



1761

De aequationibus differentialibus secundi gradus

Leonhard Euler

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DE

AEQVATIONIBVS
DIFFERENTIALIBVS SECVNDI
GRADVS.

Auctore

L. EVLERO.

I.

Omnium quaestionum, quae quidem in Mathesi suscipiuntur, solutio duabus constat partibus, quarum altera in eo versatur, ut condiciones, quibus quaestio determinatur, ad aequationes analyticas perducantur, quae solutionem continere dicuntur; altera vero pars in ipsa harum aequationum resolutione occupatur. Si quaestio ad Mathesin mixtam, vel applicatam pertineat, prior pars petenda est ex principiis, quibus ista disciplina Mathematica innititur, huicque scientiae quasi est propria; pars autem posterior semper ad Analysin puram est referenda, cum tota in resolutione aequationum versetur. Ita si quaestio, vel ex Mechanica, vel ex Hydrodynamica, vel ex Astronomia, fuerit desumpta, ex principiis cuique harum disciplinarum propriis quaestionem primum ad aequationes reduci oportet, tum vero istarum aequationum resolutio artificijs, quae quidem in Analysi comperta habemus, vnice est relinquenda. Ex quo satis est manifestum, quanti sit momenti Analysis per cunctas Matheos partes.

X 2

2. Pria-

2. Principia autem fere omnium Matheſeos applicatae partium iam ita ſunt euoluta, vt nulla prope modum quaefſtio eo pertinens proferri poſſit, cuius ſolutio non aequationibus comprehendi queat. Siue enim quaefſtio ſit de aequilibrio, ſiue de motu corporum cuiuſcunq; indolis, tam ſolidorum, quam fluidorum, cum ab aliis, tum a me, principia certiffima ſunt ſtabilita, quorum ope ſemper ad aequationes peruenire licet; atque ſi corpora coeleſtia viribus quibuſcunq; in ſe inuicem agere ſtatuantur, omnes perturbaciones, quae inde in eorum motibus efficiuntur, non difficulter ad aequationes reuocantur; ita vt ſi has aequationes reſolvere valeremus, nihil amplius ſupereffet, quod in hiſ ſcientiis deſiderari poſſet. Quocirca omne ſtudium, quod in Matheſin confertur, vtilius impendi nequit, quam ſi in limitibus Analyſeos promouendis elaboremus.

3. Quoties autem problema ad Matheſin applicatam pertinens tractatur, rariffime in aequationes algebraicas incidimus, quarum reſolutio, etiamſi nondum ultra quartum gradum ſit perducta, tamen ope approximationum ita exacte perfici poteſt, vt pro perfecta ſit habenda. Perpetuo autem fere deuoluimur ad aequationes differentiales, et quidem maximam partem ad differentiales ſecundi ordinis; principia quippe mechanica ſtatim differentialia ſecundi gradus implicant: ita vt ſine Analyſeos infinitorum ſubſidio, nihil fere in hiſ ſcientiis praefitari liceat. Cum autem in reſolutione aequationum differentialium primi gradus non admodum ſimus profecti, multo minus eſt mirandum, ſi aqua nobis

nobis haereat, quando quaestiones ad aequationes differentiales secundi gradus reducuntur. Regulae enim, quae pro huiusmodi aequationum resolutione sunt inventae, et quas mihi equidem vindicare possum, ita sunt limitatae, ut certis tantum casibus, qui non admodum frequenter occurrunt, in usum vocari queant. Huiusmodi autem regulas plures exposui in Comment. Acad. Petrop. et Vol. VII. Miscell. Berol.

4. Interim tamen iam saepius eiusmodi se mihi obtulerunt casus aequationum differentialium secundi gradus, quas tamen ope regularum illarum tractare non licuerit, tamen aliunde earum integralia habuerim perspecta; neque vlla via directa patebat, qua haec integralia erui possent. Huiusmodi casus eo magis sunt notatu digni, quod comparatio illarum aequationum cum suis integralibus tutissimam viam patefacere videatur, earum resolutionem per certas methodos perficiendi. In quo negotio, si euentus spem non fefellerit, nullum est dubium, quin methodi hunc in finem detectae, multo latius pateant, ac nostram facultatem, aequationes differentiales secundi gradus tractandi, non mediocriter promoueant. His ergo, quos huiusmodi studia iuuant, non ingratum fore arbitror, si casus illos mihi oblatos commemorauero, ut occasionem inde adipiscantur, in hac parte Analysis amplificandi, tum vero ipse methodos exponam, quas horum casuum contemplatio mihi suppeditauit.

5. Primum huiusmodi exemplum mihi occurrit in *Mechanicae* meae Tom. I. pag. 465. vbi
 X 3 ad

ad hanc perueni aequationem differentialem secundi gradus :

$$2 B x d d x - 4 B d x^2 = x^{n+5} d p^2 (1 + p p)^{\frac{n-1}{2}}$$

in qua differentiale dp sumtum est constans. Eius autem integrale aliunde mihi constabat in hac forma contineri :

$$x^{n+5} d p^2 (1 + p p)^{\frac{n-1}{2}} + C d s^2 = 0$$

existente $d s^2 = (1 + p p) d x^2 + 2 p x d p d x + x x d p^2$. Poteram etiam notare valorem huius constantis C esse $= -(n+1)B$. Diu tum temporis operam inutiliter perdidit in methodo directa indaganda, cuius ope istam aequationem integralem ex illa differentiali secundi gradus eruere possem, neque vllum artificium cognitum huc deducere est visum. Caeterum notari conuenit, integrale hic exhibitum tantum esse particulare, quia non continet quantitatem constantem ab arbitrio nostro pendentem, quae per integrationem esset introducta, infra autem ostendam ob talem constantem adiaci posse huiusmodi terminum $E x^4 d p^2$.

6. In aliud simile exemplum incidi in opusculorum meorum prima collectione pag. 82, vbi motum corporum in superficiebus mobilibus sum perferutatus: perueni autem in evolutione certi cuiusdam casus ad hanc aequationem differentialem secundi gradus:

$$\frac{d d r}{r} + \frac{(F + M k k)^2 \theta^2 d u^2}{(M k k r r + F + 2 G u + H u u)^2} = 0$$

vbi differentiale du sumtum est constans, litterae autem $F, G, H, M k k$ et θ denotant quantitates constantes quascunque. Nullo modo quoque huius aequationis
inte-

integrale erucere poteram, aliunde autem noueram, eius integrale esse:

$$\frac{(F + Mkk)\theta^2 du^2}{Mkkr + F + Gu + Huv} + \frac{dr^2}{r^2} (F + 2Gu + Huv) - \frac{adu^2}{r} (G + H\theta) + H du^2 = \frac{H du^2}{r^2} + \frac{(F + Mkk)\theta^2 du^2}{r^2}$$

quod quidem etiam est particulare, et quia tantopere est complicatum, multo minus patet, quomodo per integrationem ex illa aequatione erui queat. Deinceps vero monstrabo, hoc integrale completum reddi, si loco termini $\frac{H du^2}{r^2}$, adiciatur $\frac{C du^2}{r^2}$, ita vt C designet quantitatem constantem, a reliquis, quae in aequatione differentiali secundi gradus insunt, plane non dependentem.

7. Deinde etiam alia problemata tractans, perductus fui ad huiusmodi aequationes differentiales secundi gradus, quarum integratio non parum recondita videbatur. Veluti huius aequationis differentialis secundi gradus:

$$r r ddr + r dr^2 = n^2 s ds^2$$

sumto elemento ds constante, integrale particulare quidem inueni esse:

$$r dr + nr ds + n^2 s ds = 0$$

quae quidem aequatio, quia binae variables r et s ubique earundem dimensionum, per methodum a me olim exhibitam, tractari possent. Porro quoque se mihi obtulit haec aequatio differentio-differentialis:

$$ds^2 (\alpha ss + \beta s + \gamma) = r r ddr + 2 r^2 ddr$$

sumto elemento ds constante, cuius integrale completum deprehendi esse:

$$C = -\frac{1}{2} \left(\frac{r dr^2}{ds^2} + \frac{\alpha ss + \beta s + \gamma}{s} \right) + \frac{2 r dr (\alpha ss + \beta s)}{ds} - 2 dr^2$$

quod,

quod, quomodo inde elici queat, haud facile patet. Quin etiam ipsa aequatio integralis, etsi est differentialis primi tantum gradus, parum adiumenti afferre videtur, ob insignem variabilium implicationem.

8. Haec quatuor exempla sufficiunt, ad ostendendum, plures adhuc methodos deesse, quibus aequationes differentiales secundi gradus integrari queant, simul autem, quoniam his quidem casibus integralia constant, de earum inuentione non esse desperandum. Equidem post varia tentamina, quibus has aequationes tractavi, comperi, totum negotium eo redire, vt idonea quaeratur quantitas, per quam istae aequationes multiplicatae integrationem admittant; tali autem multiplicatore inuento, integratio nulla amplius laborat difficultate. Quemadmodum enim omnium aequationum differentialium primi gradus integratio eo reduci potest, vt investiganda sit functio quaequam binarum variabilium, per quam aequatio multiplicata euadat integrabilis, ita etiam, pro omnibus aequationibus differentialibus secundi gradus, hanc regulam non dubito tanquam generalem in medium afferre, vt statuam semper eiusmodi functionem variabilium dari, per quam aequatio multiplicata reddatur integrabilis.

9. Loquor autem hic de eiusmodi tantum aequationibus, quae duas solum variables inuoluunt, et quae iam eo sint perductae, vt differentialia supremi gradus unicam dimensionem obtineant. Ponamus x et y esse ambas variables, et posito $dy = p dx$; $dp = q dx$; $dq = r dx$, $dr = s dx$, etc. omnes aequationes differentiales

tiales cuiusque gradus ad formas sequentes reduci posse constat :

I. Forma generalis aequationum differentialium primi gradus

$$p = \text{funct. } (x \text{ et } y)$$

II. Forma generalis aequationum differ. secundi gradus

$$q = \text{funct. } (x, y \text{ et } p)$$

III. Forma generalis aequationum differ. tertii gradus

$$r = \text{funct. } (x, y, p \text{ et } q)$$

IV. Forma generalis aequationum differ. quarti gradus

$$s = \text{funct. } (x, y, p, q, \text{ et } r)$$

et ita porro de sequentibus altiorum graduum.

10. Cum igitur proposita quacunque aequatione differentiali primi gradus $p = \text{funct. } (x \text{ et } y)$, semper detur eiusmodi functio ipsarum x et y , per quam illa aequatio multiplicata reddatur integrabilis, etiamsi saepe numero hanc functionem assignare non valeamus, nullum est dubium, quin etiam pro aequationibus differentialibus secundi gradus $q = \text{funct. } (x, y \text{ et } p)$ eiusmodi multiplicator existat, qui eas reddat integrabiles, ideoque ad differentialia primi gradus reducat. Iam vero hic casus distingui oportet, quibus iste multiplicator vel binarum tantum variarum x et y functio existat, vel insuper quantitatem p , seu rationem differentialium $\frac{dy}{dx}$ inuoluat: ob hoc enim discrimen ipsa multiplicatoris inuentio modo facilior, modo difficilior

euadet. Casus autem euolutu facillimus habebitur, si multiplicator alterius tantum variabilis solius fuerit functio.

11. Si igitur litterae P, Q, R, S, T sumantur ad designandas quascunque functiones ipsarum variabilium x et y , sequentes ordines simpliciores multiplicatorum pro aequationibus differentialibus secundi gradus constituentur:

Multiplicator ordinis primi .. P

Multiplicator ordinis secundi .. $Pdx + Qdy$

Multiplicator ordinis tertii .. $Pdx^2 + Qdxdy + Rdy^2$

Multiplicator ordinis quarti .. $Pdx^3 + Qdx^2dy + Rdxdy^2 + Sdy^3$
etc.

Hi quidem sunt ordines simpliciores, quibus $p = \frac{dy}{dx}$, vel ad nullam, vel ad vnam, vel duas, vel tres dimensiones affurgit: facile autem colligitur fieri posse, vt littera p vel per fractiones, vel irrationalia, vel adeo transcendentia, multiplicatorem afficiat, cuiusmodi casus ingentem campum nouarum inuestigationum aperiunt. Hic quidem tantum in formis expositis versari constitui, quia eae sufficiunt exemplis allatis expediendis, simulque nos ad aequationes multo generaliores earum ope resolubiles manudent.

12. Proposita ergo aequatione quacunque differentiali secundi gradus, $q = \text{funct.}(x, y \text{ et } p)$, quae sumto dx constanti ad hanc formam redigetur $d^2y = dx^2 \text{ funct.}(x, y \text{ et } \frac{dy}{dx})$, tentetur primo multiplicator primae formae P, num eius ope integratio succedat? si minus,

minus, sumatur multiplicator formae secundae $Pdx + Qdy$, qui nisi negotium conficiat, recurratur ad multiplicatorem formae tertiae, tum quartae, etc. mox autem colligere licebit, vtrum per factores harum formarum integratio absolui queat, nec ne? quo posteriori casu, ad formas magis complicatas erit confugiendum, ac dummodo huiusmodi calculo fuerimus affueti, facultatem nobis comparabimus, pro quouis casu oblato idoneam multiplicatoris formam dignoscendi: ad quem scopum euolutio propositorum exemplorum erit accommodata.

Problema 1.

13. Proposita aequatione differentiali secundi gradus:

$$2ayddy - 4ady^2 - y^{n+5}dx^2(1+xx)^{\frac{n-1}{2}} = 0$$

in qua differentiale dx sumtum est constans, eius integrale inuenire.

Solutio.

Factorem primae formae P tentanti mox patebit, negotium non succedere, nisi sit $n = -2$, quo quidem casu foret $P = \frac{1}{y^3}$ et aequationis $\frac{2ayddy - 4ady^2}{y^3}$

$-\frac{dx^2}{(1+xx)\sqrt{1+xx}} = 0$ integrale esset

$$\frac{2ady}{yy} + \frac{xdx}{\sqrt{1+xx}} = \alpha dx, \text{ denuoque integrando haberetur}$$

$$-\frac{2a}{y} + V(1+xx) = \alpha x + \beta;$$

ita vt hic casus specialis nullam habeat difficultatem. In genere igitur pro valore quocunque exponentis n ,

tentetur factor formae secundae $Pdx + Qdy$, et aequatione ad hanc speciem reducta

$$2ad dy - \frac{ady^2}{y} - y^{n+1} dx^2 (1 + xx)^{\frac{n-1}{2}} = 0$$

productum erit :

$$\left. \begin{aligned} + 2aP dx ddy - \frac{aP dx dy^2}{y} - P y^{n+1} dx^3 (1 + xx)^{\frac{n-1}{2}} \\ + 2aQ dy ddy - \frac{aQ dy^3}{y} - Q y^{n+1} dx^2 dy (1 + xx)^{\frac{n-1}{2}} \end{aligned} \right\} = 0$$

quam per hypothefin integrabilem esse oportet. Duo autem primi termini, qualescunque P et Q sint functiones ipsorum x et y , non nisi ex differentiatione horum $2aP dx dy + aQ dy^2$ oriri potuerunt; vnde habebimus

Primam partem integralis $2aP dx dy + aQ dy^2$.

Huius ergo differentiale subtrahamus a nostra aequatione et ob $dP = dx \left(\frac{dP}{dx} \right) + dy \left(\frac{dP}{dy} \right)$; $dQ = dx \left(\frac{dQ}{dx} \right) + dy \left(\frac{dQ}{dy} \right)$, aequatio ordinata erit :

$$\left. \begin{aligned} - P y^{n+1} dx^3 (1 + xx)^{\frac{n-1}{2}} - Q y^{n+1} dx^2 dy (1 + xx)^{\frac{n-1}{2}} - \frac{aP dx dy^2}{y} - \frac{aQ dy^3}{y} \\ - 2a dx^2 dy \left(\frac{dP}{dx} \right) - 2a dx dy^2 \left(\frac{dP}{dy} \right) - a dy^3 \left(\frac{dQ}{dy} \right) \\ - a dx dy^2 \left(\frac{dQ}{dx} \right) \end{aligned} \right\} = 0$$

quae ob dx sumtum constans nullo modo integrabilis esse potest, nisi termini per dy^3 et dy^2 affecti seorsim se tollant. Necessè ergo est, sit :

$$\frac{dQ}{y} + \left(\frac{dQ}{dy} \right) = 0, \text{ seu } 4Q dy + y dy \left(\frac{dQ}{dy} \right) = 0$$

$$\text{et } \frac{dP}{y} + 2 \left(\frac{dP}{dy} \right) + \left(\frac{dQ}{dx} \right) = 0$$

Iam vt ex aequatione priori valorem ipsius Q eruamus, spectemus x vt constans, eritque $dy \left(\frac{dQ}{dy} \right) = dQ$, denotat

notat enim $dy \left(\frac{dQ}{dy} \right)$ incrementum ipsius Q ex solius y variabilitate ortum, vnde cum fit $4Qdy + ydQ = 0$, obtinebimus integrando

$Qy^4 = K$ functioni ipsius x tantum

ita vt fit $Q = -\frac{K}{y^4}$ et $\left(\frac{dQ}{dx} \right) = -\frac{1}{y^4} \left(\frac{dK}{dx} \right)$

vbi $\left(\frac{dK}{dx} \right)$ erit functio ipsius x . Nunc in altera aequatione quoque x sumatur constans, fietque:

$$4Pdy + 2y dP + \frac{dy}{y^3} \left(\frac{dK}{dx} \right) = 0$$

quae per y multiplicata et integrata dat:

$$2Pyy - \frac{1}{y} \left(\frac{dK}{dx} \right) = 2L, \text{ ideoque}$$

$$P = \frac{L}{yy} + \frac{1}{2y^3} \left(\frac{dK}{dx} \right)$$

vbi L denotat functionem ipsius x tantum. Destructis ergo istis membris, ob $\left(\frac{dP}{dx} \right) = \frac{1}{yy} \left(\frac{dL}{dx} \right) + \frac{1}{2y^3} \left(\frac{d^2K}{dx^2} \right)$ erit altera pars integralis:

$$-dx^2 f \left((1+xx)^{\frac{n-1}{2}} \left(Ly^{n+2} dx + \frac{1}{2} y^{n+1} dx \left(\frac{dK}{dx} \right) + Ky^n dy \right) \right. \\ \left. - 2adxx^2 f \left(\frac{dy}{yy} \left(\frac{dL}{dx} \right) + \frac{dy}{2y^3} \left(\frac{d^2K}{dx^2} \right) \right) \right)$$

quae cum conficit duobus membris, pro priori esse

debet $L=0$, et membri $f \left((1+xx)^{\frac{n-1}{2}} \left(\frac{1}{2} y^{n+1} dx \left(\frac{dK}{dx} \right) + Ky^n dy \right) \right)$

integrale erit $\frac{Ky^{n+1}}{n+1} (1+xx)^{\frac{n-1}{2}}$. Superest ergo vt red-

$$\text{datur } \frac{y^{n+1} dK}{n+1} (1+xx)^{\frac{n-1}{2}} + \frac{(n-1)Ky^{n+1}x dx}{n+1} (1+xx)^{\frac{n-3}{2}}$$

$$= \frac{1}{2} y^{n+1} dK (1+xx)^{\frac{n-1}{2}}, \text{ seu } 2(n-1)Kx dx = (n-1) dK (1+xx).$$

Atque hinc elicitur $K = 1 + xx$; ita vt alterius partis integralis membrum prius fit $-\frac{1}{n+1}y^{n+1}dx^2(1+xx)^{\frac{n+1}{2}}$; at membrum posterius ob $L=0$ et $(\frac{dK}{dx})=2$ fiet

$$-2adx^2 \int \frac{dy}{y^3} = \frac{adx^2}{yy}$$

cuius integratio cum sponte successerit, totum negotium est confectum, et integralis pars altera erit:

$$-\frac{1}{n+1}y^{n+1}dx^2(1+xx)^{\frac{n+1}{2}} + \frac{adx^2}{yy}$$

Cum deinde sit $L=0$ et $K=1+xx$, erit $(\frac{dK}{dx})=2x$, hincque fiet: $P=\frac{x}{y^3}$ et $Q=\frac{1+xx}{y^4}$; ex quo integralis pars prima habebitur

$$\frac{2axdx dy}{y^3} + \frac{a(1+xx)dy^2}{y^4}$$

Quocirca aequationis differentio-differentialis propositae adhibito termino constante Cdx^2 integrale completum erit:

$$\frac{adx^2}{yy} + \frac{2axdx dy}{y^3} + \frac{a(1+xx)dy^2}{y^4} - \frac{1}{n+1}y^{n+1}dx^2(1+xx)^{\frac{n+1}{2}} = Cdx^2;$$

seu per y^4 multiplicando:

$$\frac{1}{n+1}y^{n+5}dx^2(1+xx)^{\frac{n+1}{2}} = a(yydx^2 + 2xydx dy + (1+xx)dy^2) - Cy^4dx^2$$

quod egregie conuenit cum eo, quod ante per methodum indirectam eram affectus.

Coroll. I.

14. Aequatio ergo differentio-differentialis

$$2ad dy - \frac{ady^2}{y} - y^{n+1}dx^2(1+xx)^{\frac{n-1}{2}} = 0$$

integrabilis redditur, si multiplicetur per hunc factorem

$$\frac{2dx}{y^3} + \frac{(1+xx)dy}{y^4}$$

qui

qui si aliunde cognosci potuisset, integratio sine vlla difficultate perfecta fuisset.

Coroll. 2.

15. Vicissim ergo si aequatio integralis inuenta

$$\frac{ayydx^2 + 2axydx dy + a(1+xx)dy^2}{y^4} - \frac{1}{n+1} y^{n+1} dx^2 (1+xx)^{\frac{n+1}{2}} = C dx^2$$

sumto elemento dx constante differentietur, quo pacto constans C ex calculo egreditur, differentiale erit diuisibile per hanc formulam $\frac{x dx}{y^3} + \frac{(1+xx) dy}{y^4}$, seu hanc $xy dx + (1+xx) dy$, et diuisione instituta ipsa demum aequatio differentio-differentialis proposita proueniet.

Coroll. 3.

16. Si aequatio proposita per $\frac{\sqrt{1+xx}}{y^4}$ multiplicetur, vt habeatur

$$2a \left(ddy - \frac{2dy^2}{y} \right) \frac{\sqrt{1+xx}}{y^4} - y^n dx^2 (1+xx)^{\frac{n}{2}} = 0$$

multiplicator eam reddens integrabilem erit:

$$\frac{xy dx}{\sqrt{1+xx}} + dy \sqrt{1+xx} = d.y \sqrt{1+xx}$$

Quare si ponatur $y \sqrt{1+xx} = z$, haec obtinebitur aequatio:

$$\frac{2adz(1+xx)^2}{z^4} - \frac{4adz^2(1+xx)^2}{z^5} + \frac{4axdx dz(1+xx)}{z^4} - \frac{zadz^2}{z^3} - z^n dx^2 = 0$$

quae per dz multiplicata integrationem admittit. Erit enim integrale:

$$\frac{adz^2(1+xx)^2}{z^4} + \frac{adz^2}{z^3} - \frac{1}{n+1} z^{n+1} dx^2 = C dx^2.$$

Coroll.

Coroll. 4.

17. Hinc ergo patet, quomodo per idoneam substitutionem integratio subleuari queat; cum enim aequatio proposita per substitutionem $y = \frac{x}{\sqrt{1+xx}}$ in hanc posteriorem formam fuerit transmutata, non amplius foret difficile integrationem peragere. Sed praeterquam quod talis substitutio non facile occurrat, si multiplicator fuerit ordinis tertii, vel altioris, huiusmodi reductio ne locum quidem habere poterit.

Scholion.

18. In hac solutione usus sum singulari specie calculi, qua ad plurium litterarum introductionem vitandam differentiale functionis P duarum variabilium x et y expressi per

$$dP = dx \left(\frac{dP}{dx} \right) + dy \left(\frac{dP}{dy} \right)$$

vbi more iam satis usitato, $dx \left(\frac{dP}{dx} \right)$ denotat incrementum ipsius P ex sola variabilitate ipsius x oriundum, et $dy \left(\frac{dP}{dy} \right)$ eius incrementum, quod ex variabilitate solius y nascitur; constat autem haec duo incrementa addita praebere completum differentiale ipsius P ex vtra variabili x et y natum. Hinc formulae $\left(\frac{dP}{dx} \right)$ et $\left(\frac{dP}{dy} \right)$ denotabunt functiones finitas variabilium x et y , quippe quae per differentiationem omissis differentialibus habentur, ita si sit $P = y\sqrt{1+xx}$, erit $\left(\frac{dP}{dx} \right) = \frac{xy}{\sqrt{1+xx}}$ et $\left(\frac{dP}{dy} \right) = \sqrt{1+xx}$. Tum vero cognita altera parte huiusmodi differentialis veluti $dx \left(\frac{dP}{dx} \right)$, ipsa quantitas P inde ex parte cognoscitur. Spectata enim sola x vt
variabili

variabili erit $P = \int dx \left(\frac{dP}{dx} \right) + Y$, denotante Y functionem ipsius y tantum, atque ex hoc fonte in solutione valores quantitatum P et Q determinavi. Manifestum est quoque, si K fuerit functio ipsius x tantum, tum $dx \left(\frac{dK}{dx} \right)$ eius completum differentiale iam significare, ita ut sit $dx \left(\frac{dK}{dx} \right) = dK$: porro autem haec scriptio $\left(\frac{d^2 K}{dx^2} \right)$ denotat idem quod $\left(\frac{d \left(\frac{dK}{dx} \right)}{dx} \right)$, seu si ponatur $\left(\frac{dK}{dx} \right) = k$, erit $\left(\frac{d^2 K}{dx^2} \right) = \left(\frac{dk}{dx} \right)$. Erit enim pariter k functio ipsius x tantum; ita si sit $K = \sqrt{(1+xx)}$, erit $\left(\frac{dK}{dx} \right) = \frac{x}{\sqrt{(1+xx)}}$ et $\left(\frac{d^2 K}{dx^2} \right) = \frac{-1}{(1+xx)^{3/2}}$: hocque modo ulterius progredi licebit, ut sit $\left(\frac{d^3 K}{dx^3} \right) = \frac{-3x}{(1+xx)^2 \sqrt{(1+xx)}}$, atque haec ad intelligentiam tam huius solutionis, quam sequentium annotasse necesse est visum. Caeterum consideratio huius solutionis facile deducit ad sequens Theorema generalius.

Theorema I.

19. Ista aequatio differentialis secundi gradus, posito dx constante,

$$a ddy - \frac{m a dy^2}{y} + y^n dx^2 (\alpha + 2\beta x + \gamma xx)^{\frac{n-2m+1}{2m-2}} = 0$$

integrabilis redditur, si multiplicetur per hunc factorem:

$$\frac{(\beta + \gamma x) dx}{(m-1) y^{2m-1}} + \frac{(\alpha + 2\beta x + \gamma xx) dy}{y^{2m}}$$

atque aequatio integralis erit:

$$\frac{a\gamma y^2 dx^2 + 2(m-1)a(\beta + \gamma x) y dx dy + (m-1)^2 a(\alpha + 2\beta x + \gamma xx) dy^2}{2(m-1)^2 y^{2m}} + \frac{y^{n-2m+1} dx^2}{n-2m+1} (\alpha + 2\beta x + \gamma xx)^{\frac{n-2m+1}{2m-2}} = C dx^2.$$

Coroll. 1.

20. Si fuerit $n=1$, prodibit ista aequatio differentialis secundi gradus:

$$a dy - \frac{m a d y^2}{y} + \frac{y d x^2}{(\alpha + \beta x + \gamma x x)^2} = 0$$

quae ergo multiplicata per $\frac{(\beta + \gamma x) dx}{(m-1)y^{2m-1}} + \frac{(\alpha + 2\beta x + \gamma x x) dy}{y^{2m}}$

fit integrabilis, eius integrali existente:

$$\frac{\alpha \gamma y y d x^2 + 2(m-1)a(\beta + \gamma x) y d x d y + (m-1)^2 a(\alpha + 2\beta x + \gamma x x) d y^2}{2(m-1)^2 y^{2m}} - \frac{y y d x^2}{2(m-1)y^{2m}(\alpha + 2\beta x + \gamma x x)} = C d x^2.$$

Coroll. 2.

21. Posito $m-1=\mu$, si statuamus $y=e^{sv dx}$, aequatio nostra fiet differentialis primi ordinis:

$$a dv - \mu a v v dx + \frac{d x}{(\alpha + \beta x + \gamma x x)^2} = 0$$

cuius ergo integralis erit

$$\alpha \gamma y y d x^2 + 2 \mu a (\beta + \gamma x) y d x d y + \mu^2 a (\alpha + 2\beta x + \gamma x x) d y^2 - \frac{\mu y y d x^2}{\alpha + \beta x + \gamma x x} = 2 \mu \mu C y^{2m} d x^2$$

seu pro y valore suo substituto

$$\alpha \gamma + 2 \mu a (\beta + \gamma x) v + \mu^2 a (\alpha + 2\beta x + \gamma x x) v v - \frac{\mu}{\alpha + \beta x + \gamma x x} = 2 \mu \mu C e^{2\mu s v dx}.$$

Coroll. 3.

22. Statim ergo aequationis differentialis propositae:

$$a dv - \mu a v v dx + \frac{d x}{(\alpha + \beta x + \gamma x x)^2} = 0$$

posito

posito $C=0$, habemus aequationem integralem particularem, quae est:

$0 = a\gamma + 2\mu a(\beta + \gamma x)v + \mu^2 a(\alpha + 2\beta x + \gamma xx)vv - \frac{\mu}{\alpha + 2\beta x + \gamma xx}$
 ex qua per methodum a me alias expositam integrale completum erui potest. Quin etiam, si illa aequatio differentialis per hanc formam integralem diuidatur, integrabilis reddetur.

Problema 2.

23. Proposita aequatione differentiali secundi gradus:

$$\frac{dy}{y} + \frac{a dx^2}{(x + 2\beta x + \gamma xx + cy)^2} = 0$$

In qua differentiale dx sumtum est constans, eius integrale inuenire.

Solutio.

Tentetur iterum integratio per factorem $Pdx + Qdy$, ac posito breuitatis gratia $\alpha + 2\beta x + \gamma xx + cy = Z$, conuertatur aequatio in hanc formam:

$$ddy + \frac{ay dx^2}{ZZ} = 0$$

quae per $Pdx + Qdy$ multiplicata praebet:

$$Pdxddy + Qdyddy + \frac{aPy dx^2}{ZZ} + \frac{aQy dx^2 dy}{ZZ} = 0.$$

Quae cum integrabilis esse debeat, dabit statim

$$I. \text{ primam integralis partem} = Pdx dy + \frac{1}{2} Qdy^2;$$

superest ergo, vt integrabilis reddatur sequens expressio:

$$-\frac{1}{2} dy^2 \left(\frac{dQ}{dy} \right) - \frac{1}{2} dx dy^2 \left(\frac{dQ}{dx} \right) + \frac{aQy dx^2 dy}{ZZ} + \frac{aPy dx^2}{ZZ} \\ = dx dy^2 \left(\frac{dP}{dy} \right) - dx^2 dy \left(\frac{dP}{dx} \right).$$

Z 2.

Primum

Primum ergo necesse est, vt sit $(\frac{dQ}{dy}) = 0$, vnde fit Q functio ipsius x tantum, quae fit $Q = K$; tum vero etiam termini dy^2 inuoluentes destruendi sunt, ex quibus fit:

$$(\frac{dK}{dy}) + 2(\frac{dP}{dx}) = 0$$

feu sumto solo y pro variabili:

$$dy(\frac{dK}{dx}) + 2dP = 0.$$

cuius integrale est

$$P = L - \frac{1}{2}y(\frac{dK}{dx})$$

denotante L quoque functionem ipsius x . Quare ob

$$(\frac{dP}{dx}) = (\frac{dL}{dx}) - \frac{1}{2}y(\frac{d^2K}{dx^2})$$

et dx sumtum constans, altera pars integralis erit:

$$dx^2 / \frac{ay}{ZZ} (L dx - \frac{1}{2}y dx (\frac{dK}{dx}) + K dy) - dx^2 f dy ((\frac{dL}{dx}) - \frac{1}{2}y (\frac{d^2K}{dx^2}))$$

$$\text{at est } \int \frac{aKy dy}{ZZ} = aK \int \frac{y dy}{(\alpha + \beta x + \gamma xx + cy)^2}$$

vnde pro integrali nascitur

$$II. \text{ pars} = -\frac{a}{2c} \cdot \frac{K dx}{\alpha + \beta x + \gamma xx + cy}$$

ideoque debet esse:

$$\frac{ay}{ZZ} (L dx - \frac{1}{2}y dK) = -\frac{a}{2c} \cdot \frac{(\alpha + \beta x + \gamma xx + cy)dK - 2Kdx(\beta + \gamma x)}{ZZ}$$

feu

$$acLydx - \frac{1}{2}acyy dK = aKdx(\beta + \gamma x) - \frac{1}{2}adK(\alpha + 2\beta x + \gamma xx + cy)$$

$$\text{vel } acLydx = aKdx(\beta + \gamma x) - \frac{1}{2}adK(\alpha + 2\beta x + \gamma xx)$$

Perpicuum ergo est, esse debere

$$L = 0 \text{ et } K = \alpha + 2\beta x + \gamma xx.$$

Quare ob $(\frac{d^2K}{dx^2}) = 2\gamma$ erit

$$III. \text{ultima pars integralis} = +\frac{1}{2}\gamma yy dx^2.$$

Cum

Cum igitur fit :

$P = -\gamma(\beta + \gamma x)$ et $Q = \alpha + 2\beta x + \gamma xx$
erit noster multiplicator :

$$-y dx (\beta + \gamma x) + dy (\alpha + 2\beta x + \gamma xx)$$

et integrale quaesitum habebitur :

$$-y dx dy (\beta + \gamma x) + \frac{1}{2} dy^2 (\alpha + 2\beta x + \gamma xx) - \frac{\alpha(\alpha + 2\beta x + \gamma xx) dx^2}{2(\alpha + 2\beta x + \gamma xx + cyy)} + \frac{1}{2} \gamma yy dx^2 = C dx^2$$

At si ponatur $C = \frac{-a}{2c} + C$, erit hoc integrale :

$$\frac{1}{2} \gamma yy dx^2 - y dx dy (\beta + \gamma x) + \frac{1}{2} dy^2 (\alpha + 2\beta x + \gamma xx) + \frac{\alpha yy dx^2}{2(\alpha + 2\beta x + \gamma xx + cyy)} = C dx^2.$$

Quae forma conuenit cum ea, quam supra exhibui.

Theorema 2.

24. Ista aequatio differentialis secundi gradus posito dx constante

$$ddy + \frac{ay^{n+1} dx^2}{(\alpha + 2\beta x + \gamma xx + cyy)^{\frac{n+1}{2}}} = 0$$

integrabilis reddetur per multiplicatorem :

$$-y dx (\beta + \gamma x) + dy (\alpha + 2\beta x + \gamma xx)$$

et integrale erit :

$$\frac{1}{2} \gamma yy dx^2 - y dx dy (\beta + \gamma x) + \frac{1}{2} dy^2 (\alpha + 2\beta x + \gamma xx) + \frac{ay^{n+1} dx^2}{(n+2)(\alpha + 2\beta x + \gamma xx + cyy)^{\frac{n+2}{2}}} = C dx^2.$$

Coroll. 1.

25. Casus problematis nascitur ex Theoremate hoc, si ponatur $n=0$. Ceterum integrale in Theore-

mate exhibitum simili modo elicitur, quo solutionem problematis expediuimus; vnde superfluum foret eius demonstrationem adiicere.

Coroll. 2.

26. Si ponatur $c=0$, casus habebitur, quem etiam ex Theoremate primo deriuare licet, si ibi ponatur $m=0$. Dum enim pro a scribitur $\frac{x}{a}$ et $n+1$ loco n ; integrale ibi datum perfecte congruit cum hoc, quod istud Theorema suppeditat pro casu $c=0$.

Coroll. 3.

27. Hoc autem Theorema adeo primum in se complectitur: aequatio enim primi

$$a d d y - \frac{m a d y^2}{y} + y^n d x^2 (\alpha + 2 \beta x + \gamma x x)^{\frac{n-4m+3}{2m-2}} = 0$$

si ponatur $y = z^{\frac{1}{1-m}}$ abit in hanc:

$$\frac{a}{1-m} z^{\frac{m}{1-m}} d d z + z^{\frac{n}{1-m}} d x^2 (\alpha + 2 \beta x + \gamma x x)^{\frac{n-4m+3}{2m-2}} = 0$$

$$\text{seu } \frac{a d d z}{1-m} + z^{\frac{n-m}{1-m}} d x^2 (\alpha + 2 \beta x + \gamma x x)^{\frac{n-4m+3}{2m-2}} = 0.$$

Quod si iam statuatur $\frac{n-m}{1-m} = n+1$, vt fiat $n=1-n(m-1)$ aequatio haec abibit in istam formam:

$$\frac{a d d z}{1-m} + z^{n+1} d x^2 (\alpha + 2 \beta x + \gamma x x)^{\frac{-n-4}{2}} = 0$$

quae est casus particularis praesentis Theorematis, ex quo quippe nascitur, ponendo $c=0$.

Coroll.

Coroll. 4.

28. Praefens ergo Theorema latissime patet, atque eiusmodi casus difficillimos in se complectitur, qui nullo alio modo resolui posse videntur. Si enim $c=0$, fortasse reperietur methodus negotium conficiens, propterea quod variables non sunt inuicem permixtae: at si c non $=0$, ob permixtionem variabilium nulla methodus cognita hic cum successu in usum vocabitur.

Coroll. 5.

29. Casus hic imprimis notatu dignus hic occurrit, si $\alpha=0$, $\beta=0$, $\gamma=c=1$, quo habetur haec aequatio:

$$ddy + \frac{ay^{n+2}dx^2}{(xx+yy)^{\frac{n+2}{2}}} = 0$$

cuius ergo integrale est:

$$\frac{1}{2}(ydx - xdy)^2 + \frac{ay^{n+2}dx}{(n+2)(xx+yy)^{\frac{n+2}{2}}} = Cdx^2$$

Ponatur $y=ux$, erit $ydx - xdy = -xxdu$, fietque integrale:

$$\frac{1}{2}x^2du^2 + \frac{ax^{n+2}dx^2}{(n+2)(1+uu)^{\frac{n+2}{2}}} = Cdx^2$$

ideoque $\frac{dx}{xx} = \frac{du(1+uu)^{\frac{n+2}{2}}}{\sqrt{(2C(1+uu)^{\frac{n+2}{2}} - \frac{2a}{n+2}u^{n+2})}}$

quae ob variables separatas denuo integrari potest.

Scholion.

Scholion.

30. Hic quoque multiplicatoris forma substitutionem idoneam praebet, cuius ope aequatio differentio-differentialis in aliam tractatu faciliorem transformabitur. Statui scilicet oportet

$$y = z\sqrt{\alpha + 2\beta x + \gamma xx}$$

Hanc vero ipsam substitutionem suadet formulae indoles

$$(\alpha + 2\beta x + \gamma xx + cy) \frac{dy}{y^2}$$

quia hoc pacto vnica variabilis in vinculo relinquitur. At per hanc substitutionem ipsa aequatio multo magis fit perplexa, ita vt, etiamsi per factorem simpliciter

$dz(\alpha + 2\beta x + \gamma xx)^{\frac{3}{2}}$ ad integrabilitatem reuocetur, id tamen minus pateat. Verum si multiplicator fuerit ordinis tertii, seu altioris, ne huiusmodi quidem substitutio commode inueniri potest, vti in duobus reliquis exemplis vsu venit.

Problema 3.

31. Proposita aequatione differentiali secundi gradus:

$$yyddy + mydy^2 = axdx^2$$

in qua differentiale dx sumtum est constans, eius integrale inuenire.

Solutio.

Quia multiplicator neque primi, neque secundi ordinis succedit, ex ordine tertio desumatur. Perducta ergo aequatione ad hanc formam:

$$ddy + \frac{m}{y} dy^2 - \frac{ax}{yy} dx^2 = 0$$

multi-

multiplicetur ea per $Pdx^2 + 2Qdxdy + 3Rdy^2$, vnde statim habebitur:

I. *prima pars integralis* $Pdx^2dy + Qdxdy^2 + Rdy^3$

et integrando relinquitur haec forma:

$$\frac{1}{y} \frac{dPx^2}{dy} + \frac{2}{y} \frac{dQxdy}{dy} + \frac{3}{y} \frac{dRdy^2}{dy} + \frac{mPx^2dy^2}{y} + \frac{2mQdxdy^2}{y} + \frac{3mRdy^3}{y} - dx^2dy \left(\frac{dP}{dx}\right) - dx^2dy^2 \left(\frac{dP}{dy}\right) - dx^2dy^3 \left(\frac{dQ}{dy}\right) - dy^4 \left(\frac{dR}{dy}\right) - dx^2dy^2 \left(\frac{dQ}{dx}\right) - dx^2dy^3 \left(\frac{dR}{dx}\right).$$

Haec autem forma integrabilis esse nequit, nisi membra, quae dy^2, dy^3 et dy^4 implicant, destruantur. Primum ergo pro dy^4 habebimus:

$$\frac{3mR}{y} - \left(\frac{dR}{dy}\right) = 0, \text{ seu } 3mRdy = ydR$$

vbi x sumitur pro constante, vnde fit $R = Ky^{3m}$, denotante K functionem ipsius x tantum, sicque erit: $\left(\frac{dR}{dy}\right) = 3m \left(\frac{dK}{dx}\right)$. Iam pro destructione terminorum dy^2 continentium, fiet:

$$\frac{2mQ}{y} - \left(\frac{dQ}{dy}\right) - y^3 m \left(\frac{dK}{dx}\right) = 0$$

seu sumto x constante:

$$2mQdy - ydQ = y^{3m+1} dy \left(\frac{dK}{dx}\right)$$

quae diuisa per y^{3m+1} et integrata dat:

$$\frac{-Q}{y^{2m}} = \frac{1}{m+1} y^{m+1} \left(\frac{dK}{dx}\right) - L.$$

Sumta denuo L pro functione ipsius x , ita vt sit

$$Q = Ly^{2m} - \frac{1}{m+1} y^{3m+1} \left(\frac{dK}{dx}\right), \text{ ideoque } \left(\frac{dQ}{dx}\right) = y^{2m} \left(\frac{dL}{dx}\right) - \frac{1}{m+1} y^{3m+1} \left(\frac{ddK}{dx^2}\right).$$

Destruantur denique etiam termini dy^2 continentés, unde prodit :

$$-3aKy^{2m-2}x - y^{2m} \left(\frac{dL}{dx} \right) + \frac{1}{m+1} y^{2m+1} \left(\frac{d^2K}{dx^2} \right) + \frac{mP}{y} - \left(\frac{dP}{dy} \right) = 0$$

quae sumta x constante per $y dy$ multiplicata praebet :

$$-3aKxy^{2m-1} dy - y^{2m+1} dy \left(\frac{dL}{dx} \right) + \frac{1}{m+1} y^{2m+2} dy \left(\frac{d^2K}{dx^2} \right) + mP dy - y dP = 0$$

quae per y^{m+1} diuisa et integrata dat :

$$\frac{-3a}{2m+1} K x y^{2m-1} - \frac{1}{m+1} y^{m+1} \left(\frac{dL}{dx} \right) + \frac{1}{2(m+1)^2} y^{2m+2} \left(\frac{d^2K}{dx^2} \right) - \frac{P}{y^m} + M = 0$$

denotante M functionem ipsius x tantum. Ergo fit

$$P = M y^m - \frac{3a}{2m+1} K x y^{2m-1} - \frac{1}{m+1} y^{m+1} \left(\frac{dL}{dx} \right) + \frac{1}{2(m+1)^2} y^{2m+2} \left(\frac{d^2K}{dx^2} \right)$$

ideoque

$$\left(\frac{dP}{dx} \right) = y^m \left(\frac{dM}{dx} \right) - \frac{3a}{2m+1} K y^{2m-1} - \frac{3ax}{2m+1} y^{2m-1} \left(\frac{dK}{dx} \right) - \frac{1}{m+1} y^{2m+1} \left(\frac{d^2L}{dx^2} \right) + \frac{1}{2(m+1)^2} y^{2m+2} \left(\frac{d^3K}{dx^3} \right)$$

Nunc termini $\frac{2ax dx dy}{yy} - dx^2 dy \left(\frac{d^2P}{dx^2} \right)$, integrati, x pro constante sumta, suppeditabunt.

II. alteram integralis partem :

$$-2ax dx^2 \left(\frac{1}{2m+1} L y^{2m-1} - \frac{1}{3m(m+1)} y^{2m} \left(\frac{d^2K}{dx^2} \right) \right) - N dx^2 - dx^2 \left(\frac{1}{m+1} y^{m+1} \left(\frac{dM}{dx} \right) - \frac{a}{m(2m+1)} K y^{2m} - \frac{ax}{m(2m+1)} y^{2m} \left(\frac{dK}{dx} \right) - \frac{1}{2(m+1)^2} y^{2m+2} \left(\frac{d^2L}{dx^2} \right) + \frac{1}{6(m+1)^3} y^{2m+3} \left(\frac{d^3K}{dx^3} \right) \right)$$

Huius ergo differentiale posito y constante sumtum aequale esse debet residuae parti $\frac{-aP-x dx^2}{yy}$; unde per dx^2 diuiso habebimus sequentem aequationem :

aMuy

$$\begin{aligned}
 M x y^{m-2} - \frac{2 a x x}{2 m-1} K y^{2 m-3} - \frac{a x}{m+1} y^{2 m-1} \left(\frac{d L}{d x} \right) + \frac{a x}{2(m+1)^2} y^{2 m} \left(\frac{d d K}{d x^2} \right) \\
 - \frac{2 a}{2 m-1} L y^{2 m-1} + \frac{2 a}{2 m(m+1)} y^{2 m} \left(\frac{d K}{d x} \right) - \frac{2 a x}{2 m-1} y^{2 m-1} \left(\frac{d L}{d x} \right) + \frac{2 a x}{2 m(m+1)} x \\
 y^{2 m} \left(\frac{d d K}{d x^2} \right) - \frac{1}{m+1} y^{m+1} \left(\frac{d d M}{d x^2} \right) + \frac{a}{m(2 m-1)} y^{2 m} \left(\frac{d K}{d x} \right) + \frac{a}{m(2 m-1)} y^{2 m} \left(\frac{d K}{d x} \right) \\
 + \frac{a x}{m(2 m-1)} y^{2 m} \left(\frac{d d K}{d x^2} \right) + \frac{1}{2(m+1)^2} y^{2 m+2} \left(\frac{d^2 L}{d x^2} \right) - \frac{1}{6(m+1)} y^{2 m+2} \\
 \left(\frac{d^4 K}{d x^4} \right) = \text{functioni ipsius } x = \left(\frac{d N}{d x} \right).
 \end{aligned}$$

Hic iam singulae diuersae ipsius y potestates seorsim ad nihilum redigantur, et quia $y^m - 2$ et y^{2m-3} semel occurrunt, nisi sit vel $m=2$, vel $m=1$, habebimus $M=0$, et $K=0$; et supererunt tantum termini per L affecti, inter quos solitarius est y^{2m+2} ; vnde esse debet $\left(\frac{d^2 L}{d x^2} \right) = 0$, ideoque $L = a + 2\beta x + \gamma x x$, reliqui per y^{2m-1} affecti dant:

$$\frac{-2 a x (\beta + \gamma x)}{m+1} - \frac{2 a (\alpha + 2 \beta x + \gamma x x)}{2 m-1} - \frac{4 a x (\beta + \gamma x)}{2 m-1} = 0.$$

Hinc debet esse $\alpha=0$, et $\frac{\beta + \gamma x}{m+1} + \frac{4 \beta + 3 \gamma x}{2 m-1} = 0$.

Quibus conditionibus in genere satisfieri nequit; constituendi ergo sunt casus sequentes:

I. Si $\alpha=0$, et $\gamma=0$, fiet $m = -\frac{1}{2}$, ita vt aequatio proposita sit:

$$y y d d y - \frac{2}{3} y d y^2 = a x d x^2$$

seu

$$d d y - \frac{d y^2}{2 y} - \frac{a x d x^2}{y y} = 0.$$

Cum igitur sit $K=0$, $L=x$, $M=0$, erit:

$$R=0; Q = \frac{x}{y}; \text{ et } P = -2$$

et noster multiplicator erit: $-2 d x^2 + \frac{x d x d y}{y}$

ideoque integrale quaesitum:

$$-2 d x^2 d y + \frac{x d x d y^2}{y} + \frac{a x x d x^2}{y y} = C d x^2,$$

feu per dx diuidendo

$$axxdx^2 + xydy^2 - 2yydxdy = Cy y dx^2$$

II. Sit $\alpha = 0$; $\beta = 0$; erit $m = -\frac{2}{3}$; et aequatio differentio-differentialis proposita :

$$ddy - \frac{2dy^2}{3y} - \frac{axdx^2}{yy} = 0.$$

Cum igitur fit $K = 0$, $L = xx$, et $M = 0$, erit

$$R = 0; Q = xxy^{-\frac{4}{3}}; P = -\frac{10}{3}xy^{\frac{1}{3}}$$

vnde noster multiplicator fiet :

$$-\frac{10}{3}xy^{\frac{1}{3}}dx^2 + 2xxy^{-\frac{4}{3}}dxdy$$

et integrale quaesitum

$$-\frac{10}{3}xy^{\frac{1}{3}}dx^2 dy + xxy^{-\frac{4}{3}}dxdy^2 + \frac{10}{9}ax^2y^{-\frac{2}{3}}dx^3 + \frac{25}{9}y^{\frac{4}{3}}dx^3 = Cdx^{\frac{3}{2}}$$

feu per dx diuidendo, et $y^{\frac{2}{3}}$ multiplicando,

$$-\frac{10}{3}xy y dx dy + xxy dy^2 + \frac{10}{9}ax^2 dx^2 + \frac{25}{9}y^2 dx^2 = Cy^{\frac{2}{3}} dx^2$$

III. Ante vero iam duos casus commemorauimus, quibus est vel $m = 1$, vel $m = 2$. Sit ergo primo $m = 1$ et aequatio proposita

$$ddy + \frac{dy^2}{y} - \frac{axdx^2}{yy} = 0$$

ac fieri debet

$$\begin{aligned} \left(\frac{dN}{dx}\right) &= \frac{aMx}{y} - 3axxK - \frac{1}{2}axy\left(\frac{dL}{dx}\right) + \frac{1}{3}axy^2\left(\frac{d^2K}{dx^2}\right) \\ &- 2aLy + \frac{1}{3}ay^3\left(\frac{dK}{dx}\right) - 2axy\left(\frac{dL}{dx}\right) + \frac{1}{3}axy^2\left(\frac{d^2K}{dx^2}\right) \\ &- \frac{1}{2}yy\left(\frac{d^2M}{dx^2}\right) + 2ay^2\left(\frac{dK}{dx}\right) + axy^3\left(\frac{d^2K}{dx^2}\right) + \frac{1}{3}y^4\left(\frac{d^3L}{dx^3}\right) - \frac{1}{24}y^6\left(\frac{d^4K}{dx^4}\right) \end{aligned}$$

vnde obtinemus $M = 0$; $N = -3aa\int Kxxdx$; et

$$\begin{aligned} -\frac{1}{3}x\left(\frac{dL}{dx}\right) - 2L &= 0; \frac{25}{24}x\left(\frac{d^2K}{dx^2}\right) + \frac{1}{3}\left(\frac{dK}{dx}\right) = 0 \\ \left(\frac{d^3L}{dx^3}\right) &= 0; \left(\frac{d^4K}{dx^4}\right) = 0. \end{aligned}$$

His

His conditionibus satisfit, si sumatur :

$$L=0; K=1; M=0; \text{ et } N=-aax^3$$

vnde fit: $R=y^3; Q=0; P=-3axy^2$.

Quare noster multiplicator erit :

$$-3axy^2dx^2 + 3y^3dy^2$$

et integrale quaesitum :

$$-3axy^2dx^2dy + y^3dy^3 + xy^2dx^2 + aax^3dx^2 = Cdx^3.$$

IV. Sit iam $m=2$, vt aequatio nostra fiat

$$ddy + \frac{2dy^2}{y} - \frac{axdx^2}{yy} = 0$$

ac satisfieri debet huic aequationi :

$$\left(\frac{dN}{dx}\right) = aMx - aKxy^2 - \frac{2}{3}aLy^3 - axy^2\left(\frac{dL}{dx}\right) - \frac{1}{3}y\left(\frac{dM}{dx^2}\right) \\ + \frac{1}{3}ay^6\left(\frac{d^2K}{dx^2}\right) + \frac{1}{18}y^6\left(\frac{d^3L}{dx^3}\right) + \frac{1}{3}axy^6\left(\frac{d^2K}{dx^2}\right) - \frac{1}{18}y^6\left(\frac{d^4K}{dx^4}\right).$$

Erit ergo $N = a \int Mx dx$, ac statui potest $L=0; K=0; M=1$, vnde fit $N = \frac{1}{2}axx$. Hinc vero fit :

$$R=0; Q=0; P=y^2$$

ita vt multiplicator futurus sit ydx^2 et integrale :

$$yydx^2dy - \frac{1}{2}axxdx^2 = Cdx^2, \text{ seu} \\ 2yydy - axxdx = Cdx.$$

Coroll. I.

§ 2. Casus ergo vltimus, quo $m=2$, est omnium facillimus, cum per multiplicatorem adeo primi ordinis confici possit, quin primo intuitu aequationis

$$yyddy + 2ydy^2 = axdx^2$$

integrale $yydy = \frac{1}{2}axxdx + Cdx$ patet. Casus autem primus et secundus, quibus est $m=-\frac{1}{2}$ et $m=-\frac{2}{3}$ per multi-

multiplicatorem formae secundae, ob $R=0$, resolui potuissent.

Coroll. 2.

33. Solus ergo casus tertius, quo est $m=1$, resolutu est difficillimus, quia requirit multiplicatorem formae tertiae. Quare notetur, sequentem aequationem differentialem secundi gradus

$$yyddy + ydy^2 - axdx^2 = 0$$

integrabilem reddi, si multiplicetur per

$$3ydy^2 - 3axdx^2$$

et integrale esse:

$$y^3 dy^3 - 3axydydx^2 + ay^3 dx^3 + aax^3 dx^3 = Cdx^3.$$

Coroll. 3.

34. Porro autem notandum est, hanc expressionem in tres factores simplices resolui posse. Si enim ponatur breuitatis gratia $a=c^2$ et $\mu = -\frac{1+\sqrt{-3}}{2}$ et $\nu = -\frac{1-\sqrt{-3}}{2}$, aequatio haec integralis ita repraesentari potest:

$$(ydy + cydx + c^2x dx)(ydy + \mu cydx + \nu c^2x dx)(ydy + \nu cydx + \mu c^2x dx) = Cdx^3.$$

Coroll. 4.

35. Hinc si constans C sumatur $=0$, tres statim prodeunt aequationes integrales particulares:

$$ydy + cydx + c^2x dx = 0$$

$$ydy + \mu cydx + \nu c^2x dx = 0$$

$$ydy + \nu cydx + \mu c^2x dx = 0$$

quarum

quarum prima continet casum iam supra (7) indicatum
duae reliquae vero sunt imaginariae.

Scholion.

36. Restat ergo quartum exemplum, quod erat

$$ds^2(\alpha s + \beta s + \gamma) = r r dr^2 + 2r^2 ddr$$

quod posito

$$r = y^{\frac{2}{3}}; \text{ vt. fit. } dr = \frac{2}{3} y^{-\frac{1}{3}} dy, \text{ et } ddr = \frac{2}{3} y^{-\frac{1}{3}} ddy - \frac{2}{3} y^{-\frac{4}{3}} dy^2$$

abit in hanc formam :

$$\frac{4}{3} y^{\frac{5}{3}} ddy = ds^2(\alpha s + \beta s + \gamma).$$

In genere autem obseruo, si habeatur huiusmodi aequatio:

$$S ds^2 = m r^m dr^2 + n r^{m+1} ddr$$

eam per substitutionem $r = y^{\frac{m}{m+1}}$ reduci ad hanc formam simpliciore[m] :

$$S ds^2 = \frac{n n}{m+1} y^{\frac{m n - m + n}{m+1}} ddy.$$

Huiusmodi ergo aequationes omnes complecti licet in hac forma generali : $ddy = y^n X dx^2$. Videamus ergo quibusnam casibus tam exponentis n , quam functionis X , haec aequatio integrari queat per nostram methodum.

Problema 4.

37. Casus pro exponente n et naturam functionis X inuenire, quibus haec aequatio differentialis secundi gradus

$$ddy + y^n X dx^2 = \sigma,$$

vbi dx est constans, integrari queat.

Solutio

Solutio I.

Sumatur primo multiplicator primi ordinis P, et integranda erit haec aequatio:

$$Pddy + y^n PX dx^2 = 0$$

ac integralis pars prima erit $= Pdy$, et integranda restat haec expressio:

$$y^n PX dx^2 - dx dy \left(\frac{dP}{dx} \right) - dy^2 \left(\frac{dP}{dy} \right)$$

vnde necesse est, sit $\left(\frac{dP}{dy} \right) = 0$, ideoque P functio ipsius x tantum. Sit ergo $P = K$, et integrari oportet ob dx constans:

$$dx (y^n K X dx - dy \left(\frac{dK}{dx} \right))$$

cuius integrale nequit esse, nisi $-y dx \left(\frac{dK}{dx} \right) = -y dK$. Oportet autem sit $y^n K X dx^2 + y d dK = 0$,

quod fieri nequit, nisi sub his conditionibus:

$$n = 1 \text{ et } X = -\frac{d dK}{K dx^2}$$

ac tum aequatio integralis erit:

$$K dy - y dK = C dx.$$

Solutio II.

Sumto multiplicatore secundae formae $P dx + 2Q dy$, integrabilis efficienda est haec aequatio:

$$2Q dy ddy + P dx ddy + y^n X dx^2 (P dx + 2Q dy) = 0$$

vnde integralis pars prima colligitur $P dx dy + Q dy^2$.

Supereft ergo, vt integretur:

$$y^n PX dx^2 + 2y^n QX dx^2 dy$$

$$- dx^2 dy \left(\frac{dP}{dx} \right) - dx dy^2 \left(\frac{dP}{dy} \right)$$

$$- dx dy^2 \left(\frac{dQ}{dx} \right) - dy^3 \left(\frac{dQ}{dy} \right).$$

Hinc

Hinc quo termini tollantur, quibus dy plus vna habet dimensione, oportet esse

$$\left(\frac{dQ}{dy}\right) = 0; \text{ ideoque } Q = K \text{ functioni ipsius } x.$$

Deinde habebimus

$$\left(\frac{dP}{dy}\right) + \left(\frac{dQ}{dx}\right) = 0, \text{ seu } dP + dy\left(\frac{dK}{dx}\right) = 0$$

vnde fit:

$$P = L - y\left(\frac{dK}{dx}\right) \text{ et } \left(\frac{dP}{dx}\right) = \left(\frac{dL}{dx}\right) - y\left(\frac{d^2K}{dx^2}\right).$$

Iam altera pars integralis erit:

$$dx^2 \int (y^n P X dx + 2y^n Q X dy - dy\left(\frac{dP}{dx}\right)) \text{ siue}$$

$$dx^2 \int \left\{ \begin{array}{l} + y^n L X dx \quad + 2y^n K X dy \\ - y^{n+1} X dx \left(\frac{dK}{dx}\right) - dy\left(\frac{dL}{dx}\right) + y dy\left(\frac{d^2K}{dx^2}\right) \end{array} \right\}$$

ex variabilitate ipsius y ergo concluditur altera pars integralis.

$$\text{II. } dx^2 \left(\frac{2}{n+1} y^{n+1} K X - y\left(\frac{dL}{dx}\right) + \frac{1}{2} y y\left(\frac{d^2K}{dx^2}\right) + M \right).$$

Ac variabilitas ipsius x postulat, vt fit:

$$y^n L X - y^{n+1} X \left(\frac{dK}{dx}\right) = \frac{2}{n+1} y^{n+1} K \left(\frac{dX}{dx}\right) + \frac{2}{n+1} y^{n+1} X \left(\frac{dK}{dx}\right) - y\left(\frac{d^2L}{dx^2}\right) + \frac{1}{2} y y\left(\frac{d^2K}{dx^2}\right) + \left(\frac{dM}{dx}\right).$$

Si n velimus indefinitum relinquere; esse debet

$$L = 0; \left(\frac{d^2K}{dx^2}\right) = 0 \text{ et } \left(\frac{dM}{dx}\right) = 0; \text{ tum vero}$$

$$\frac{2}{n+1} K \left(\frac{dX}{dx}\right) + \frac{n+2}{n+1} X \left(\frac{dK}{dx}\right) = 0$$

vnde colligitur $K^{\frac{n+2}{n+1}} X = A$ constanti: at ob $\left(\frac{d^2K}{dx^2}\right) = 0$

erit $K = \alpha + 2\beta x + \gamma xx$, ideoque $X = \frac{A}{(\alpha + \beta x + \gamma xx)^{\frac{n+2}{2}}}$ et

$Q = \alpha + 2\beta x + \gamma xx$; $P = -2y(\beta + \gamma x)$. Quocirca multiplicator erit:

$$-2y dx(\beta + \gamma x) + 2dy(\alpha + 2\beta x + \gamma xx)$$

et huius aequationis differentio-differentialis

$$d^2y + \frac{Ay^n dx^2}{(\alpha + 2\beta x + \gamma xx)^2} = 0$$

integrale erit:

$$-2y dx dy (\beta + \gamma x) + dy^2 (\alpha + 2\beta x + \gamma xx) + \frac{A}{y^{n+1}} \frac{dx^2}{(\alpha + 2\beta x + \gamma xx)^2} = C dx^2.$$

Superfunt autem casus, quibus est vel $n=1$, vel $n=2$.

I. Sit $n=1$; et condiciones praecedentes postulant

$$LX + \left(\frac{d^2L}{dx^2}\right) = 0; \quad \frac{2}{n+1}K\left(\frac{dX}{dx}\right) + \frac{n+3}{n+1}X\left(\frac{dK}{dx}\right) + \frac{1}{2}\left(\frac{d^2K}{dx^2}\right) = 0$$

$$\text{feu } LX dx^2 + ddL = 0 \text{ et } 2K dX + 4X dK + dx\left(\frac{d^2K}{dx^2}\right) = 0$$

hinc fit $2KKX + \int \frac{Kd^2K}{dx^2} = \text{Const.}$ ideoque

$$2KKX dx^2 + K d dK - \frac{1}{2} dK^2 = C dx^2$$

$$\text{et } X = \frac{E dx^2 + \frac{1}{2} dK^2 - K d dK}{2KK}$$

pro priori conditione autem ponatur $L=0$. Quare erit

$$Q=K; \quad P=-y\left(\frac{dK}{dx}\right); \quad \text{atque huius aequationis}$$

$$d^2y + yX dx^2 = 0.$$

$$\text{Existente } X = \frac{E dx^2 + \frac{1}{2} dK^2 - K d dK}{2KK dx^2}, \text{ quaecunque fun-}$$

ctio ipsius x sumatur pro K , erit integrale:

$$-y dx dy \left(\frac{dK}{dx}\right) + K dy^2 + yyKX dx^2 + \frac{1}{2} yy dx^2 \left(\frac{d^2K}{dx^2}\right) = C dx^2$$

II. Sit $n=2$; et condiciones postulant:

$$2K dX + 5X dK = 0; \quad LX = \frac{1}{2}\left(\frac{d^2K}{dx^2}\right); \quad \left(\frac{d^2L}{dx^2}\right) = 0.$$

Prima

Prima dat $X = AK^{-\frac{5}{2}}$, qui in altera substitutus praebet

$$2ALK^{-\frac{5}{2}}dx^2 = d^2K;$$

verum, ob $(\frac{dL}{dx}) = 0$, erit $L = \alpha + \beta x$, vnde, posito

$$K = (\alpha + \beta x)^\mu, \text{ erit } 2A(\alpha + \beta x)^{\mu-\frac{5}{2}} = \mu(\mu-1)(\mu-2) \\ (\alpha + \beta x)^{\mu-3}\beta^2$$

et $\mu = \frac{5}{2}$; hincque $2A = \frac{43}{343}\beta^2$; et $X = \frac{A}{(\alpha + \beta x)^{\frac{20}{7}}} = \frac{21\beta^2}{343(\alpha + \beta x)^{\frac{20}{7}}}$

Porro $Q = (\alpha + \beta x)^{\frac{6}{7}}$; $P = \alpha + \beta x - \frac{8}{7}\beta y(\alpha + \beta x)^{\frac{1}{7}}$

Consequenter huius aequationis differentio-differentialis

$$d^2y + y^2 X dx^2 = 0$$

existente $X = \frac{21\beta^2}{343(\alpha + \beta x)^{\frac{20}{7}}}$, integrale est

$$dx dy (\alpha + \beta x - \frac{8}{7}\beta y(\alpha + \beta x)^{\frac{1}{7}}) + dy^2 (\alpha + \beta x)^{\frac{6}{7}} - \frac{116\beta^2 y^2 dx^2}{343(\alpha + \beta x)^{\frac{15}{7}}} \\ - \beta y dx^2 + \frac{4\beta^2 y^2 dx^2}{49(\alpha + \beta x)^{\frac{6}{7}}} = C dx^2$$

II. Si $n = 2$, adhuc casus notari meretur, quo $L = \alpha$, et posito

$$K = x^\mu, \text{ erit } 2\alpha Ax^{\mu-\frac{5}{2}} = \mu(\mu-1)(\mu-2)x^{\mu-3}, \text{ vnde fit } \mu = \frac{5}{2}$$

et $2\alpha A = \frac{6 \cdot 1 \cdot 3}{343}$; ideoque $\alpha = \frac{24}{343A}$. Quare erit

$$K = x^{\frac{5}{2}}; L = \frac{24}{343A}; X = \frac{A}{x^{\frac{15}{2}}}; \text{ ac porro}$$

$$Q = x^{\frac{6}{7}}; P = \frac{24}{343A} - \frac{16y}{7x^{\frac{1}{2}}}$$

Consequenter huius aequationis :

$$d^2y + \frac{Ay^2 dx^2}{x^{\frac{15}{2}}} = 0$$

integrale erit

$$\frac{2dx dy}{s+2A} - \frac{ay dx dy}{2ax^2} + x^{\frac{6}{7}} dy^2 + \frac{2Ay^2 dx^2}{2ax^7} - \frac{yy dx^2}{49x^7} = C dx^2.$$

Solutio III.

Sumto multiplicatore $P dx^2 + 2 Q dx dy + 3 R dy^2$, prima integralis pars existit $P dx^2 dy + Q dx dy^2 + R dy^3$, et reliqua expressio integranda

$$\begin{aligned} y^n P X dx^2 + 2y^n Q X dx dy + 3y^n R X dx^2 dy^2 \\ - dx^2 dy \left(\frac{dP}{dx} \right) - dx^2 dy^2 \left(\frac{dP}{dy} \right) \\ - dx^2 dy^2 \left(\frac{dQ}{dx} \right) - dx dy^3 \left(\frac{dQ}{dy} \right) \\ - dx dy^3 \left(\frac{dR}{dx} \right) - dy^4 \left(\frac{dR}{dy} \right) \end{aligned}$$

vnde statim, vt ante concludimus, $R = K$ functioni ipsius x tum vero $Q = L - y \left(\frac{dK}{dx} \right)$, ergo $\left(\frac{dQ}{dx} \right) = \left(\frac{dL}{dx} \right) - y \left(\frac{d^2 K}{dx^2} \right)$. Deinde destructio terminorum per dy^2 affectorum praebet:

$$\begin{aligned} 3y^n K X - \left(\frac{dP}{dy} \right) - \left(\frac{dL}{dx} \right) + y \left(\frac{d^2 K}{dx^2} \right) = 0, \text{ ex quo fit} \\ P = M - y \left(\frac{dL}{dx} \right) + \frac{1}{2} y y \left(\frac{d^2 K}{dx^2} \right) + \frac{2}{n+1} y^{n+1} K X. \end{aligned}$$

Cum ergo fit

$$\left(\frac{dP}{dx} \right) = \left(\frac{dM}{dx} \right) - y \left(\frac{d^2 L}{dx^2} \right) + \frac{1}{2} y y \left(\frac{d^3 K}{dx^3} \right) + \frac{2}{n+1} y^{n+1} \left(\frac{d^2 K X}{dx^2} \right)$$

$$\text{ob } 2y^n Q X dx dy = 2X dx^2 (y^n L dy - y^{n+1} dy \left(\frac{dK}{dx} \right))$$

termini per dy affecti praebent alteram integralis partem

$$dx^2 \left\{ \frac{2}{n+1} L X y^{n+1} - \frac{2}{n+2} y^{n+2} X \left(\frac{dK}{dx} \right) - y \left(\frac{dM}{dx} \right) + \frac{1}{2} y y \left(\frac{d^2 L}{dx^2} \right) \right\} \\ - \frac{1}{2} y^2 \left(\frac{d^3 K}{dx^3} \right) - \frac{2}{(n+1)(n+2)} y^{n+2} \left(\frac{d^2 K X}{dx^2} \right) + N \left\{$$

Iam

Iam vero, ob primum terminum $y^n P X dx^4$, esse oportet:

$$\begin{aligned} 0 &= y^n M X - y^{n+1} X \left(\frac{dL}{dx} \right) + \frac{1}{2} y^{n+2} X \left(\frac{ddK}{dx^2} \right) + \frac{3}{n+1} y^{2n+1} K X X \\ &- \frac{2}{n+1} y^{n+1} \left(\frac{dLX}{dx} \right) + \frac{2}{n+2} y^{n+2} X \left(\frac{ddK}{dx^2} \right) + \frac{2}{n+2} y^{n+2} \left(\frac{dX}{dx} \right) \left(\frac{dK}{dx} \right) \\ &+ y \left(\frac{ddM}{dx^2} \right) - \frac{1}{2} y' \left(\frac{d^3L}{dx^3} \right) + \frac{1}{6} y^3 \left(\frac{d^4K}{dx^4} \right) + \frac{3}{(n+1)(n+2)} y^{n+2} \left(\frac{ddKK}{dx^2} \right) - \frac{dN}{dx} \end{aligned}$$

Hic autem, si n determinare nolimus, esse debet $L=0$, ideoque $R=0$, vnde hic casus ad praecedentem deducetur. Consideremus ergo casus sequentes:

I. Sit $n=1$; eritque $N=0$; $M X + \left(\frac{ddM}{dx^2} \right) = 0$; vnde ne X ad primam solutionem reuocetur, fieri debet $M=0$; tum vero habebitur:

$$\begin{aligned} -X \left(\frac{dL}{dx} \right) - \left(\frac{dLX}{dx} \right) - \frac{1}{2} \left(\frac{d^3L}{dx^3} \right) &= 0 \text{ et} \\ \frac{1}{2} X \left(\frac{ddK}{dx^2} \right) + \frac{3}{2} K X X + \frac{2}{3} X \left(\frac{ddK}{dx^2} \right) &+ \frac{2}{3} \left(\frac{dX}{dx} \right) \left(\frac{dK}{dx} \right) \\ + \frac{1}{6} \left(\frac{d^4K}{dx^4} \right) + \frac{1}{2} \left(\frac{ddKK}{dx^2} \right) &= 0. \end{aligned}$$

Ac ne X ad modum casus praecedentis definiatur, quod erat $n=1$, ponatur $L=0$; vnde X ex hac aequatione defini debet:

$$\frac{2}{3} K X X dx^2 + \frac{3}{2} X dx^2 ddK + \frac{1}{2} dx^2 dK dX + \frac{1}{2} K dx^2 ddX + \frac{1}{6} d^4K = 0$$

II. Sit $n=\frac{1}{2}$; eritque $2K X X - \frac{1}{2} \left(\frac{d^3L}{dx^3} \right) = 0$; $M=0$; $N=0$:

$$\begin{aligned} -X \left(\frac{dL}{dx} \right) - \frac{1}{2} \left(\frac{dLX}{dx} \right) &= 0; \left(\frac{d^3L}{dx^3} \right) = 0; \text{ et} \\ \frac{1}{2} X \left(\frac{ddK}{dx^2} \right) + \frac{1}{2} \left(\frac{dX}{dx} \right) \left(\frac{dK}{dx} \right) &+ \frac{1}{2} \left(\frac{ddKK}{dx^2} \right) = 0 \end{aligned}$$

$$\text{seu } \frac{2}{3} X ddK + \frac{1}{2} dK dX + \frac{1}{2} K ddX = 0$$

sed huiusmodi casibus non immoror.

Solutio IV.

Tentetur etiam factor tertii ordinis

$$P dx^3 + 2 Q dx^2 dy + 3 R dx dy^2 + 4 S dy^3$$

vnde nascitur *integralis pars prima* :

$$P dx^3 dy + Q dx^2 dy^2 + R dx dy^3 + S dy^4$$

et reliqua expressio integranda erit :

$$\begin{aligned} y^n P X dx^3 + 2 y^n Q X dx^2 dy + 3 y^n R X dx dy^2 + 4 y^n S X dx^2 dy^3 \\ - dx^4 dy \left(\frac{dP}{dx} \right) - dx^3 dy^2 \left(\frac{dP}{dy} \right) \\ - dx^3 dy^2 \left(\frac{dQ}{dx} \right) - dx^2 dy^3 \left(\frac{dQ}{dy} \right) \\ - dx^2 dy^3 \left(\frac{dR}{dx} \right) - dx dy^4 \left(\frac{dR}{dy} \right) \\ - dx dy^4 \left(\frac{dS}{dx} \right) - dy^5 \left(\frac{dS}{dy} \right). \end{aligned}$$

Erit ergo $S = K$; $R = L - y \left(\frac{dK}{dx} \right)$; atque

$$4 y^n K X dy - dQ - dy \left(\frac{dL}{dx} \right) + y dy \left(\frac{dK}{dx^2} \right) = 0$$

Ne hic in calculos nimis molestos delabamur, ponamus

$$K = A; L = B; \text{ vt fit } S = A \text{ et } R = B; \text{ iam}$$

ob $\left(\frac{dL}{dx} \right) = 0$ et $\left(\frac{dK}{dx^2} \right) = 0$, erit $Q = \frac{4A}{n+1} y^{n+1} X$

Tum vero habebimus :

$$3 B y^n X - \left(\frac{dP}{dy} \right) - \frac{4A}{n+1} y^{n+1} \left(\frac{dX}{dx} \right) = 0$$

$$\text{ergo } P = \frac{3}{n+1} B X y^{n+1} - \frac{4A}{(n+1)(n+2)} y^{n+2} \left(\frac{dX}{dx} \right)$$

$$\text{et } \left(\frac{dP}{dx} \right) = \frac{3B}{n+1} y^{n+1} \left(\frac{dX}{dx} \right) - \frac{4A}{(n+1)(n+2)} y^{n+2} \left(\frac{d^2 X}{dx^2} \right).$$

Hinc ergo nascitur *altera integralis pars* :

$$dx^4 \left(\frac{4A}{(n+1)^2} X X y^{2n+2} - \frac{3B}{(n+1)(n+2)} y^{n+2} \left(\frac{dX}{dx} \right) - \frac{4A}{(n+1)(n+2)(n+3)} y^{n+3} \left(\frac{d^2 X}{dx^2} \right) \right)$$

effeque debet

$$0 = \frac{3B}{n+1} X^2 y^{2n+1} - \frac{4A}{(n+1)(n+2)} X y^{2n+2} \left(\frac{dX}{dx} \right) - \frac{3A}{(n+1)^2} X y^{2n+2} \left(\frac{d^2 X}{dx^2} \right)$$

$$+ \frac{3B}{(n+1)(n+2)} y^{n+2} \left(\frac{d^2 X}{dx^2} \right) - \frac{4A}{(n+1)(n+2)(n+3)} y^{n+3} \left(\frac{d^3 X}{dx^3} \right).$$

Cui

Cui aequationi vt satisfiat, ponatur $B=0$; et $(\frac{d^2x}{dx^2})=0$
 seu

$$X = \alpha + 2\beta x + \gamma xx, \text{ fiatque } \frac{4A}{(n+1)(n+2)} + \frac{8A}{(n+1)^2} = 0.$$

siue $n = -\frac{8}{3}$.

vnde erit :

$$S = A; R = 0; Q = -6Ay^{-\frac{2}{3}}(\alpha + 2\beta x + \gamma xx) \text{ et}$$

$$P = 36Ay^{\frac{1}{3}}(\beta + \gamma x). \text{ Quare haec aequatio differentio-}$$

differentialis :

$$ddy + y^{-\frac{2}{3}}dx^2(\alpha + 2\beta x + \gamma xx) = 0$$

fit integrabilis, si multiplicetur per

$$36y^{\frac{1}{3}}(\beta + \gamma x)dx^2 - 12y^{-\frac{2}{3}}(\alpha + 2\beta x + \gamma xx)dx^2dy + 4dy^2$$

et integrale erit

$$36y^{\frac{1}{3}}(\beta + \gamma x)dx^2dy - 6y^{-\frac{2}{3}}(\alpha + 2\beta x + \gamma xx)dx^2dy^2 + dy^3$$

$$+ 9y^{-\frac{4}{3}}(\alpha + 2\beta x + \gamma xx)^2dx^2 - 27\gamma y^{\frac{4}{3}}dx^4 = Cdx^4$$

atque in hac solutione continetur exemplum quartum

Coroll. I.

38. Quartum ergo exemplum supra allatum aequationem differentialem maxime memorabilem continet, propterea quod ea non nisi per factorem tertii ordinis ad integrabilitatem perducitur potest, vnde eius integratio multo minus ab aliis methodis expectari potest.

Coroll.

Coroll. 2.

39. Si vicissim ergo ponamus $y = fz^{\frac{5}{3}}$; vt fit
 $y^{\frac{3}{5}} = z^{\frac{1}{3}}\sqrt[5]{f}$ et $y^{\frac{5}{3}} = fz^{\frac{5}{3}}\sqrt[5]{ff}$; erit $dy = \frac{1}{3}fz^{\frac{1}{3}}dz$ et $ddy = \frac{1}{3}fz^{\frac{1}{3}}ddz + \frac{1}{3}fz^{-\frac{1}{3}}dz^2$

et aequatio proposita :

$$\frac{1}{3}fz^{\frac{1}{3}}ddz + \frac{1}{3}fz^{-\frac{1}{3}}dz^2 + \frac{dx^2(\alpha + 2\beta x + \gamma xx)}{fz^{\frac{5}{3}}\sqrt[5]{ff}} = 0$$

fit integrabilis, si multiplicetur per

$$36z^{\frac{1}{3}}(\beta + \gamma x)dx^3\sqrt[5]{f} - \frac{18(\alpha + 2\beta x + \gamma xx)dx^2dz}{z^{\frac{1}{3}}}\sqrt[5]{f} + \frac{1}{3}fz^{\frac{5}{3}}dz^2$$

et integrale erit :

$$54fz(\beta + \gamma x)dx^3dz\sqrt[5]{f} - \frac{1}{2}f(\alpha + 2\beta x + \gamma xx)dx^2dz^2\sqrt[5]{f} + \frac{1}{15}f^{\frac{11}{3}}z^{\frac{5}{3}}dz^3$$

$$+ \frac{9(\alpha + 2\beta x + \gamma xx)^2dx^4}{fz^{\frac{5}{3}}\sqrt[5]{f}} - 27\gamma fzzdx^4\sqrt[5]{f} = Cdx^4$$

Coroll. 3.

40 Ponatur $ff\sqrt[5]{ff} = \frac{1}{3}$, vt habeatur haec aequatio:

$$2z^3ddz + z^2dz^2 + dx^2(\alpha + 2\beta x + \gamma xx) = 0$$

haecque fiet integrabilis, si multiplicetur per:

$$\frac{2(\beta + \gamma x)dx^3}{z^2} - \frac{(\alpha + 2\beta x + \gamma xx)dx^2dz}{z^2} + \frac{dz^3}{z}$$

eritque integrale :

$$4z(\beta + \gamma x)dx^3dz - (\alpha + 2\beta x + \gamma xx)dx^2dz^2 + \frac{1}{3}z^3dz^3$$

$$+ \frac{(\alpha + 2\beta x + \gamma xx)^2dx^4}{2z^2} - 2\gamma z^2dx^4 = Cdx^4$$

quae

quae aequatio etiam hoc modo repraesentari potest :

$$((\alpha + 2\beta x + \gamma x x) dx^2 - z z dz^2)^2 + 8 z^2 (\beta + \gamma x) dx^2 dz - 4 \gamma z^4 = E z z dx^4.$$

Coroll. 4.

41. Si fit $\alpha = 0$; $\beta = 0$; et $\gamma = a^2$, seu ista aequatio integranda proponatur :

$$2 z^2 d dz + z z dz^2 + a a x x dx^2 = 0,$$

ea integrabilis reddetur per hunc multiplicatorem :

$$\frac{2 a a x dx^2}{z z} - \frac{a a x x dx^2 dz}{z^3} + \frac{dz^2}{z}$$

et aequatio integralis erit :

$$(a a x x dx^2 - z z dz^2)^2 + 8 a a x z^2 dx^2 dz - 4 a a z^4 dx^4 = E z z dx^4.$$

$$\text{seu } (a a x x dx^2 + z z dz^2)^2 - 4 a a (z dx - x dz)^2 z z dx^2 = E z z dx^4.$$

Coroll. 5.

42. Posita ergo constante $E = 0$, pro hoc casu gemina aequatio integralis particularis habebitur :

$$\text{I. } a a x x dx^2 + z z dz^2 - 2 a z dx (z dx - x dz) = 0$$

$$\text{II. } a a x x dx^2 + z z dz^2 + 2 a z dx (z dx - x dz) = 0$$

quarum illa resoluitur in $a x dx + z dz = \pm z dx \sqrt{2 a}$

haec vero in . . . $a x dx - z dz = \pm z dx \sqrt{-2 a}$

Scholion.

43. Evolutio horum exemplorum ita est comparata, vt non parum utilitatis in resolutione aequationum differentialium secundi gradus afferre videatur; cum enim haec exempla, si nonnullos casus faciliores excipiamus, ope methodorum adhuc vſitatarum expedi

diri nequeant, noua haec methodus, qua negotium per multiplicatores conficitur, non solum optimo cum successu adhibetur, sed etiam nullum est dubium, quin ea, si vberius excolatur, multo maiora commoda sit allatura. Pari autem quoque successu ad aequationes differentiales tertii et altiorum graduum extendi poterit, siquidem certum est, quacunque proposita aequatione differentiali cuiuscunque gradus, inter duas variables, semper dari eiusmodi quantitatem, per quam, si aequatio multiplicetur, reddatur integrabilis. Quod cum etiam verum sit in aequationibus differentialibus primi gradus, et harum resolutio per methodum tales factores inuestigandi non mediocriter promoueri poterit; vbi quidem totum negotium eo reducitur, vt quouis casu oblato idoneus multiplicator inueniatur; atque in aequationibus quidem differentialibus primi gradus hic factor semper erit functio ipsarum x et y tantum, verum ob hoc ipsum quod diuersitas ordinum locum non habet, eius inuestigatio multo difficilior videtur, imprimis quando iste factor est functio transcendens. Cum autem haec ratio integrandi naturae aequationum sit maxime consentanea, non sine eximio fructu studium in ea excollenda collocabitur.
