup of D_8, hence cannot be a radical element. This completes the proof of the Lemma. □

Lemma 4.5. *Let $C \subseteq Q$ be cyclic of order 4, and $R = \mathrm{Res}_C^Q H^3(Q, \mathbb{C}^*)$. Then*

$$|R| = \begin{cases} 2 & \text{if } l \geq 2, \\ 4 & \text{if } l = 1. \end{cases}$$

Proof. First assume that $Q \cong Q_8$. Then a generator of $H^3(Q, \mathbb{C}^*)$ (cf. Lemma 4.1a)) restricts to a generator of $H^3(C, \mathbb{C}^*)$ for any subgroup $C \subseteq Q$ ([CE]), and in particular $|R| = 4$ in this case.

Next assume that $l \geq 2$. There is a unique nontrivial square in $H^3(Q, \mathbb{C}^*)$, call it η. Then η is described in Lemma 4.3, and that result shows that for any subgroup $Q_1 \subseteq Q$ satisfying $Q_1 \cong Q_8$, the element $\mathrm{Res}_{Q_1}^Q \eta$ is *nontrivial*. Now let $\zeta \in H^3(Q, \mathbb{C}^*)$ have order 4. Then $\zeta^2 = \eta$, and it follows from what we have said that $\mathrm{Res}_{Q_1}^Q \zeta$ continues to have order 4. From the first paragraph of the proof it follows that $\mathrm{Res}_C^Q \zeta$ has order 2 whenever $C \subseteq Q_1$ is cyclic of order 4. On the other hand, since $Q \not\cong D_8$, any cyclic subgroup of Q of order 4 is contained in some Q_1. This completes the proof of the Lemma in the case $l \geq 2$.

It remains to deal with the case $Q \cong D_8$. We use an argument based on the Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence

$$1 \to \mathbb{Z}_4 \to D_8 \to \mathbb{Z}_2 \to 1.$$

The E_2-term is

$$E_2^{pq} = H^p(\mathbb{Z}_2, H^q(\mathbb{Z}_4, \mathbb{Z})).$$

Now

$$\begin{aligned} E_2^{31} &= E_2^{13} = 0, \\ E_2^{04} &= H^4(\mathbb{Z}_4, \mathbb{Z})^{\mathbb{Z}_2} = H^4(\mathbb{Z}_4, \mathbb{Z}) = \mathbb{Z}_4, \\ E_2^{22} &= H^2(\mathbb{Z}_2, H^2(\mathbb{Z}_4, \mathbb{Z})) = \mathbb{Z}_2, \\ E_2^{40} &= H^4(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2. \end{aligned}$$

Using Lemma 4.1b) we see that $|H^4(D_8, \mathbb{Z})| = |E_2^{04}||E_2^{22}||E_2^{40}| = 16$, whence $E_2^{04} = E_\infty^{04}, E_2^{22} = E_\infty^{22}, E_2^{40} = E_\infty^{40}$. In particular

$$\mathrm{Im}(\mathrm{Res}\colon H^4(D_8, \mathbb{Z}) \to H^4(\mathbb{Z}_4, \mathbb{Z})) = E_\infty^{04} = H^4(\mathbb{Z}_4, \mathbb{Z}).$$

This completes the proof of the Lemma. □

We now apply these results to twisted quantum doubles. To begin with, by combining Lemma 4.2 with Theorem 2.1 we immediately obtain

Theorem 4.6. *Let Q be an extra-special 2-group of width l with $Q \ncong D_8$, and let E be an elementary abelian 2-group of order 2^{2l+1}. For any $\eta \in \Omega H^3(Q, \mathbb{C}^*)$, there are $\mu \in H^3(E, \mathbb{C}^*)$ and an equivalence of braided tensor categories*

$$D^{\mu}(E)\text{-}\mathbf{Mod} \cong D^{\eta}(Q)\text{-}\mathbf{Mod}. \tag{18}$$

After the last Theorem the question arises: what can one say about $D^{\eta}(Q)$ in the case where η has order *greater* than 2? We will show that an equivalence of the type (18) is not possible in this case. For this we need to compute some gauge invariants of the twisted quantum doubles. We use the *Frobenius-Schur exponent*, which is particularly accessible.

First recall ([MN2], [NS1], [NS2]) the Frobenius-Schur index $\nu(\chi)$ and their higher order analogs $\nu^{(n)}(\chi)$ associated to an irreducible character χ of a quasi-Hopf algebra and a positive integer n. These are indeed gauge invariants (loc. cit.) The smallest positive integer f such that $\nu_{\chi}^{(f)} = \chi(1)$ for all irreducible characters χ of $D^{\omega}(G)$ is called the the *Frobenius-Schur exponent* of $D^{\omega}(G)$. The existence of f for general quasi-Hopf algebras is shown in [NS3], and it is therefore also a gauge invariant. The following explicit formula for the Frobenius-Schur exponent $f = f(G, \omega)$ of a twisted quantum double $D^{\omega}(G)$ is known (loc. cit.):

$$f = \mathrm{lcm}\,(\ |C|\ |\omega_C|\ )\,. \tag{19}$$

The notation is as follows: C ranges over cyclic subgroups of G, ω_C is the restriction of ω to C, and $|\omega_C|$ is the order of the corresponding cohomology class. We note that in [N2], Natale defined the *modified exponent* $\exp_{\omega} G$ of a group G endowed with a 3-cocycle ω by same formula (19), and proved that $\exp_{\omega} G$ is a gauge invariant of $D^{\omega}(G)$ for groups of odd order. It has been proved in [NS3] that f is always a gauge invariant of $D^{\omega}(G)$ and that $f = \exp(D^{\omega}(G))$ or $2\exp(D^{\omega}(G))$.

We now have

Theorem 4.7. *The following hold for* any 3-*cocycle ω on one of the indicated groups (Q is an extra-special group of width l):*

$$a)\ f(\mathbb{Z}_2^l, \omega) = \begin{cases} 4 & \text{if } \omega \neq 1, \\ 2 & \text{if } \omega = 1,\ l \geq 1; \end{cases}$$

$$b)\ f(Q, \omega) = \begin{cases} 4 & \text{if } \omega^2 = 1,\ l \geq 2, \\ 8 & \text{if } \omega^2 \neq 1,\ l \geq 2; \end{cases}$$

$$c)\ f(Q_8, \omega) = \begin{cases} 4 & \text{if } \omega^2 = 1, \\ 8 & \text{if } \omega \text{ has order } 4, \\ 16 & \text{if } \omega \text{ has order } 8; \end{cases}$$

$$d)\ f(D_8, \omega) = \begin{cases} 4 & \text{if } \omega \in \mathrm{Infl}_V^{D_8} H^3(V, \mathbb{C}^*), \\ 8 & \text{if } \omega \in \Omega H^3(D_8, \mathbb{C}^*) \setminus \mathrm{Infl}_V^{D_8} H^3(V, \mathbb{C}^*), \\ 16 & \text{if } \omega \text{ has order } 4. \end{cases}$$

Proof. If ω is trivial then by (19) f coincides with the exponent of the group in question. Suppose that ω has order 2 in the elementary abelian 2-group case. Then from [M2] that there is a subgroup C of order 2 such that ω_C is nontrivial. Part a) now follows. Parts b) and c) follow immediately from Lemma 4.5.

It remains to handle the case $Q = D_8$. Let C be the cyclic subgroup of order 4 and let $I = \mathrm{Infl}_V^{D_8} H^3(V, \mathbb{C}^*)$. By Lemma 4.4 we have $H^3(D_8, \mathbb{C}^*) = I\langle\zeta\rangle$ for any ζ of order 4. If $\omega \in I$ then ω_C is necessarily trivial, and therefore $f(D_8, \omega) = 4$ for such ω. If ω has order 4 then ω_C also has order 4 (Lemma 4.5), so $f = 16$ in this case. Finally, ζ_C^2 has order 2 for ζ of order 4. Since $\zeta^2 \notin I$ we conclude that ω_C has order 2 whenever $\omega \in \Omega H^3(D_8, \mathbb{C}^*) \setminus I$, so that $f = 8$ in this case. This completes the proof of all parts of the Theorem. □

5 Twisted Quantum Doubles of Dimension 64

We have seen in the previous Section that the properties of the groups $H^3(Q, \mathbb{C}^*)$ for $Q = Q_8, D_8$ are exceptional in several ways. Thus the same is true for the corresponding twisted quantum doubles. In this Section we consider the problem of understanding the gauge equivalence classes of the quantum doubles $D^\omega(G)$ where G has order 8. In [MN1], the same question was treated for *abelian* groups G and *abelian* cocycles, that is cocycles ω for which $D^\omega(G)$ is commutative. Here we consider the case when $D^\omega(G)$ is *noncommutative*. This precludes the two groups $\mathbb{Z}_8$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$, leaving the groups $E_8 = \mathbb{Z}_2^3, Q_8$ and D_8 to be considered. In this Section we denote these three groups by E, Q and D respectively. Now E has 64 nonabelian 3-cohomology classes, while D and Q have 16 and 8 classes respectively (cf. Remark 4.1). So our task is to sort the 88 resulting twisted quantum doubles into gauge equivalence classes. ~~This remains out of reach at present.~~ We will see that there are at least 8, and no more than 20, gauge equivalence classes.

A significant reduction in the problem is achieved by considering automorphisms of the three groups in question. That is because it is easy to see ([MN1], Remark 2.1(iii)) that automorphisms preserve gauge equivalence classes. Consider first the group E. One knows that $H = H^3(E, \mathbb{C}^*)$ has order 2^7 and exponent 2. Of the 128 cohomology classes in H, the abelian classes form a subgroup of order 64 generated by the Chern classes of characters of E ([MN1], Proposition 7.5).

The automorphism group of E is the simple group $SL(3,2)$ of order 168, and we regard H as a 7-dimensional $\mathbb{F}_2 SL(3,2)$-module. We can understand the structure of H as follows (for more details, see [M2]). A cohomology class $\omega \in H$ is characterized by the subset of elements $1 \neq g \in E$ for which the restriction $\mathrm{Res}_{\langle g\rangle}^E \omega$ is *nontrivial*. Let us call this set of elements the *support* of ω, denoted supp ω. The *weight* of ω is the cardinality $|\text{supp } \omega|$ of its support. There is an isomorphism of $SL(3,2)$-modules

$$H \xrightarrow{\cong} \mathbb{F}_2^7, \quad \omega \mapsto \text{supp } \omega. \tag{20}$$

So H is a permutation module for $SL(3,2)$ corresponding to the permutation action of $SL(3,2)$ on the nonzero elements of E. Thus there is a decomposition

$$H = 3 \oplus \overline{3} \oplus 1 \tag{21}$$

into simple modules. The abelian cocycles are those in the unique submodule of codimension 1, and they may therefore be alternately characterized as those classes of *even* weight. As for the nonabelian classes, the possible weights are $1, 3, 5, 7$, and the number of cohomology classes of each type is $7, 35, 21, 1$ respectively. Those of weight $1, 5$ or 7 form single $SL(3,2)$-orbits of size $7, 21, 1$ respectively. Those of weight 3 split into two orbits according as supp ω is a set of linearly dependent (7 of them) or linearly independent (28 of them) elements of E. We will utilize this information to label the cohomology classes of E.

Based on what we have said so far, there are (at most) 5 gauge equivalence classes of noncommutative twisted quantum doubles $D^{\omega}(E_8)$, with representative cocycles $\omega_1, \omega_{3i}, \omega_{3d}, \omega_5, \omega_7$. Here, the numerical subscript is the weight of the cocycle, and the additional subscript i or d at weight 3 indicates whether supp ω is a linearly independent or dependent set respectively.

Next we consider the quantum doubles $D(D)$ and $D(Q)$. Let ϵ_Q, ϵ_D denote elements of $H^2(V, \mathbb{Z}_2)$ defining Q and D respectively as central extensions (15). If h_1, h_2 are generators of V and t a generator of $\mathbb{Z}_2$ then we may take both ϵ_Q and ϵ_D to be (multiplicatively) bilinear, and

$$\epsilon_D(h_1,h_1) = \epsilon_D(h_1,h_2) = \epsilon_D(h_2,h_2) = 1, \epsilon_D(h_2,h_1) = t; \tag{22}$$
$$\epsilon_Q(h_1,h_1) = \epsilon_Q(h_2,h_1) = \epsilon_Q(h_2,h_2) = t, \epsilon_Q(h_1,h_2) = 1. \tag{23}$$

There is a useful isomorphism analogous to (20) (cf. [M2]), namely

$$H^3(V, \mathbb{C}^*) \longrightarrow \mathbb{F}_2^3, \quad \zeta \mapsto \text{supp } \zeta. \tag{24}$$

Identifying E with $\langle h_1, h_2 \rangle \times \langle t \rangle$, we find that the 3-cocycles $\omega_D, \omega_Q \in H^3(E, \mathbb{C}^*)$ associated to ϵ_D, ϵ_Q respectively by (11) satisfy

$$\begin{aligned} \text{supp } \omega_D &= \{(h_1h_2, t)\}, \\ \text{supp } \omega_Q &= \{(h_1, t), (h_2, t), (h_1h_2, t)\}. \end{aligned}$$

By the case $\zeta = 1$ of Theorem 2.1 we conclude that

$$D(D) \sim D^{\omega_1}(E), \tag{25}$$
$$D(Q) \sim D^{\omega_{3i}}(E). \tag{26}$$

With the notation of Lemma 4.4, that result shows that I is generated by two classes $\alpha_i, i = 1, 2$ such that

$$\text{supp } \alpha_i = \{h_i\}.$$

Now set $\alpha'_i = \mathrm{Infl}_{E_4}^{E_8} \alpha_i$, and note that

$$\begin{aligned}
\text{supp } \alpha'_1\omega_D &= \{(h_1,1),(h_1,t),(h_1h_2,t)\}, \\
\text{supp } \alpha'_2\omega_D &= \{(h_2,1),(h_2,t),(h_1h_2,t)\}, \\
\text{supp } \alpha'_1\alpha'_2\omega_D &= \{(h_1,1),(h_1,t),(h_2,1),(h_2,t),(h_1h_2,t)\}.
\end{aligned}$$

According to Theorem 2.1 we can conclude that

$$D^{\alpha_1}(D) \sim D^{\alpha_2}(D) \sim D^{\omega_{3i}}(E), \tag{27}$$
$$D^{\alpha_1\alpha_2}(D) \sim D^{\omega_5}(E). \tag{28}$$

Note that the gauge equivalence $D^{\alpha_1}(D) \sim D^{\alpha_2}(D)$ also follows from the observation that an involutorial automorphism a of V lifts to an automorphism of D, also denoted as a, which commutes with inflation. Then a exchanges α_1 and α_2 and therefore induces a gauge equivalence between the corresponding twisted quantum doubles. Indeed, it can be verified that there is an $\langle a\rangle$-equivariant decomposition

$$H^3(D,\mathbb{C}^*) = \langle\alpha_1\rangle \oplus \langle\alpha_2\rangle \oplus \langle\alpha_3\rangle$$

where α_3 is a class of order 4 which is a-invariant (cf. Lemma 4.4). As a result, we get the following additional gauge equivalences induced by the action of a on cohomology:

$$\begin{aligned}
D^{\alpha_1\alpha_3}(D_8) &\sim D^{\alpha_2\alpha_3}(D_8), \\
D^{\alpha_1\alpha_3^2}(D_8) &\sim D^{\alpha_2\alpha_3^2}(D_8), \\
D^{\alpha_1\alpha_3^3}(D_8) &\sim D^{\alpha_2\alpha_3^3}(D_8).
\end{aligned} \tag{29}$$

The automorphism group of Q acts *trivially* on the relevant cohomology group, so that no new gauge equivalences can be realized from automorphisms of Q. Let γ be a generator of $H^3(Q,\mathbb{C}^*)$ (cf. Lemma 4.1c)). By Lemma 4.2 we have $\mathrm{Infl}_{E_4}^{Q} H^3(E_4,\mathbb{C}^*) = \langle\gamma^4\rangle$. We then obtain, after a calculation, the following gauge equivalence that arises from application of Theorem 2.1:

$$D^{\gamma^4}(Q_8) \sim D^{\omega_{3d}}(E_8). \tag{30}$$

We have now established

Theorem 5.1. *The 88 noncommutative twisted quantum doubles of E, D and Q fall into* at most *20 gauge equivalence classes, namely (25), (26), (28), (29) and (30) together with the class of $D^{\omega_7}(E_8)$, the 6 classes $D^{\alpha}(D_8)$ with $\alpha \neq 1$ or $\alpha_1\alpha_2$ in $\langle\alpha_1\alpha_2,\alpha_3\rangle$, and the classes $D^{\gamma^m}(Q), m = 1,2,3,5,6,7$.*

It is possible that there are *less* than 20 distinct gauge equivalence classes. We will see that there *at least* 8 such classes. To describe this, consider the following sets of cohomology classes:

$$E : \begin{cases} \mu_1 : \omega_1, \omega_7, \\ \mu_2 : \omega_{3i}, \omega_{3d}, \omega_5 \end{cases}$$

$$D : \begin{cases} \eta_0 : 1, \\ \eta_1 : \alpha_1, \alpha_2, \alpha_1\alpha_2, \\ \eta_2 : \alpha_1\alpha_3, \alpha_2\alpha_3, \alpha_1\alpha_3^3, \alpha_2\alpha_3^3, \\ \eta_3 : \alpha_3, \alpha_3^3, \alpha_1\alpha_2\alpha_3, \alpha_1\alpha_2\alpha_3^3, \\ \eta_4 : \alpha_1\alpha_3^2, \alpha_2\alpha_3^2, \alpha_1\alpha_2\alpha_3^2, \\ \eta_5 : \alpha_3^2. \end{cases}$$

$$Q : \begin{cases} \gamma_0 : 1, \gamma^4, \\ \gamma_1 : \gamma^2, \gamma^6, \\ \gamma_2 : \gamma, \gamma^3, \gamma^5, \gamma^7. \end{cases}$$

Here, we have partitioned representatives for the orbits of $\mathrm{Aut}\, G$ ($G = E, D, Q$), acting on the relevant 3-cohomology classes of G, into certain subsets.

Theorem 5.2. *If G is one of the groups E, D, Q, a pair of twisted quantum doubles $D^\alpha(G), D^\beta(G)$ have the* same *sets of (higher) Frobenius-Schur indicators if, and only if, α and β both lie in one of the sets μ_i, η_j, γ_k. Between them, there are just 8 distinct sets of higher Frobenius-Schur indictators.*

Proof. The complete sets of indicators are given in Appendix below. The Theorem follows from this data. □

We note in particular the following table of Frobenius-Schur exponents. The entries of this table follow from Theorem 4.7.

Twisted Quantum Doubles	Frobenius-Schur exponents
$D^{\mu_1}(E_8), D^{\eta_2}(E_8), D^{\eta_0}(D_8), D^{\eta_1}(D_8), D^{\gamma_0}(Q_8)$	4
$D^{\eta_4}(D_8), D^{\eta_5}(D_8), D^{\gamma_1}(Q_8)$	8
$D^{\eta_2}(D_8), D^{\eta_3}(D_8), D^{\gamma_2}(Q_8)$	16

Finally, we briefly consider some bialgebra isomorphisms. Let $\sigma \in H = H^3(E, \mathbb{C}^*)$ be a cohomology class inflated, as in Theorem 4.1, from the nontrivial class of E/F where F is a subgroup of E of index 2. Hence, supp σ consists of the elements in $E \setminus F$. By Theorem 4.1, there is an isomorphism of *bialgebras*

$$D^\omega(E) \cong D^{\omega\sigma}(E)$$

for all cohomology classes $\omega \in H$. Using this, easy arguments lead to the following bialgebra isomorphisms:

$$D^{\omega_1}(E) \cong D^{\omega_5}(E) \cong D^{\omega_{3i}}(E),$$
$$D^{\omega_7}(E) \cong D^{\omega_{3d}}(E).$$

As we have seen, $D^{\omega_7}(E)$ and $D^{\omega_{3d}}(E)$, for example, are not gauge equivalent because they have different sets of Frobenius-Schur indicators. Thus they afford an example of a pair of twisted doubles which are isomorphic as bialgebras but not gauge equivalent.

6 Invariance of Ribbon Structure

Let $(\mathcal{C}, \otimes, I, \Phi)$ be a left rigid braided monoidal category, where I is the neutral object and Φ the associativity isomorphism. Here, we assume that the neutral object I is *strict*, i.e. $I \otimes V = V \otimes I = V$ for $V \in \mathcal{C}$. For $V \in \mathcal{C}$, we use the notation $V^\vee$ for the left dual of V with the dual basis map $\mathrm{db}_V : I \to V \otimes V^\vee$ and the evaluation map $\mathrm{ev}_V : V^\vee \otimes V \to I$. Note that $(-)^\vee$ can be extended to a contravariant monoidal equivalence of $\mathcal{C}$ with $I^\vee = I$, $\mathrm{ev}_I = \mathrm{db}_V = \mathrm{id}_I$. Thus $(-)^{\vee\vee}$ is a monoidal equivalence on $\mathcal{C}$. See [K] for more details on right monoidal category and monoidal functor. We follow the notation and terminology introduced in [NS1] for the discussion to come.

For any braiding c on the left rigid monoidal category $\mathcal{C}$, the associated Drinfeld isomorphism u is defined by

$$u_V = \Big(V \xrightarrow{\mathrm{db}_{V^\vee} \otimes \mathrm{id}} (V^\vee \otimes V^{\vee\vee}) \otimes V \xrightarrow{c \otimes \mathrm{id}} (V^{\vee\vee} \otimes V^\vee) \otimes V \xrightarrow{\Phi} V^{\vee\vee} \otimes (V^\vee \otimes V) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} V^{\vee\vee}\Big)$$

for $V \in \mathcal{C}$. In particular, if $\mathcal{C}$ is *strict*, the Drinfeld isomorphism satisfies the equation

$$u_{V \otimes W} = (u_V \otimes u_W) c_{V,W}^{-1} c_{W,V}^{-1}$$

for $V, W \in \mathcal{C}$.

Suppose j is a *pivotal structure* on $\mathcal{C}$ (not necessarily strict), i.e. $j : \mathrm{Id} \to (-)^{\vee\vee}$ is an isomorphism of monoidal functors. Then $v = u^{-1} \circ j$ is a *twist* of the braided monoidal category $(\mathcal{C}, c)$, i.e. a natural automorphism of the identity functor Id of $\mathcal{C}$ such that

$$v_{V \otimes W} = (v_V \otimes v_W) c_{W,V} c_{V,W}$$

for all $V, W \in \mathcal{C}$. If $\mathcal{C}$ is a *spherical fusion category over* $\mathbb{C}$, then v defines a ribbon structure on $\mathcal{C}$ (cf. [NS3]). We will use the notation $(\mathcal{C}, c, j)$ to denote a pivotal braided monoidal category $\mathcal{C}$ with the braiding c and the pivotal structure j.

Lemma 6.1. *Let $(\mathcal{C}, c, j)$, $(\mathcal{D}, c, j)$ be pivotal braided monoidal categories. If $(\mathcal{F}, \xi) : \mathcal{C} \to \mathcal{D}$ is a braided monoidal equivalence which preserves the pivotal structures, then*

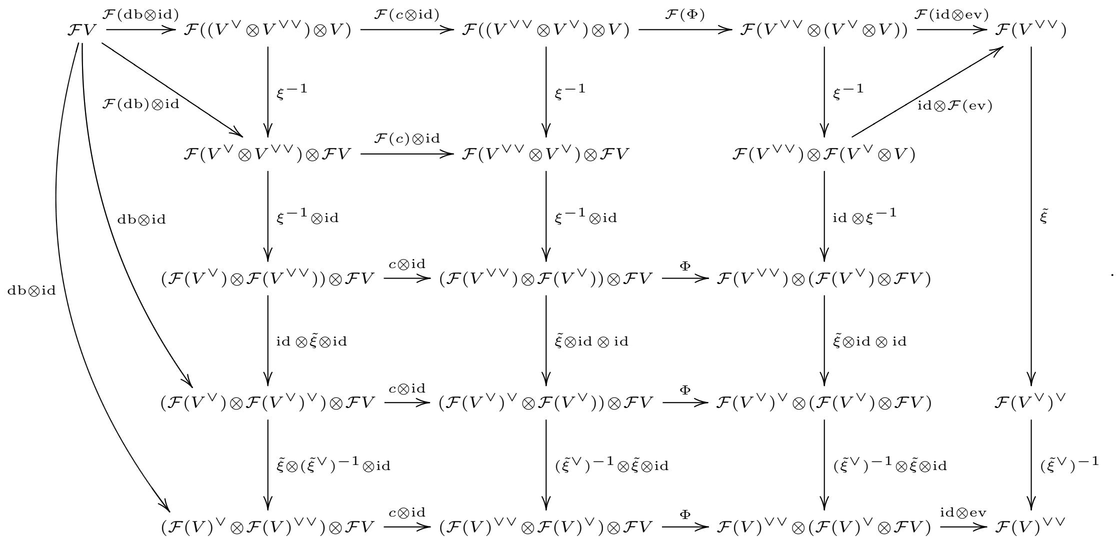
$$\mathcal{F}(v_V) = v_{\mathcal{F}V}$$

for $V \in \mathcal{C}$.

Proof. From [NS1] it follows that the duality transformation $\tilde{\xi} : \mathcal{F}(V^\vee) \to \mathcal{F}(V)^\vee$ of $(\mathcal{F}, \xi)$ is determined by either of the commutative diagrams

$$\begin{array}{ccc} \mathcal{F}(I) & = \!=\!=\!=\!= & I \\ \downarrow{\scriptstyle \mathcal{F}(\mathrm{db})} & & \\ \mathcal{F}(V \otimes V^\vee) & & \downarrow{\scriptstyle \mathrm{db}} \\ \downarrow{\scriptstyle \xi^{-1}} & & \\ \mathcal{F}(V) \otimes \mathcal{F}(V^\vee) & \xrightarrow{\mathrm{id} \otimes \tilde{\xi}} & \mathcal{F}(V) \otimes \mathcal{F}(V)^\vee \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(V^\vee \otimes V) & \xrightarrow{\mathcal{F}(\mathrm{ev})} & \mathcal{F}(I) \\ \downarrow{\scriptstyle \xi^{-1}} & & \\ \mathcal{F}(V^\vee) \otimes \mathcal{F}(V) & & \| \\ \downarrow{\scriptstyle \tilde{\xi} \otimes \mathrm{id}} & & \\ \mathcal{F}(V)^\vee \otimes \mathcal{F}(V) & \xrightarrow{\mathrm{ev}} & I \end{array} \quad . \qquad (31)$$

We observe that following diagram is commutative:



Commutativity of the middle rectangles are consequences of either the properties of the braided monoidal equivalence $(\mathcal{F}, \xi)$, or the naturality of c and Φ. Commutativity of the two triangles at the upper corners follows from properties of the coherence map ξ, and that of the lower left triangle follows from properties of left duality. The commutativity of remaining two polygons on both sides follow from (31).

Note that the top edge is $\mathcal{F}(u_V)$. The commutativity of the above diagram implies that

$$(\tilde{\xi^\vee})^{-1}\tilde{\xi}\mathcal{F}(u_V) = u_{\mathcal{F}(V)}\,.$$

Since $(\mathcal{F}, \xi)$ also preserves pivotal structures (cf. [NS3]),

$$(\tilde{\xi^\vee})^{-1}\tilde{\xi} = j\mathcal{F}(j^{-1})$$

and hence

$$\mathcal{F}(v_V) = v_{\mathcal{F}V}\,. \quad \square$$

Suppose that $\mathcal{C} = H\text{-}\mathbf{Mod}_{\mathsf{fin}}$, the tensor category of finite-dimensional modules over a semisimple quasi-Hopf algebra H over $\mathbb{C}$. Then $\mathcal{C}$ admits a canonical pivotal structure given by a trace element g of H, namely

$$j_V(x)(f) = f(g^{-1}x)$$

for $x \in V$ and $f \in V^\vee$ (cf. [MN2], [NS2], [ENO]). If, in addition, H admits a universal $\mathcal{R}$-matrix, then $\mathcal{C}$ is a braided spherical fusion category. The Drinfeld isomorphism $u_V : V \to V^{\vee\vee}$ is given by

$$u_V(x)(f) = f(ux)$$

where u is the Drinfeld element and the associated ribbon structure v_V is given by the multiplication of the element $v = (gu)^{-1}$.

Proposition 6.2. *Let H, K be semisimple, braided, quasi-Hopf algebras. If $(\mathcal{F}, \xi)$: H-$\mathbf{Mod}_{\mathsf{fin}} \to K$-$\mathbf{Mod}_{\mathsf{fin}}$ is a braided monoidal equivalence, then*

$$\mathcal{F}(v_V) = v_{\mathcal{F}(V)}$$

for $V \in H$-$\mathbf{Mod}$.

Proof. By [NS2], $(\mathcal{F}, \xi)$ preserves the canonical pivotal structures. Thus the result follows immediately from Lemma 6.1. □

Note that v_V is a scalar for any simple H-module V. Thus if $(\mathcal{F}, \xi) : H$-$\mathbf{Mod}_{\mathsf{fin}} \to K$-$\mathbf{Mod}_{\mathsf{fin}}$ is an equivalence of $\mathbb{C}$-linear braided monoidal categories, then

$$v_V = v_{\mathcal{F}(V)} \quad \text{and} \quad \nu_n(V) = \nu_n(\mathcal{F}(V))$$

for all positive integer n and simple objects $V \in H$-$\mathbf{Mod}_{\mathsf{fin}}$.

In case $H = D^\omega(G)$, it is shown in [MN2] that the trace element g of $D^\omega(G)$ is given by

$$g = \sum_{x \in G} \omega(x, x^{-1}, x) e(x) \otimes 1.$$

By [AC],

$$u = \sum_{x \in G} \omega(x, x^{-1}, x)^{-2} e(x) \otimes x^{-1},$$

is the Drinfeld element for the associated braiding of $D^\omega(G)$. Hence

$$v = (gu)^{-1} = \sum_{x \in G} \omega(x, x^{-1}, x)^{-1} e(x) \otimes x^{-1} = \sum_{x \in G} e(x) \otimes x$$

defines the twist or ribbon structure associated with the underlying canonical pivotal structure and braiding of $D^\omega(G)$-$\mathbf{Mod}_{\mathsf{fin}}$. With a different convention, the formula of the ribbon element of $D^\omega(G)$ is also shown [AC].

We can now use the formula for v to compute the corresponding scalar v_V for each simple module V of a 64-dimensional twisted double H. If χ is the irreducible character afforded by V, the scalar $v_\chi = \chi(v)/\chi(1)$ is equal to v_V. The sequence of higher indicators $\nu_\chi^{(n)}$ and v_χ for each irreducible character χ of H are presented in Tables in the Appendix. It follows from this data that the 20 classes of twisted doubles H of dimension 64 identified in Section 5 have *distinct* sets of sequences. Since both higher indicators and v_χ are preserved by a braided tensor equivalence, we have

Theorem 6.3. *There are exactly 20 gauge equivalence classes of quasi-triangular quasi-bialgebras among the 64-dimensional noncommutative twisted quantum doubles of finite groups.* □

7 Appendix

In the following tables, m denotes the multiplicity of the sequence, and κ is a primitive 16th complex root of 1 such that $\kappa^4 = i$.

Frobenius-Schur Exponent 16

H	$\chi(1)$	$\nu^{(2)}$	$\nu^{(3)}$	$\nu^{(4)}$	$\nu^{(5)}$	$\nu^{(6)}$	$\nu^{(7)}$	$\nu^{(8)}$	$\nu^{(9)}$	$\nu^{(10)}$	$\nu^{(11)}$	$\nu^{(12)}$	$\nu^{(13)}$	$\nu^{(14)}$	$\nu^{(15)}$	v_χ	m
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\gamma}(Q_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	$-i$	4
	2	-1	0	2	0	-1	0	2	0	-1	0	2	0	-1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	i	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}^7$	3
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^3$	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^5	3
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\gamma^3}(Q_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	i	4
	2	-1	0	2	0	-1	0	2	0	-1	0	2	0	-1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	$-i$	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}^5$	3
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ^3	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}$	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^7	3
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\gamma^5}(Q_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	$-i$	4
	2	-1	0	2	0	-1	0	2	0	-1	0	2	0	-1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	i	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}^3$	3
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ^5	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^7$	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ	3
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\gamma^7}(Q_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	i	4
	2	-1	0	2	0	-1	0	2	0	-1	0	2	0	-1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	$-i$	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}$	3
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ^7	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^5$	3
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^3	3

H	$\chi(1)$	$\nu^{(2)}$	$\nu^{(3)}$	$\nu^{(4)}$	$\nu^{(5)}$	$\nu^{(6)}$	$\nu^{(7)}$	$\nu^{(8)}$	$\nu^{(9)}$	$\nu^{(10)}$	$\nu^{(11)}$	$\nu^{(12)}$	$\nu^{(13)}$	$\nu^{(14)}$	$\nu^{(15)}$	v_χ	m
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\alpha_2\alpha_3}(D_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
$D^{\alpha_1\alpha_3}(D_8)$	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	$-i$	4
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}^7$	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}^3$	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ^5	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	i	1
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	-1	2
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	2
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	$-i$	2
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	i	2
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\alpha_1\alpha_2\alpha_3}(D_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	$-i$	4
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^7$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^3$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^5	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	i	1
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	-1	4
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	4
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\alpha_3}(D_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	$-i$	4
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^7$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^3$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^5	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	i	1
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	$-i$	4
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	i	4
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\alpha_2\alpha_3^3}(D_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
$D^{\alpha_1\alpha_3^3}(D_8)$	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	i	4
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}^5$	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	$\overline{\kappa}$	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ^7	1
	2	-1	0	1	0	-1	0	0	0	-1	0	1	0	-1	0	κ^3	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	$-i$	1
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	-1	2
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	2
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	$-i$	2
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	i	2

H	$\chi(1)$	$\nu^{(2)}$	$\nu^{(3)}$	$\nu^{(4)}$	$\nu^{(5)}$	$\nu^{(6)}$	$\nu^{(7)}$	$\nu^{(8)}$	$\nu^{(9)}$	$\nu^{(10)}$	$\nu^{(11)}$	$\nu^{(12)}$	$\nu^{(13)}$	$\nu^{(14)}$	$\nu^{(15)}$	v_χ	m
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\alpha_1\alpha_2\alpha_3^3}(D_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	i	4
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^5$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^7	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^3	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	$-i$	1
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	-1	4
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	4
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$D^{\alpha_3^3}(D_8)$	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	3
	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	i	4
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}^5$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	$\overline{\kappa}$	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^7	1
	2	1	0	1	0	1	0	0	0	1	0	1	0	1	0	κ^3	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	1	1
	2	1	0	2	0	1	0	2	0	1	0	2	0	1	0	$-i$	1
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	$-i$	4
	2	0	0	1	0	0	0	0	0	0	0	1	0	0	0	i	4

Frobenius-Schur Exponent 8

H	$\chi(1)$	$\nu^{(2)}$	$\nu^{(3)}$	$\nu^{(4)}$	$\nu^{(5)}$	$\nu^{(6)}$	$\nu^{(7)}$	v_χ	m
$D^{\alpha_2\alpha_3^2}(D_8)$ $D^{\alpha_1\alpha_3^2}(D_8)$	1	1	1	1	1	1	1	1	1
	1	1	0	1	0	1	0	1	3
	1	1	0	1	0	1	0	-1	4
	2	-1	0	0	0	-1	0	$\overline{\kappa}^6$	1
	2	-1	0	0	0	-1	0	$\overline{\kappa}^2$	1
	2	-1	0	0	0	-1	0	κ^6	1
	2	-1	0	0	0	-1	0	κ^2	1
	2	-1	0	0	0	-1	0	$-i$	2
	2	-1	0	0	0	-1	0	i	2
	2	1	0	0	0	1	0	-1	2
	2	1	0	0	0	1	0	1	2
	2	1	0	2	0	1	0	1	2
$D^{\alpha_1\alpha_2\alpha_3^2}(D_8)$	1	1	1	1	1	1	1	1	1
	1	1	0	1	0	1	0	1	3
	1	1	0	1	0	1	0	-1	4
	2	1	0	0	0	1	0	$\overline{\kappa}^6$	1
	2	1	0	0	0	1	0	$\overline{\kappa}^2$	1
	2	1	0	0	0	1	0	κ^6	1
	2	1	0	0	0	1	0	κ^2	1
	2	1	0	2	0	1	0	1	2
	2	-1	0	0	0	-1	0	$-i$	4
	2	-1	0	0	0	-1	0	i	4
$D^{\alpha_3^2}(D_8)$	1	1	1	1	1	1	1	1	1
	1	1	0	1	0	1	0	1	3
	1	1	0	1	0	1	0	-1	4
	2	1	0	0	0	1	0	$\overline{\kappa}^6$	1
	2	1	0	0	0	1	0	$\overline{\kappa}^2$	1
	2	1	0	0	0	1	0	κ^6	1
	2	1	0	0	0	1	0	κ^2	1
	2	1	0	2	0	1	0	1	2
	2	1	0	0	0	1	0	-1	4
	2	1	0	0	0	1	0	1	4
$D^{\gamma^6}(Q_8)$	1	1	1	1	1	1	1	1	1
	1	1	0	1	0	1	0	1	3
	1	1	0	1	0	1	0	-1	4
	2	-1	0	2	0	-1	0	1	2
	2	-1	0	0	0	-1	0	$\overline{\kappa}^6$	3
	2	-1	0	0	0	-1	0	κ^2	3
	2	1	0	0	0	1	0	$\overline{\kappa}^2$	3
	2	1	0	0	0	1	0	κ^6	3
$D^{\gamma^2}(Q_8)$	1	1	1	1	1	1	1	1	1
	1	1	0	1	0	1	0	1	3
	1	1	0	1	0	1	0	-1	4
	2	-1	0	2	0	-1	0	1	2
	2	-1	0	0	0	-1	0	$\overline{\kappa}^2$	3
	2	-1	0	0	0	-1	0	κ^6	3
	2	1	0	0	0	1	0	$\overline{\kappa}^6$	3
	2	1	0	0	0	1	0	κ^2	3

Frobenius-Schur Exponent 4

H	$\chi(1)$	$\nu^{(2)}$	$\nu^{(3)}$	v_χ	m
$D(Q_8)$ $D^{\alpha_1}(D_8)$ $D^{\alpha_2}(D_8)$ $D^{\omega_{3i}}(E_8)$	1	1	1	1	1
	1	1	0	1	7
	2	-1	0	-1	1
	2	-1	0	1	1
	2	-1	0	$-i$	3
	2	-1	0	i	3
	2	1	0	-1	3
	2	1	0	1	3
$D^{\alpha_1\alpha_2}(D_8)$ $D^{\omega_5}(E_8)$	1	1	1	1	1
	1	1	0	1	7
	2	1	0	$-i$	1
	2	1	0	i	1
	2	1	0	-1	2
	2	1	0	1	2
	2	-1	0	$-i$	4
	2	-1	0	i	4
$D(D_8)$ $D^{\omega_1}(E_8)$	1	1	1	1	1
	1	1	0	1	7
	2	1	0	$-i$	1
	2	1	0	i	1
	2	1	0	-1	6
	2	1	0	1	6
$D^{\gamma^4}(Q_8)$ $D^{\omega_{3d}}(E_8)$	1	1	1	1	1
	1	1	0	1	7
	2	1	0	$-i$	3
	2	1	0	i	3
	2	-1	0	-1	4
	2	-1	0	1	4
$D^{\omega_7}(E_8)$	1	1	1	1	1
	1	1	0	1	7
	2	1	0	$-i$	7
	2	1	0	i	7

References

[AC] D. Altschüler and A. Coste, Quasi-quantum groups, knots, three-manifolds, and topological field theory, *Comm. Math. Phys.* **150** (1992), no. 1, 83–107.

[BC] D. Benson and J. Carlson, The cohomology of extra-special groups, *Bull. Lond. Math. Soc.* No. 108, **24** (3), 1992.

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Landmarks in Mathematics, Princeton Univ. Press, Princeton, 1973.

[DM] C. Dong and G. Mason, Vertex operator algebras and moonshine: a survey, in *Advanced Studies in Pure Mathematics 24, Progress in Algebraic Combinatorics*, Math. Soc. of Japan, Kinokuniya, Tokyo, 1996.

[DPR] R. Dijkgraaf, V. Pasquier, and P. Roche, Quasi-Hopf Algebras, Group Cohomology and Orbifold Models, *Nucl. Phys. B (Proc. Suppl.)* **18B**, 1990, 60–72.

[D] V. Drinfeld, Quasi-Hopf algebras, *Leningrad Mathematical Journal* **1**, 1990, 1419–1457.

[DGM] L. Dolan, P. Goddard and P. Montague, Conformal field theory of twisted vertex operators, *Nucl. Phys.* **B338**(1990), 529-601.

[FLM] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, New York, 1988.

[EG] P. Etinghof and S. Gelaki, On families of triangular Hopf algebras, *Int. Math. Res. Not.* **14** (2002), 757-768.

[ENO] P. Etingof, D. Nikshych, and V. Ostrik, On fusion categories, *Ann. of Math. (2)* **162** (2005), no. 2, 581–642.

[HK] M. Harada and A. Kono, On the integral cohomology of extra-special 2-groups, *J. Pure and Appl. Math.* **44** (1987), 215-219.

[K] C. Kassel, *Quantum Groups*, Springer-Verlag, New York, 1995.

[Ma] Shahn Majid, *Tannaka-Kreĭn theorem for quasi-Hopf algebras and other results*, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), Amer. Math. Soc., Providence, RI, 1992, pp. 219–232.

[Ma1] Shahn Majid, Quantum double for quasi-Hopf algebras, *Lett. Math. Phys.* **45** (1998), no. 1, 1–9.

[M1] G. Mason, The Quantum Double of a Finite Group and its Role in Conformal Field Theory, *London Math. Society Lecture Note Series* **212**, v.2, 1995, 405–417.

[M2] G. Mason, Reed-Muller codes, the fourth cohomology of a finite group, and the β-invariant, to appear in J. Alg.

[MN1] G. Mason and S.-H. Ng, Group cohomology and gauge equivalence of some twisted quantum doubles, *Trans. Amer. Math. Soc.* **353** (2001), no. 9, 3465–3509.

[MN2] G. Mason and S.-H. Ng, Central invariants and Frobenius-Schur indicators for semisimple quasi-Hopf algebras, *Adv. Math*. **190** (2005), no. 1, 161-195.

[N1] S. Natale, On group theoretical Hopf algebras and exact factorizations of finite groups, *J. Alg.* **270** (2003), no. 1, 199-211.

[N2] S. Natale, On the exponent of tensor categories coming from finite groups, arXiv:math.QA/0511123.

[NS1] S.-H. Ng and P. Schauenberg, Higher Frobenius-Schur indicators for pivotal categories, arXiv:math.QA/0503167, to appear in *Contemporary Mathematics*.

[NS2] S.-H. Ng and P. Schauenberg, Central invariants and higher indicators for semisimple quasi-Hopf algebras, arXiv:math.QA/0508140, to appear in *Trans. Amer. Math. Soc.*

[NS3] S.-H. Ng and P. Schauenberg, Frobenius-Schur indicators and exponents of spherical categories, arXiv:math.QA/0601012, *Adv. Math.* (2006) doi:10.1016/j.aim.2006.07.017.

[S1] P. Schauenburg, The monoidal center construction and bimodules, *J. Pure Appl. Algebra* **158** (2001), no. 2-3, 325–346.

[S2] P. Schauenburg, Hopf bimodules, coquasibialgebras, and an exact sequence of Kac, *Adv. Math.* **165** (2002), no. 2, 194–263.

[Q] D. Quillen, The mod 2 cohomology rings of extra-special 2-groups and the spinor groups, *Math. Ann.* **194** (1971), 197-212.