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De aequationibus differentialibus secundi gradus

Leonhard Euler

DE AEQVATIONIBVS DIFFERENTIALIBVS SECVNDI GRADVS.

Auctore

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1.

Omnium quaestionum, quae quidem in Mathesi suscipiuntur, solutio duabus constat partibus, quarum altera in eo versatur, vt conditiones, quibus quaestio determinatur, ad aequationes analyticas perducantur, quae solutionem continere dicuntur, altera vero pars in ipsa harum aequationum resolutione occupatur. Si quaestio ad Mathesin mixtam, vel applicatam pertineat, prior pars petenda est ex principiis, quibus ista disciplina Mathematica innititur, huicque scientiae quasi est propria; pars autem posterior semper ad Analysin puram est referenda, cum tota in resolutione aequationum versetur. Ita si quaestio, vel ex Mechanica, vel ex Hydrodynamica, vel ex Astronomia, fuerit desumta, ex principiis cuique harum disciplinarum propriis quaestionem primum ad aequationes reduci oportet, tum vero istarum aequationum resolutio artificiis, quae quidem in Analysi comperta habemus, vnice est relinquenda. Ex quo satis est manifestum, quanti sit momenti Analysis per cunctas Matheseos partes.

 2. Prin-

2. Principia autem fere omnium Matheseos applicatae partium iam ita sunt euoluta, vt nulla propemodum quaestio eo pertinens proferri possit, cuius solutio non aequationibus comprehendi queat. Siue enim quaestio sit de aequilibrio, siue de motu corporum cuiuscunque indolis, tam solidorum, quam fluidorum, cum ab aliis, tum a me, principia certissima sunt stabilita, quorum ope semper ad aequationes peruenire licet: atque si corpora coelestia viribus quibuscunque in se inuicem agere statuantur, omnes perturbationes, quae inde in eorum motibus efficiuntur, non difficulter ad aequationes reuocantur; ita vt si has aequationes resoluere valeremus, nihil amplius superesset, quod in his scientiis desiderari posset. Quocirca omne studium, quod in Mathesin confertur, vtilius impendi nequit, quam si in limitibus Analyseos promouendis elaboremus.

3. Quoties autem problema ad Mathesin applicatam pertinens tractatur, rarissime in aequationes algebraicas incidimus, quarum resolutio, etiamsi nondum vltra quartum gradum sit perducta, tamen ope approximationum ita exacte perfici potest, vt pro perfecta sit habenda. Perpetuo autem fere deuoluimur ad aequationes differentiales, et quidem maximam partem ad differentiales secundi ordinis; principia quippe mechanica statim differentialia secundi gradus implicant: ita vt sine Analyseos infinitorum subsidio, nihil fere in his scientiis praestari liceat. Cum autem in resolutione aequationum differentialium primi gradus non admodum simus profecti, multo minus est mirandum, si aqua nobis

nobis haereat, quando quaestiones ad aequationes differentiales secundi gradus reducuntur. Regulae enim, quae pro huiusmodi aequationum resolutione sunt inventae, et quas mihi equidem vindicare possum, ita sunt limitatae, vt certis tantum casibus, qui non admodum frequenter occurrunt, in vsum vocari queant. Huiusmodi autem regulas plures exposui in Comment. Acad. Petrop. et Vol. VII. Miscell. Berol.

4. Interim tamen iam saepius eiusmodi se mihi obtulerunt casus aequationum differentialium secundi gradus, quas tametsi ope regularum illarum tractare non licuerit, tamen aliunde earum integralia habuerim perspecta; neque vlla via directa patebat, qua haec integralia erui possent. Huiusmodi casus eo magis sunt notatu digni, quod comparatio illarum aequationum cum suis integralibus tutissimam viam patefacere videatur, earum resolutionem per certas methodos perficiendi. In quo negotio, si euentus spem non fefellerit, nullum est dubium, quin methodi hunc in finem detectae, multo latius pateant, ac nostram facultatem, aequationes differentiales secundi gradus tractandi, non mediocriter promoueant. Iis ergo, quos huiusmodi studia iuuant, non ingratum fore arbitror, si casus illos mihi oblatos commemorauero, vt occasionem inde adipiscantur, in hac parte Analysin amplificandi, tum vero ipse methodos exponam, quas horum casuum contemplatio mihi suppeditauit.

5. Primum huiusmodi exemplum mihi occurrit in Mechanicae meae Tom. I. pag. 465. vbi

ad hanc perueni aequationem differentialem secundi gradus:

$$2Bx\,ddx - 4B\,dx^2 = x^{n+5}dp^2(1+pp)^{\frac{n-1}{2}}$$

in qua differentiale dp sumtum est constans. Eius autem integrale aliunde mihi constabat in hac forma contineri:

$$x^{n+5}dp^2(1+pp)^{\frac{n+1}{2}} + C\,ds^2 = 0$$

existente $ds^2 = (1+pp)dx^2 + 2px\,dp\,dx + xx\,dp^2$. Poteram etiam notare valorem huius constantis C esse $= -(n+1)B$. Diu tum temporis operam inutiliter perdidi in methodo directa indaganda, cuius ope istam aequationem integralem ex illa differentiali secundi gradus eruere possem, neque vllum artificium cognitum huc deducere est visum. Caeterum notari conuenit, integrale hic exhibitum tantum esse particulare, quia non continet quantitatem constantem ab arbitrio nostro pendentem, quae per integrationem esset introducta, infra autem ostendam ob talem constantem adiici posse huiusmodi terminum Ex^4dp^2.

6. In aliud simile exemplum incidi in opusculorum meorum prima collectione pag. 82, vbi motum corporum in superficiebus mobilibus sum perscrutatus: perueni autem in euolutione certi cuiusdam casus ad hanc aequationem differentialem secundi gradus:

$$\frac{ddr}{r} + \frac{(F+Mkk)^2\theta^2du^2}{(Mkkrr+F+2Gu+Huu)^2} = 0$$

vbi differentiale du sumtum est constans, litterae autem F, G, H, Mkk et θ denotant quantitates constantes quascunque. Nullo modo quoque huius aequationis inte-

integrale eruere poteram, aliunde autem noueram, eius integrale esse:

$$\frac{(F+Mkk)^2\theta^2du^2}{Mkkrr+F+2Gu+Huu}+\frac{dr^2}{r^2}(F+2Gu+Huu)-\frac{2dudr}{r}(G+Hu)$$
$$+H\,du^2=\frac{H\,du^2}{r^2}+\frac{(F+Mkk)\theta^2du^2}{rr}$$

quod quidem etiam est particulare, et quia tantopere est complicatum, multo minus patet, quomodo per integrationem ex illa aequatione erui queat. Deinceps vero monstrabo, hoc integrale completum reddi, si loco termini $\frac{H\,du^2}{rr}$, adiiciatur $\frac{C\,du^2}{rr}$, ita vt C designet quantitatem constantem, a reliquis, quae in aequatione differentiali secundi gradus insunt, plane non pendentem.

7. Deinde etiam alia problemata tractans, perductus fui ad huiusmodi aequationes differentiales secundi gradus, quarum integratio non parum recondita videbatur. Veluti huius aequationis differentialis secundi gradus:

$$rr\,ddr+r\,dr^2=n^2s\,ds^2$$

sumto elemento ds constante, integrale particulare quidem inueni esse:

$$r\,dr+nr\,ds+nns\,ds=0$$

quae quidem aequatio, quia binae variabiles r et s vbique earundem dimensionum, per methodum a me olim exhibitam, tractari posset. Porro quoque se mihi obtulit haec aequatio differentio-differentialis:

$$ds^2(\alpha ss+\beta s+\gamma)=rr\,dr^2+2r^3\,ddr$$

sumto elemento ds constante, cuius integrale completum deprehendi esse:

$$C=-\tfrac{1}{2}\left(\frac{rdr^2}{ds^2}+\frac{\alpha ss+\beta s+\gamma}{r}\right)^2+\frac{2rdr(2\alpha s+\beta)}{ds}-2\alpha rr$$

quod,

quod, quomodo inde elici queat, haud facile patet. Quin etiam ipsa aequatio integralis, etsi est differentialis primi tantum gradus, parum adiumenti afferre videtur, ob insignem variabilium implicationem.

8. Haec quatuor exempla sufficiunt, ad ostendendum, plures adhuc methodos deesse, quibus aequationes differentiales secundi gradus integrari queant, simul autem, quoniam his quidem casibus integralia constant, de earum inuentione non esse desperandum. Equidem post varia tentamina, quibus has aequationes tractaui, comperi, totum negotium eo redire, vt idonea quaeratur quantitas, per quam istae aequationes multiplicatae integrationem admittant; tali autem multiplicatore inuento, integratio nulla amplius laborat difficultate. Quemadmodum enim omnium aequationum differentialium primi gradus integratio eo reduci potest, vt inuestiganda sit functio quaepiam binarum variabilium, per quam aequatio multiplicata euadat integrabilis, ita etiam, pro omnibus aequationibus differentialibus secundi gradus, hanc regulam non dubito tanquam generalem in medium afferre, vt statuam semper eiusmodi functionem variabilium dari, per quam aequatio multiplicata reddatur integrabilis.

9. Loquor autem hic de eiusmodi tantum aequationibus, quae duas solum variabiles inuoluunt, et quae iam eo sint perductae, vt differentialia supremi gradus vnicam dimensionem obtineant. Ponamus x et y esse ambas variabiles, et posito $dy = p\,dx$; $dp = q\,dx$; $dq = r\,dx$, $dr = s\,dx$, etc. omnes aequationes differentiales

tiales cuiusque gradus ad formas sequentes reduci posse constat:

I. Forma generalis aequationum differentialium primi gradus

$$p = \text{funct.}\ (x \text{ et } y)$$

II. Forma generalis aequationum differ. secundi gradus

$$q = \text{funct.}\ (x, y \text{ et } p)$$

III. Forma generalis aequationum differ. tertii gradus

$$r = \text{funct.}\ (x, y, p \text{ et } q)$$

IV. Forma generalis aequationum differ. quarti gradus

$$s = \text{funct.}\ (x, y, p, q, \text{ et } r)$$

et ita porro de sequentibus altiorum graduum.

10. Cum igitur proposita quacunque aequatione differentiali primi gradus $p = \text{funct.}\ (x \text{ et } y)$, semper detur eiusmodi functio ipsarum x et y, per quam illa aequatio multiplicata reddatur integrabilis, etiamsi saepe numero hanc functionem assignare non valeamus, nullum est dubium, quin etiam pro aequationibus differentialibus secundi gradus $q = \text{funct.}\ (x, y \text{ et } p)$ eiusmodi multiplicator existat, qui eas reddat integrabiles, ideoque ad differentialia primi gradus reducat. Iam vero hic casus distingui oportet, quibus iste multiplicator vel binarum tantum variabilium x et y functio existat, vel insuper quantitatem p, seu rationem differentialium $\frac{dy}{dx}$ inuoluat: ob hoc enim discrimen ipsa multiplicatoris inuentio modo facilior, modo difficilior

 euadet.

euadet. Casus autem euolutu facillimus habebitur, si multiplicator alterius tantum variabilis solius fuerit functio.

11. Si igitur litterae P, Q, R, S, T sumantur ad designandas quascunque functiones ipsarum variabilium x et y, sequentes ordines simpliciores multiplicatorum pro aequationibus differentialibus secundi gradus constituentur:

Multiplicator ordinis primi .. P

Multiplicator ordinis secundi .. $Pdx+Qdy$

Multiplicator ordinis tertii .. $Pdx^2+Qdxdy+Rdy^2$

Multiplicator ordinis quarti .. $Pdx^3+Qdx^2dy+Rdxdy^2+Sdy^3$

etc.

Hi quidem sunt ordines simpliciores, quibus $p=\frac{dy}{dx}$, vel ad nullam, vel ad vnam, vel duas, vel tres dimensiones assurgit: facile autem colligitur fieri posse, vt littera p vel per fractiones, vel irrationalia, vel adeo transcendentia, multiplicatorem afficiat, cuiusmodi casus ingentem campum nouarum inuestigationum aperiunt. Hic quidem tantum in formis expositis versari constitui, quia eae sufficiunt exemplis allatis expediendis, simulque nos ad aequationes multo generaliores earum ope resolubiles manuducent.

12. Proposita ergo aequatione quacunque differentiali secundi gradus, $q=$ funct. (x, y et p), quae sumto dx constanti ad hanc formam redigetur $ddy=dx^2$ funct. (x, y et $\frac{dy}{dx}$), tentetur primo multiplicator primae formae P, num eius ope integratio succedat? sin minus,

minus, ſumatur multiplicator formae ſecundae $Pdx + Qdy$, qui niſi negotium conficiat, recurratur ad multiplicatorem formae tertiae, tum quartae, etc. mox autem colligere licebit, vtrum per factores harum formarum integratio abſolui queat, nec ne? quo poſteriori caſu, ad formas magis complicatas erit confugiendum, ac dummodo huiusmodi calculo fuerimus aſſueti, facultatem nobis comparabimus, pro quouis caſu oblato idoneam multiplicatoris formam dignoſcendi: ad quem ſcopum euolutio propoſitorum exemplorum erit accommodata.

Problema 1.

13. Propoſita aequatione differentiali ſecundi gradus:

$$2\,ay\,ddy - 4\,a\,dy^2 - y^{n+5}dx^2(1+xx)^{\frac{n-1}{2}} = 0$$

in qua differentiale dx ſumtum eſt conſtans, eius integrale inuenire.

Solutio.

Factorem primae formae P tentanti mox patebit, negotium non ſuccedere, niſi ſit $n = -2$, quo quidem caſu foret $P = \frac{1}{y^3}$ et aequationis $\frac{2\,ay\,ddy - 4\,a\,dy^2}{y^3}$ $- \frac{dx^2}{(1+xx)\sqrt{(1+xx)}} = 0$ integrale eſſet

$$\frac{2ady}{yy} + \frac{xdx}{\sqrt{(1+xx)}} = \alpha dx,$$ denuoque integrando haberetur

$$-\frac{2a}{y} + \sqrt{(1+xx)} = \alpha x + \beta;$$

ita vt hic caſus ſpecialis nullam habeat difficultatem. In genere igitur pro valore quocunque exponentis n,

 tentetur

tentetur factor formae secundae $Pdx + Qdy$, et aequatione ad hanc speciem reducta

$$2\,a\,ddy - \frac{4\,a\,dy^2}{y} - y^{n+4}\,dx^2\,(1 + xx)^{\frac{n-1}{2}} = 0$$

productum erit:

$$\left.\begin{aligned} &+ 2\,a\,P\,dx\,ddy - \frac{4aPdxdy^2}{y} - P\,y^{n+4}dx^3(1+xx)^{\frac{n-1}{2}} \\ &+ 2\,a\,Q\,dy\,ddy - \frac{4aQdy^3}{y} - Q\,y^{n+4}dx^2dy(1+xx^{\frac{n-1}{2}} \end{aligned}\right\} = 0$$

quam per hypothesin integrabilem esse oportet. Duo autem primi termini, qualescunque P et Q sint functiones ipsarum x et y, nonnisi ex differentiatione horum $2\,a\,P\,dx\,dy + a\,Q\,dy^2$ oriri potuerunt; vnde habebimus

Primam partem integralis $2\,a\,P\,dx\,dy + a\,Q\,dy^2$.

Huius ergo differentiale subtrahamus a nostra aequatione et ob $dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right)$; $dQ = dx\left(\frac{dQ}{dx}\right) + dy\left(\frac{dQ}{dy}\right)$, aequatio ordinata erit:

$$\left.\begin{aligned} -Py^{n+4}dx^3(1+xx)^{\frac{n-1}{2}} - Q\,y^{n+4}dx^2dy(1+xx)^{\frac{n-1}{2}} - \frac{4aPdxdy^2}{y} - \frac{4aQdy^3}{y} \\ - 2\,a\,dx^2\,dy\left(\frac{dP}{dx}\right) - 2\,a\,dx\,dy^2\left(\frac{dP}{dy}\right) - a\,dy^3\left(\frac{dQ}{dy}\right) \\ - a\,dx\,dy^2\left(\frac{dQ}{dx}\right) \end{aligned}\right\} = 0$$

quae ob dx sumtum constans nullo modo integrabilis esse potest, nisi termini per dy^3 et dy^2 affecti seorsim se tollant. Necesse ergo est, sit:

$$\frac{4Q}{y} + \left(\frac{dQ}{dy}\right) = 0, \text{ seu } 4Q\,dy + y\,dy\left(\frac{dQ}{dy}\right) = 0$$

et $\frac{4P}{y} + 2\left(\frac{dP}{dy}\right) + \left(\frac{dQ}{dx}\right) = 0$

Iam vt ex aequatione priori valorem ipsius Q eruamus, spectemus x vt constans, eritque $dy\left(\frac{dQ}{dy}\right) = dQ$, denotat

notat enim $dy\left(\frac{dQ}{dy}\right)$ incrementum ipsius Q ex solius y variabilitate ortum, vnde cum sit $4Qdy + ydQ = 0$, obtinebimus integrando

$$Qy^4 = K \text{ functioni ipsius } x \text{ tantum}$$

ita vt sit $Q = -\frac{K}{y^4}$ et $\left(\frac{dQ}{dx}\right) = \frac{1}{y^4}\left(\frac{dK}{dx}\right)$

vbi $\left(\frac{dK}{dx}\right)$ erit functio ipsius x. Nunc in altera aequatione quoque x sumatur constans, fietque:

$$4Pdy + 2ydP + \frac{dy}{y^3}\left(\frac{dK}{dx}\right) = 0$$

quae per y multiplicata et integrata dat:

$$2Pyy - \frac{1}{y}\left(\frac{dK}{dx}\right) = 2L, \text{ ideoque}$$

$$P = \frac{L}{yy} + \frac{1}{2y^3}\left(\frac{dK}{dx}\right)$$

vbi L denotat functionem ipsius x tantum. Destructis ergo istis membris, ob $\left(\frac{dP}{dx}\right) = \frac{1}{yy}\left(\frac{dL}{dx}\right) + \frac{1}{2y^3}\left(\frac{ddK}{dx^2}\right)$ erit altera pars integralis:

$$-dx^2\int\left((1+xx)^{\frac{n-1}{2}}\left(Ly^{n+2}dx + \tfrac{1}{2}y^{n+1}dx\left(\tfrac{dK}{dx}\right) + Ky^n dy\right)\right.$$
$$\left. - 2a\,dx^2\int\left(\frac{dy}{yy}\left(\frac{dL}{dx}\right) + \frac{dy}{2y^3}\left(\frac{ddK}{dx^2}\right)\right)\right)$$

quae cum constet duobus membris, pro priori esse debet $L=0$, et membri $\int(1+xx)^{\frac{n-1}{2}}\left(\frac{1}{2}y^{n+1}dx\left(\frac{dK}{dx}\right) + Ky^n dy\right)$ integrale erit $\frac{Ky^{n+1}}{n+1}(1+xx)^{\frac{n-1}{2}}$. Superest ergo vt reddatur $\frac{y^{n+1}dK}{n+1}(1+xx)^{\frac{n-1}{2}} + \frac{(n-1)Ky^{n+1}x\,dx}{n+1}(1+xx)^{\frac{n-3}{2}}$ $= \frac{1}{2}y^{n+1}dK(1+xx)^{\frac{n-1}{2}}$, seu $2(n-1)Kx\,dx = (n-1)\,dK\,(1+xx)$.

Atque hinc elicitur $K = 1 + xx$; ita vt alterius partis integralis membrum prius fit $-\frac{1}{n+1} y^{n+1} dx^2 (1+xx)^{\frac{n+1}{2}}$: at membrum posterius ob $L = 0$ et $\left(\frac{ddK}{dx^2}\right) = 2$ fiet

$$-2adx^2 \int \frac{dy}{y^3} = \frac{adx^2}{yy}$$

cuius integratio cum sponte successerit, totum negotium est confectum, et integralis pars altera erit:

$$-\frac{1}{n+1} y^{n+1} dx^2 (1+xx)^{\frac{n+1}{2}} + \frac{adx^2}{yy}.$$

Cum deinde sit $L = 0$ et $K = 1 + xx$, erit $\left(\frac{dK}{dx}\right) = 2x$, hincque fiet: $P = \frac{x}{y^3}$ et $Q = \frac{1+xx}{y^4}$; ex quo integralis pars prima habebitur

$$\frac{2axdxdy}{y^3} + \frac{a(1+xx)dy^2}{y^4}$$

Quocirca aequationis differentio-differentialis propositae adhibito termino constante Cdx^2 integrale completum erit:

$$\frac{adx^2}{yy} + \frac{2axdxdy}{y^3} + \frac{a(1+xx)dy^2}{y^4} - \frac{1}{n+3} y^{n+1} dx^2 (1+xx)^{\frac{n+1}{2}} = Cdx^2;$$

seu per y^4 multiplicando:

$$\frac{1}{n+1} y^{n+5} dx^2 (1+xx)^{\frac{n+1}{2}} = a(yydx^2 + 2xydxdy + (1+xx)dy^2) - Cy^4 dx^2$$

quod egregie conuenit cum eo, quod ante per methodum indirectam eram assecutus.

Coroll. I.

14. Aequatio ergo differentio-differentialis

$$2addy - \frac{4ady^2}{y} - y^{n+4} dx^2 (1+xx)^{\frac{n-1}{2}} = 0$$

integrabilis redditur, si multiplicetur per hunc factorem

$$\frac{2dx}{y^3} + \frac{(1+xx)dy}{y^4}$$

qui

qui si aliunde cognosci potuisset, integratio sine vlla difficultate perfecta fuisset.

Coroll. 2.

15. Vicissim ergo si aequatio integralis inuenta

$$\frac{ayydx^2+2axydxdy+a(1+xx)dy^2}{y^4}-\frac{1}{n+1}y^{n+1}dx^2(1+xx)^{\frac{n+1}{2}}=Cdx^2$$

sumto elemento dx constante differentietur, quo pacto constans C ex calculo egreditur, differentiale erit diuisibile per hanc formulam $\frac{xdx}{y^3}+\frac{(1+xx)dy}{y^4}$, seu hanc $xydx+(1+xx)dy$, et diuisione instituta ipsa demum aequatio differentio-differentialis proposita proueniet.

Coroll. 3.

16. Si aequatio proposita per $\frac{\sqrt{(1+xx)}}{y^4}$ multiplicetur, vt habeatur

$$2a\left(ddy-\frac{2dy^2}{y}\right)\frac{\sqrt{(1+xx)}}{y^4}-y^n dx^2(1+xx)^{\frac{n}{2}}=0$$

multiplicator eam reddens integrabilem erit:

$$\frac{xydx}{\sqrt{(1+xx)}}+dy\sqrt{(1+xx)}=d.y\sqrt{(1+xx)}$$

Quare si ponatur $y\sqrt{(1+xx)}=z$, haec obtinebitur aequatio:

$$\frac{2addz(1+xx)^2}{z^4}-\frac{4adz^2(1+xx)^2}{z^5}+\frac{4axdxdz(1+xx)}{z^4}-\frac{2adx^2}{z^3}-z^n dx^2=0$$

quae per dz multiplicata integrationem admittit. Erit enim integrale:

$$\frac{adz^2(1+xx)^2}{z^4}+\frac{adx^2}{zz}-\frac{1}{n+1}z^{n+1}dx^2=Cdx^2.$$

Coroll.

Coroll. 4.

17. Hinc ergo patet, quomodo per idoneam ſubſtitutionem integratio ſubleuari queat; cum enim aequatio propoſita per ſubſtitutionem $y = \frac{z}{\sqrt{(1+zz)}}$ in hanc poſteriorem formam fuerit transmutata, non amplius foret difficile integrationem peragere. Sed praeterquam quod talis ſubſtitutio non facile occurrat, ſi multiplicator fuerit ordinis tertii, vel altioris, huiusmodi reductio ne locum quidem habere poterit.

Scholion.

18. In hac ſolutione vſus ſum ſingulari ſpecie calculi, qua ad plurium litterarum introductionem vitandam differentiale functionis P duarum variabilium x et y expreſſi per

$$dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right)$$

vbi more iam ſatis vſitato, $dx(\frac{dP}{dx})$ denotat incrementum ipſius P ex ſola variabilitate ipſius x oriundum, et $dy(\frac{dP}{dy})$ eius incrementum, quod ex variabilitate ſolius y naſcitur; conſtat autem haec duo incrementa addita praebere completum differentiale ipſius P ex vtra variabili x et y natum. Hinc formulae $(\frac{dP}{dx})$ et $(\frac{dP}{dy})$ denotabunt functiones finitas variabilium x et y, quippe quae per differentiationem omiſſis differentialibus habentur, ita ſi ſit $P = y\sqrt{(1+xx)}$, erit $(\frac{dP}{dx}) = \frac{xy}{\sqrt{(1+xx)}}$ et $(\frac{dP}{dy}) = \sqrt{(1+xx)}$. Tum vero cognita altera parte huiusmodi differentialis veluti $dx(\frac{dP}{dx})$, ipſa quantitas P inde ex parte cognoſcitur. Spectata enim ſola x vt variabili

variabili erit $P = \int dx\left(\frac{dP}{dx}\right) + Y$, denotante Y functionem ipsius y tantum, atque ex hoc fonte in solutione valores quantitatum P et Q determinaui. Manifestum est quoque, si K fuerit functio ipsius x tantum, tum $dx\left(\frac{dK}{dx}\right)$ eius completum differentiale iam significare, ita vt sit $dx\left(\frac{dK}{dx}\right) = dK$: porro autem haec scriptio $\left(\frac{ddK}{dx^2}\right)$ denotat idem quod $\left(\frac{d.(dK:dx)}{dx}\right)$, seu si ponatur $\left(\frac{dK}{dx}\right) = k$, erit $\left(\frac{ddK}{dx^2}\right) = \left(\frac{dk}{dx}\right)$. Erit enim pariter k functio ipsius x tantum; ita si sit $K = \sqrt{(1+xx)}$, erit $\left(\frac{dK}{dx}\right) = \frac{x}{\sqrt{(1+xx)}}$ et $\left(\frac{ddK}{dx^2}\right) = \frac{1}{(1+xx)\sqrt{(1+xx)}}$: hocque modo vlterius progredi licebit, vt sit $\left(\frac{d^3K}{dx^3}\right) = \frac{-3x}{(1+xx)^2\sqrt{(1+xx)}}$, atque haec ad intelligentiam tam huius solutionis, quam sequentium annotasse necesse est visum. Caeterum consideratio huius solutionis facile deducit ad sequens Theorema generalius.

Theorema I.

19. Ista aequatio differentialis secundi gradus, posito dx constante,

$$a\,ddy - \frac{ma\,dy^2}{y} + y^n dx^2(\alpha + 2\beta x + \gamma xx)^{\frac{n-4m+3}{2m-2}} = 0$$

integrabilis redditur, si multiplicetur per hunc factorem:

$$\frac{(\beta+\gamma x)dx}{(m-1)y^{2m-1}} + \frac{(\alpha+2\beta x+\gamma xx)dy}{y^{2m}}$$

atque aequatio integralis erit:

$$\frac{a\gamma y^2dx^2 + 2(m-1)a(\beta+\gamma x)ydxdy + (m-1)^2 a(\alpha+2\beta x+\gamma xx)dy^2}{2(m-1)^2 y^{2m}}$$
$$+ \frac{y^{n-2m+1}dx^2}{n-2m+1}(\alpha+2\beta x+\gamma xx)^{\frac{n-2m+1}{2m-2}} = C dx^2.$$

 Coroll.

Coroll. 1.

20. Si fuerit $n = 1$, prodibit ista aequatio differentialis secundi gr dus:

$$a\,ddy - \frac{ma\,dy^2}{y} + \frac{y\,dx^2}{(\alpha + 2\beta x + \gamma xx)^2} = 0$$

quae ergo multiplicata per $\frac{(\beta+\gamma x)dx}{(m-1)y^{2m-1}} + \frac{(\alpha + 2\beta x + \gamma xx)dy}{y^{2m}}$

fit integrabilis, eius integrali existente:

$$\frac{a\gamma yydx^2 + 2(m-1)a(\beta+\gamma x)ydxdy + (m-1)^2 a(\alpha + 2\beta x + \gamma xx)dy^2}{2(m-1)^2 y^{2m}}$$

$$- \frac{yydx^2}{2(m-1)y^{2m}(\alpha + 2\beta x + \gamma xx)} = C dx^2.$$

Coroll. 2.

21. Posito $m - 1 = \mu$, si statuamus $y = e^{\int v dx}$, aequatio nostra fiet differentialis primi ordinis:

$$a\,dv - \mu avvdx + \frac{dx}{(\alpha + 2\beta x + \gamma xx)^2} = 0$$

cuius ergo integralis erit

$$a\gamma yydx^2 + 2\mu a(\beta+\gamma x)ydxdy + \mu^2 a(\alpha + 2\beta x + \gamma xx)dy^2$$
$$- \frac{\mu yydx^2}{\alpha + 2\beta x + \gamma xx} = 2\mu\mu C y^{2m} dx^2$$

seu pro y valore suo substituto

$$a\gamma + 2\mu a(\beta + \gamma x)v + \mu^2 a(\alpha + 2\beta x + \gamma xx)vv - \frac{\mu}{\alpha + 2\beta x + \gamma xx}$$
$$= 2\mu\mu C e^{2\mu\int vdx}.$$

Coroll. 3.

22. Statim ergo aequationis differentialis propositae:

$$a\,dv - \mu avvdx + \frac{dx}{(\alpha + 2\beta x + \gamma xx)^2} = 0$$

posito

posito $C=0$, habemus aequationem integralem particularem, quae est:

$$0=a\gamma+2\mu a(\beta+\gamma x)v+\mu^2 a(\alpha+2\beta x+\gamma xx)vv-\frac{\mu}{\alpha+2\beta x+\gamma xx}$$

ex qua per methodum a me alias expositam integrale completum erui potest. Quin etiam, si illa aequatio differentialis per hanc formam integralem diuidatur, integrabilis reddetur.

Problema 2.

23. Proposita aequatione differentiali secundi gradus:

$$\frac{ddy}{y}+\frac{a\,dx^2}{(\alpha+2\beta x+\gamma xx+cyy)^2}=0$$

In qua differentiale dx sumtum est constans, eius integrale inuenire.

Solutio.

Tentetur iterum integratio per factorem $Pdx+Qdy$, ac posito breuitatis gratia $\alpha+2\beta x+\gamma xx+cyy=Z$, conuertatur aequatio in hanc formam:

$$ddy+\frac{ay\,dx^2}{ZZ}=0$$

quae per $Pdx+Qdy$ multiplicata praebet:

$$Pdx\,ddy+Qdy\,ddy+\frac{aPy\,dx^3}{ZZ}+\frac{aQy\,dx^2\,dy}{ZZ}=0.$$

Quae cum integrabilis esse debeat, dabit statim

I. *primam integralis partem* $=Pdx\,dy+\frac{1}{2}Qdy^2$;

superest ergo, vt integrabilis reddatur sequens expressio:

$$-\tfrac{1}{2}dy^3\left(\frac{dQ}{dy}\right)-\tfrac{1}{2}dx\,dy^2\left(\frac{dQ}{dx}\right)+\frac{aQy\,dx^2\,dy}{ZZ}+\frac{aPy\,dx^3}{ZZ}$$
$$-dx\,dy^2\left(\frac{dP}{dy}\right)-dx^2\,dy\left(\frac{dP}{dx}\right).$$

Primum ergo necesse est, vt sit $\left(\frac{dQ}{dy}\right)=0$, vnde fit Q functio ipsius x tantum, quae sit $Q=K$; tum vero etiam termini dy^2 inuoluentes destruendi sunt, ex quibus fit:

$$\left(\frac{dK}{dy}\right)+2\left(\frac{dP}{dx}\right)=0$$

seu sumto solo y pro variabili:

$$dv\left(\frac{dK}{dx}\right)+2dP=0.$$

cuius integrale est

$$P=L-\tfrac{1}{2}y\left(\frac{dK}{dx}\right)$$

denotante L quoque functionem ipsis x. Quare ob

$$\left(\frac{dP}{dx}\right)=\left(\frac{dL}{dx}\right)-\tfrac{1}{2}y\left(\frac{ddK}{dx^2}\right)$$

et dx sumtum constans, altera pars integralis erit:

$$dx^2 \int \frac{ay}{ZZ}\left(L\,dx-\tfrac{1}{2}y\,dx\left(\frac{dK}{dx}\right)+K\,dy\right)-dx^2\int dy\left(\left(\frac{dL}{dx}\right)-\tfrac{1}{2}y\left(\frac{ddK}{dx^2}\right)\right)$$

at est $\int\frac{aKy\,dy}{ZZ}=aK\int\frac{y\,dy}{(\alpha+2\beta x+\gamma xx+cyy)^2}$

vnde pro integrali nascitur

II. pars $=-\frac{a}{2c}\cdot\frac{K\,dx^2}{\alpha+2\beta x+\gamma xx+cyy}$

ideoque debet esse:

$$\frac{ay}{ZZ}\left(L\,dx-\tfrac{1}{2}y\,dK\right)=-\frac{a}{2c}\cdot\frac{(\alpha+2\beta x+\gamma xx+cyy)dK-2K\,dx(\beta+\gamma x)}{ZZ}$$

seu

$$acLy\,dx-\tfrac{1}{2}acyy\,dK=aK\,dx(\beta+\gamma x)-\tfrac{1}{2}a\,dK(\alpha+2\beta x+\gamma xx+cyy)$$

vel $acLy\,dx=aK\,dx(\beta+\gamma x)-\tfrac{1}{2}a\,dK(\alpha+2\beta x+\gamma xx)$

Perspicuum ergo est, esse debere

$$L=0 \text{ et } K=\alpha+2\beta x+\gamma xx.$$

Quare ob $\left(\frac{ddK}{dx^2}\right)=2\gamma$ erit

III. ultima pars integralis $=+\tfrac{1}{2}\gamma yy\,dx^2$.

Cum

Cum igitur sit:

$$P = -y(\beta + \gamma x) \text{ et } Q = \alpha + 2\beta x + \gamma xx$$

erit noster multiplicator:

$$-y dx(\beta + \gamma x) + dy(\alpha + 2\beta x + \gamma xx)$$

et integrale quaesitum habebitur:

$$-ydxdy(\beta+\gamma x) + \tfrac{1}{2}dy^2(\alpha+2\beta x+\gamma xx) - \frac{a(\alpha+2\beta x+\gamma xx)dx^2}{2c(\alpha+2\beta x+\gamma xx+cyy)} + \tfrac{1}{2}\gamma yy\,dx^2 = C\,dx^2$$

At si ponatur $C = \frac{-a}{2c} + C$, erit hoc integrale:

$$\tfrac{1}{2}\gamma yy\,dx^2 - y\,dx\,dy(\beta + \gamma x) + \tfrac{1}{2}dy^2(\alpha+2\beta x+\gamma xx) + \frac{ayy\,dx^2}{2(\alpha+2\beta x+\gamma xx+cyy)} = C\,dx^2.$$

Quae forma conuenit cum ea, quam supra exhibui.

Theorema 2.

24. Ista aequatio differentialis secundi gradus posito dx constante

$$ddy + \frac{ay^{n+1}dx^2}{(\alpha+2\beta x+\gamma xx+cyy)^{\frac{n+4}{2}}} = 0$$

integrabilis reddetur per multiplicatorem:

$$-y\,dx(\beta+\gamma x) + dy(\alpha+2\beta x+\gamma xx)$$

et integrale erit:

$$\tfrac{1}{2}\gamma yy\,dx^2 - y\,dx\,dy(\beta+\gamma x) + \tfrac{1}{2}dy^2(\alpha+2\beta x+\gamma xx) + \frac{ay^{n+2}dx^2}{(n+2)(\alpha+2\beta x+\gamma xx+cyy)^{\frac{n+2}{2}}} = C\,dx^2.$$

Coroll. 1.

25. Casus problematis nascitur ex Theoremate hoc, si ponatur $n = 0$. Ceterum integrale in Theore-

mate exhibitum simili modo elicitur, quo solutionem problematis expediuimus; vnde superfluum foret eius demonstrationem adiicere.

Coroll. 2.

26. Si ponatur $c=0$, casus habebitur, quem etiam ex Theoremate primo deriuare licet, si ibi ponatur $m=0$. Dum enim pro a scribitur $\frac{1}{a}$ et $n+1$ loco n; integrale ibi datum perfecte congruit cum hoc, quod istud Theorema suppeditat pro casu $c=0$.

Coroll. 3.

27. Hoc autem Theorema adeo primum in se complectitur: aequatio enim primi

$$a\,ddy-\frac{m\,a\,dy^2}{y}+y^n dx^2(\alpha+2\beta x+\gamma xx)^{\frac{n-4m+3}{2m-2}}=0$$

si ponatur $y=z^{\frac{1}{1-m}}$ abit in hanc:

$$\frac{a}{1-m}z^{\frac{m}{1-m}}ddz+z^{\frac{n}{1-m}}dx^2(\alpha+2\beta x+\gamma xx)^{\frac{n-4m+3}{2m-2}}=0$$

seu $\frac{a\,ddz}{1-m}+z^{\frac{n-m}{1-m}}dx^2(\alpha+2\beta x+\gamma xx)^{\frac{n-4m+3}{2m-2}}=0.$

Quodsi iam statuatur $\frac{n-m}{1-m}=n+1$, vt fiat $n=1-n(m-1)$ aequatio haec abibit in istam formam:

$$\frac{a\,ddz}{1-m}+z^{n+1}dx^2(\alpha+2\beta x+\gamma xx)^{\frac{-n-4}{2}}=0$$

quae est casus particularis praesentis Theorematis, ex quo quippe nascitur, ponendo $c=0$.

Coroll.

Coroll. 4.

28. Praesens ergo Theorema latissime patet, atque eiusmodi casus difficillimos in se complectitur, qui nullo alio modo resolui posse videntur. Si enim $c = 0$, fortasse reperietur methodus negotium conficiens, propterea quod variabiles non sunt inuicem permixtae: at si c non $= 0$, ob permixtionem variabilium nulla methodus cognita hic cum successu in vsum vocabitur.

Coroll. 5.

29. Casus hic imprimis notatu dignus hic occurrit, si $\alpha = 0$, $\beta = 0$, $\gamma = c = 1$, quo habetur haec aequatio:

$$ddy + \frac{ay^{n+1}dx^2}{(xx+yy)^{\frac{n+4}{2}}} = 0$$

cuius ergo integrale est:

$$\tfrac{1}{2}(ydx - xdy)^2 + \frac{ay^{n+2}dx}{(n+2)(xx+yy)^{\frac{n+2}{2}}} = Cdx^2$$

Ponatur $y = ux$, erit $ydx - xdy = -xxdu$, fietque integrale:

$$\tfrac{1}{2}x^4du^2 + \frac{au^{n+2}dx^2}{(n+2)(1+uu)^{\frac{n+2}{2}}} = Cdx^2$$

ideoque $\frac{dx}{xx} = \frac{du(1+uu)^{\frac{n+2}{4}}}{\surd\left(2C(1+uu)^{\frac{n+2}{2}} - \frac{2a}{n+2}u^{n+2}\right)}$

quae ob variabiles separatas denuo integrari potest.

Scholion.

Scholion.

30. Hic quoque multiplicatoris forma ſubſtitutionem idoneam praebet, cuius ope aequatio differentio-differentialis in aliam tractatu faciliorem transformabitur. Statui ſcilicet oportet

$$y = z\sqrt{(\alpha + 2\beta x + \gamma xx)}$$

Hanc vero ipſam ſubſtitutionem ſuadet formulae indoles

$$(\alpha + 2\beta x + \gamma xx + cyy)^{\frac{[illegible]}{2}}$$

quia hoc pacto vnica variabilis in vinculo relinquitur. At per hanc ſubſtitutionem ipſa aequatio multo magis fit perplexa, ita vt, etiamſi per factorem ſimplicitem $dz(\alpha + 2\beta x + \gamma xx)^{\frac{5}{2}}$ ad integrabilitatem reuocetur, id tamen minus pateat. Verum ſi multiplicator fuerit ordinis tertii, ſeu altioris, ne huiusmodi quidem ſubſtitutio commode inueniri poteſt, vti in duobus reliquis exemplis vſu venit.

Problema 3.

31. Propoſita aequatione differentiali ſecundi gradus:

$$yyddy + mydy^2 = axdx^2$$

in qua differentiale dx ſumtum eſt conſtans, eius integrale inuenire.

Solutio.

Quia multiplicator neque primi, neque ſecundi ordinis ſuccedit, ex ordine tertio deſumatur. Perducta ergo aequatione ad hanc formam:

$$ddy + \frac{mdy^2}{y} - \frac{axdx^2}{yy} = 0$$

multi-

multiplicetur ea per $Pdx^2 + 2Qdxdy + 3Rdy^2$, vnde ſtatim habebitur:

I. prima pars integralis $Pdx^2dy + Qdxdy^2 + Rdy^3$ et integrando relinquitur haec forma:

$$-\frac{aPxdx^4}{yy} - \frac{2aQxdx^3dy}{yy} - \frac{3aRxdx^2dy^2}{yy}$$
$$+ \frac{mPdx^2dy^2}{y} + \frac{2mQdxdy^3}{y} + \frac{3mRdy^4}{y}$$
$$-dx^3dy\left(\frac{dP}{dx}\right) - dx^2dy^2\left(\frac{dP}{dy}\right) - dxdy^3\left(\frac{dQ}{dy}\right) - dy^4\left(\frac{dR}{dy}\right)$$
$$-dx^2dy^2\left(\frac{dQ}{dx}\right) - dxdy^3\left(\frac{dR}{dx}\right).$$

Haec autem forma integrabilis eſſe nequit, niſi membra, quae dy^2, dy^3 et dy^4 implicant, deſtruantur. Primum ergo pro dy^4 habebimus:

$$\frac{3mR}{y} - \left(\frac{dR}{dy}\right) = 0, \text{ ſeu } 3mRdy = ydR$$

vbi x ſumitur pro conſtante, vnde fit $R = Ky^{3m}$, denotante K functionem ipſius x tantum, ſicque erit: $\left(\frac{dR}{dx}\right) = y^{3m}\left(\frac{dK}{dx}\right)$. Iam pro deſtructione terminorum dy^3 continentium, fiet:

$$\frac{2mQ}{y} - \left(\frac{dQ}{dy}\right) - y^{3m}\left(\frac{dK}{dx}\right) = 0$$

ſeu ſumto x conſtante:

$$2mQdy - ydQ = y^{3m+1}dy\left(\frac{dK}{dx}\right)$$

quae diuiſa per y^{2m+1} et integrata dat:

$$\frac{-Q}{y^{2m}} = \frac{1}{m+1}y^{m+1}\left(\frac{dK}{dx}\right) - L.$$

Sumta denuo L pro functione ipſius x, ita vt ſit

$$Q = Ly^{2m} - \frac{1}{m+1}y^{3m+1}\left(\frac{dK}{dx}\right), \text{ ideoque}$$
$$\left(\frac{dQ}{dx}\right) = y^{2m}\left(\frac{dL}{dx}\right) - \frac{1}{m+1}y^{3m+1}\left(\frac{ddK}{dx^2}\right).$$

 Deſtru-

Deſtruantur denique etiam termini dy^2 continentes, vnde prodit:

$$-3aKy^{3m-2}x - y^{2m}\left(\frac{dL}{dx}\right) + \frac{1}{m+1}y^{3m+1}\left(\frac{ddK}{dx^2}\right) + \frac{mP}{y} - \left(\frac{dP}{dy}\right) = 0$$

quae ſumta x conſtante per ydy multiplicata praebet:

$$-3aKxy^{3m-1}dy - y^{2m+1}dy\left(\frac{dL}{dx}\right) + \frac{1}{m+1}y^{3m+2}dy\left(\frac{ddK}{dx^2}\right) + mP\,dy - y\,dP = 0$$

quae per y^{m+1} diuiſa et integrata dat:

$$\frac{-3a}{2m-1}Kxy^{2m-1} - \frac{1}{m+1}y^{m+1}\left(\frac{dL}{dx}\right) + \frac{1}{2(m+1)^2}y^{2m+2}\left(\frac{ddK}{dx^2}\right) - \frac{P}{y^m} + M = 0$$

denotante M functionem ipſius x tantum. Ergo fit

$$P = My^m - \frac{3a}{2m-1}Kxy^{3m-1} - \frac{1}{m+1}y^{2m+1}\left(\frac{dL}{dx}\right) + \frac{1}{2(m+1)^2}y^{3m+2}\left(\frac{ddK}{dx^2}\right)$$

ideoque

$$\left(\frac{dP}{dx}\right) = y^m\left(\frac{dM}{dx}\right) - \frac{3a}{2m-1}Ky^{3m-1} - \frac{3ax}{2m-1}y^{3m-1}\left(\frac{dK}{dx}\right) - \frac{1}{m+1}y^{2m+1}\left(\frac{ddL}{dx^2}\right) + \frac{1}{2(m+1)^2}y^{3m+2}\left(\frac{d^3K}{dx^3}\right).$$

Nunc termini $-\frac{2aQx\,dx^3\,dy}{yy} - dx^3dy\left(\frac{dP}{dx}\right)$, integrati, x pro conſtante ſumta, ſuppeditabunt

II. *alteram integralis partem*:

$$-2axdx^3\left(\frac{1}{2m-1}Ly^{2m-1} - \frac{1}{3m(m+1)}y^{3m}\left(\frac{dK}{dx}\right)\right) - Ndx^3$$
$$-dx^3\left(\frac{1}{m+1}y^{m+1}\left(\frac{dM}{dx}\right) - \frac{a}{m(2m-1)}Ky^{3m} - \frac{ax}{m(2m-1)}y^{3m}\left(\frac{dK}{dx}\right) - \frac{1}{2(m+1)^2}y^{2m+2}\left(\frac{ddL}{dx^2}\right) + \frac{1}{6(m+1)^3}y^{3m+3}\left(\frac{d^3K}{dx^3}\right)\right);$$

Huius ergo differentiale poſito y conſtante ſumtum aequale eſſe debet reſiduae parti $\frac{-aPx\,dx^4}{yy}$: vnde per dx^4 diuiſo habebimus ſequentem aequationem:

$$a M x y^{m-2} - \frac{3 a a x x}{2m-1} K y^{2m-3} - \frac{a x}{m+1} y^{2m-1} \left(\frac{dL}{dx}\right) + \frac{a x}{2(m+1)^2} y^{3m} \left(\frac{ddK}{dx^2}\right) - \frac{2a}{2m-1} L y^{2m-1} + \frac{2a}{3m(m+1)} y^{3m} \left(\frac{dK}{dx}\right) - \frac{2ax}{2m-1} y^{2m-1} \left(\frac{dL}{dx}\right) + \frac{2ax}{3m(m+1)} \times y^{3m} \left(\frac{ddK}{dx^2}\right) - \frac{1}{m+1} y^{m+1} \left(\frac{ddM}{dx^2}\right) + \frac{a}{m(2m-1)} y^{3m} \left(\frac{dK}{dx}\right) + \frac{a}{m(2m-1)} y^{3m} \left(\frac{dK}{dx}\right) + \frac{ax}{m(2m-1)}) y^{3m} \left(\frac{ddK}{dx^2}\right) + \frac{1}{2(m+1)^2} y^{2m+2} \left(\frac{d^3 L}{dx^3}\right) - \frac{1}{6(m+1)}) y^{3m+1} \left(\frac{d^4 K}{dx^4}\right) = \text{functioni ipsius } x = \left(\frac{dN}{dx}\right).$$

Hic iam singulae diuersae ipsius y potestates seorsim ad nihilum redigantur, et quia y^{m-2} et y^{3m-3} semel occurrunt, nisi sit vel $m=2$, vel $m=1$, habebimus $M=0$, et $K=0$; et supererunt tantum termini per L affecti, inter quos solitarius est y^{2m+2}; vnde esse debet $\left(\frac{d^3 L}{dx^3}\right)=0$, ideoque $L=\alpha+2\beta x+\gamma x x$, reliqui per y^{2m-1} affecti dant:

$$-\frac{2ax(\beta+\gamma x)}{m+1} - \frac{2a(\alpha+2\beta x+\gamma x x)}{2m-1} - \frac{4ax(\beta+\gamma x)}{2m-1} = 0.$$

Hinc debet esse $\alpha=0$, et $\frac{\beta+\gamma x}{m+1} + \frac{4\beta+3\gamma x}{2m-1} = 0$. Quibus conditionibus in genere satisfieri nequit; constituendi ergo sunt casus sequentes:

I. Si $\alpha=0$, et $\gamma=0$, fiet $m=-\frac{1}{2}$, ita vt aequatio proposita sit:

$$y y d d y - \tfrac{1}{2} y d y^2 = a x d x^2$$

seu

$$d d y - \frac{d y^2}{2y} - \frac{a x d x^2}{y y} = 0.$$

Cum igitur sit $K=0$, $L=x$, $M=0$, erit:

$$R=0;\ Q=\frac{x}{y};\ \text{et } P=-2$$

et noster multiplicator erit: $-2 d x^2 + \frac{2 x d x d y}{y}$ ideoque integrale quaesitum:

$$-2 d x^2 d y + \frac{x d x d y^2}{y} + \frac{a x x d x^3}{y y} = C d x^3,$$

 seu

feu per dx diuidendo

$$axxdx^2+xydy^2-2yydxdy=Cyydx^2$$

II. Sit $\alpha=0$; $\beta=0$; erit $m=-\frac{2}{5}$; et aequatio differentio-differentialis propofita:

$$ddy-\frac{2dy^2}{5y}-\frac{axdx^2}{yy}=0.$$

Cum igitur fit $K=0$, $L=xx$, et $M=0$, erit

$$R=0;\; Q=xxy^{-\frac{4}{5}};\; P=-\tfrac{10}{3}xy^{\frac{1}{5}}$$

vnde nofter multiplicator fiet:

$$-\tfrac{10}{3}xy^{\frac{1}{5}}dx^2+2xxy^{-\frac{4}{5}}dxdy$$

et integrale quaefitum

$$-\tfrac{10}{3}xy^{\frac{1}{5}}dx^2dy+xxy^{-\frac{4}{5}}dxdy^2+\tfrac{10}{9}ax^3y^{-\frac{9}{5}}dx^3+\tfrac{25}{9}y^{\frac{6}{5}}dx^3=Cdx^3$$

feu per dx diuidendo, et $y^{\frac{9}{5}}$ multiplicando,

$$-\tfrac{10}{3}xyydxdy+xxydy^2+\tfrac{10}{9}ax^3dx^2+\tfrac{25}{9}y^3dx^2=Cy^{\frac{9}{5}}dx^2$$

III. Ante vero iam duos cafus commemorauimus, quibus eft vel $m=1$, vel $m=2$. Sit ergo primo $m=1$ et aequatio propofita

$$ddy+\frac{dy^2}{y}-\frac{axdx^2}{yy}=0$$

ac fieri debet

$$\left(\frac{dN}{dx}\right)=\frac{aMx}{y}-3aaxxK-\tfrac{1}{2}axy\left(\frac{dL}{dx}\right)+\tfrac{1}{8}axy^3\left(\frac{ddK}{dx^2}\right)$$
$$-2aLy+\tfrac{1}{3}ay^3\left(\frac{dK}{dx}\right)-2axy\left(\frac{dL}{dx}\right)+\tfrac{1}{3}axy^3\left(\frac{ddK}{dx^2}\right)$$
$$-\tfrac{1}{2}yy\left(\frac{ddM}{dx^2}\right)+2ay^3\left(\frac{dK}{dx}\right)+axy^3\left(\frac{ddK}{dx^2}\right)+\tfrac{1}{8}y^4\left(\frac{d^3L}{dx^3}\right)-\tfrac{1}{48}y^6\left(\frac{d^4K}{dx^4}\right)$$

vnde obtinemus $M=0$; $N=-3aa\int Kxxdx$; et

$$-\tfrac{5}{2}x\left(\frac{dL}{dx}\right)-2L=0;\; \tfrac{35}{24}x\left(\frac{ddK}{dx^2}\right)+\tfrac{7}{3}\left(\frac{dK}{dx}\right)=0$$
$$\left(\frac{d^3L}{dx^3}\right)=0;\; \left(\frac{d^4K}{dx^4}\right)=0.$$

His

His conditionibus satisfit, si sumatur :

$$L=0;\ K=1;\ M=0;\ \text{et}\ N=-aax^3$$

vnde fit: $R=y^3$; $Q=0$; $P=-3axy^2$.

Quare noster multiplicator erit :

$$-3axy^2dx^2+3y^3dy^2$$

et integrale quaesitum :

$$-3axy^2dx^2dy+y^3dy^3+ay^3dx^3+aax^3dx^3=Cdx^3.$$

IV. Sit iam $m=2$, vt aequatio nostra fiat

$$ddy+\frac{2dy^2}{y}-\frac{axdx^2}{yy}=0$$

ac satisfieri debet huic aequationi :

$$\left(\frac{dN}{dx}\right)=aMx-aaKxxy^2-\tfrac{2}{3}aLy^3-axy^3\left(\frac{dL}{dx}\right)-\tfrac{1}{3}y\left(\frac{ddM}{dx^2}\right)$$
$$+\tfrac{4}{9}ay^6\left(\frac{dK}{dx}\right)+\tfrac{1}{18}y^6\left(\frac{d^3L}{dx^3}\right)+\tfrac{1}{3}axy^6\left(\frac{ddK}{dx^2}\right)-\tfrac{1}{162}y^9\left(\frac{d^4K}{dx^4}\right).$$

Erit ergo $N=a\int Mxdx$, ac statui potest $L=0$; $K=0$; $M=1$, vnde fit $N=\tfrac{1}{2}axx$. Hinc vero fit :

$$R=0;\ Q=0;\ P=y^2$$

ita vt multiplicator futurus sit $ydxy^2$ et integrale :

$$yydx^2dy-\tfrac{1}{2}axxdx^3=Cdx^3,\ \text{seu}$$
$$2yydy-axxdx=Cdx.$$

Coroll. 1.

32. Casus ergo vltimus, quo $m=2$, est omnium facillimus, cum per multiplicatorem adeo primi ordinis confici possit, quin primo intuitu aequationis

$$yyddy+2ydy^2=axdx^2$$

integrale $yydy=\tfrac{1}{2}axxdx+Cdx$ patet. Casus autem primus et secundus, quibus est $m=-\tfrac{1}{2}$ et $m=-\tfrac{2}{3}$ per

 multi-

multiplicatorem formae secundae, ob $R=0$, resolui potuissent.

Coroll. 2.

33. Solus ergo casus tertius, quo est $m=1$, resolutu est difficillimus, quia requirit multiplicatorem formae tertiae. Quare notetur, sequentem aequationem differentialem secundi gradus

$$yyddy+ydy^2-axdx^2=0$$

integrabilem reddi, si multiplicetur per

$$3ydy^2-3axdx^2$$

et integrale esse:

$$y^3dy^3-3axyydx^2dy+ay^3dx^3+aax^3dx^3=Cdx^3.$$

Coroll. 3.

34. Porro autem notandum est, hanc expressionem in tres factores simplices resolui posse. Si enim ponatur breuitatis gratia $a=c^3$ et $\mu=-\frac{1+\sqrt{-3}}{2}$ et $\nu=-\frac{1-\sqrt{-3}}{2}$, aequatio haec integralis ita repraesentari potest:

$$(ydy+cydx+c^2xdx)(ydy+\mu cydx+\nu c^2xdx)(ydy+\nu cydx+\mu c^2xdx)=Cdx^3.$$

Coroll. 4.

35. Hinc si constans C sumatur $=0$, tres statim prodeunt aequationes integrales particulares:

$$ydy+cydx+c^2xdx=0$$

$$ydy+\mu cydx+\nu c^2xdx=0$$

$$ydy+\nu cydx+\mu c^2xdx=0$$

quarum

quarum prima continet casum iam supra (7) indicatum: duae reliquae vero sunt imaginariae.

Scholion.

36. Restat ergo quartum exemplum, quod erat

$$ds^2(\alpha ss + \beta s + \gamma) = rr\,dr^2 + 2r^3\,ddr$$

quod posito

$$r = y^{\frac{2}{3}};\ \text{vt fit}\ dr = \tfrac{2}{3}y^{-\frac{1}{3}}dy,\ \text{et}\ ddr = \tfrac{2}{3}y^{-\frac{1}{3}}ddy - \tfrac{2}{9}y^{-\frac{4}{3}}dy^2$$

abit in hanc formam:

$$\tfrac{4}{3}y^{\frac{5}{3}}ddy = ds^2(\alpha ss + \beta s + \gamma).$$

In genere autem obseruo, si habeatur huiusmodi aequatio:

$$S\,ds^2 = mr^{\mu}dr^2 + nr^{\mu+1}ddr$$

eam per substitutionem $r = y^{\frac{n}{m+n}}$ reduci ad hanc formam simpliciorem:

$$S\,ds^2 = \frac{nn}{m+n}\,y^{\frac{\mu n - m + n}{m+n}}\,ddy.$$

Huiusmodi ergo aequationes omnes complecti licet in hac forma generali: $ddy = y^n X\,dx^2$. Videamus ergo quibusnam casibus tam exponentis n, quam functionis X, haec aequatio integrari queat per nostram methodum.

Problema 4.

37. Casus pro exponente n et naturam functionis X inuenire, quibus haec aequatio differentialis secundi gradus

$$ddy + y^n X\,dx^2 = 0,$$

vbi dx est constans, integrari queat.

Solutio

Solutio I.

Sumatur primo multiplicator primi ordinis P, et integranda erit haec aequatio:

$$P\,ddy + y^n PX\,dx^2 = 0$$

ac integralis pars prima erit $= P\,dy$, et integranda restat haec expressio:

$$y^n PX\,dx^2 - dx\,dy\left(\frac{dP}{dx}\right) - dy^2\left(\frac{dP}{dy}\right)$$

vnde necesse est, sit $\left(\frac{dP}{dy}\right) = 0$, ideoque P functio ipsius x tantum. Sit ergo $P = K$, et integrari oportet ob dx constans:

$$dx\left(y^n KX\,dx - dy\left(\frac{dK}{dx}\right)\right)$$

cuius integrale nequit esse, nisi $-y\,dx\left(\frac{dK}{dx}\right) = -y\,dK$. Oportet autem sit $y^n KX\,dx^2 + y\,ddK = 0$, quod fieri nequit, nisi sub his conditionibus:

$$n = 1 \text{ et } X = -\frac{ddK}{K\,dx^2}$$

ac tum aequatio integralis erit:

$$K\,dy - y\,dK = C\,dx.$$

Solutio II.

Sumto multiplicatore secundae formae $P\,dx + 2Q\,dy$, integrabilis efficienda est haec aequatio:

$$2Q\,dy\,ddy + P\,dx\,ddy + y^n X\,dx^2(P\,dx + 2Q\,dy) = 0$$

vnde integralis pars prima colligitur $P\,dx\,dy + Q\,dy^2$. Superest ergo, vt integretur:

$$y^n PX\,dx^3 + 2y^n QX\,dx^2\,dy$$
$$- dx^2\,dy\left(\frac{dP}{dx}\right) - dx\,dy^2\left(\frac{dP}{dy}\right)$$
$$- dx\,dy^2\left(\frac{dQ}{dx}\right) - dy^3\left(\frac{dQ}{dy}\right).$$

Hinc

Hinc quo termini tollantur, quibus dy plus vna habet dimensione, oportet esse

$$\left(\frac{dQ}{dy}\right)=0; \text{ ideoque } Q=K \text{ functioni ipsius } x.$$

Deinde habebimus

$$\left(\frac{dP}{dy}\right)+\left(\frac{dQ}{dx}\right)=0, \text{ seu } dP+dy\left(\frac{dK}{dx}\right)=0$$

vnde fit:

$$P=L-y\left(\frac{dK}{dx}\right) \text{ et } \left(\frac{dP}{dx}\right)=\left(\frac{dL}{dx}\right)-y\left(\frac{ddK}{dx^2}\right).$$

Iam altera pars integralis erit:

$$dx^2\int\left(y^n P X dx+2y^n Q X dy-dy\left(\frac{dP}{dx}\right)\right) \text{ siue}$$

$$dx^2\int\left\{\begin{array}{l}+y^n L X dx \qquad +2y^n K X dy\\ -y^{n+1} X dx\left(\frac{dK}{dx}\right)-dy\left(\frac{dL}{dx}\right)+y\,dy\left(\frac{ddK}{dx^2}\right)\end{array}\right\}$$

ex variabilitate ipsius y ergo concluditur *altera pars integralis.*

$$\text{II. } dx^2\left(\frac{2}{n+1}y^{n+1}KX-y\left(\frac{dL}{dx}\right)+\frac{1}{2}yy\left(\frac{ddK}{dx^2}\right)+M\right).$$

Ac variabilitas ipsius x postulat, vt sit:

$$y^n LX-y^{n+1}X\left(\frac{dK}{dx}\right)=\frac{2}{n+1}y^{n+1}K\left(\frac{dX}{dx}\right)+\frac{2}{n+1}y^{n+1}X\left(\frac{dK}{dx}\right)$$
$$-y\left(\frac{ddL}{dx^2}\right)+\frac{1}{2}yy\left(\frac{d^3K}{dx^3}\right)+\left(\frac{dM}{dx}\right).$$

Si n velimus indefinitum relinquere; esse debet

$$L=0;\ \left(\frac{d^3K}{dx^3}\right)=0 \text{ et } \left(\frac{dM}{dx}\right)=0; \text{ tum vero}$$

$$\frac{2}{n+1}K\left(\frac{dX}{dx}\right)+\frac{n+3}{n+1}X\left(\frac{dK}{dx}\right)=0$$

vnde colligitur $K^{\frac{n+3}{2}}X=A$ constanti: at ob $\left(\frac{d^3K}{dx^3}\right)=0$ erit $K=\alpha+2\beta x+\gamma xx$, ideoque $X=\frac{A}{(\alpha+2\beta x+\gamma xx)^{\frac{n+3}{2}}}$ et $Q=\alpha+2\beta x+\gamma xx$; $P=-2y(\beta+\gamma x)$. Quocirca multiplicator erit:

$$-2y\,dx(\beta+\gamma x)+2dy(\alpha+2\beta x+\gamma xx)$$

 et

et huius aequationis differentio-differentialis

$$ddy+\frac{Ay^n dx^2}{(\alpha+2\beta x+\gamma xx)^{\frac{n+3}{2}}}=0$$

integrale erit:

$$-2ydxdy(\beta+\gamma x)+dy^2(\alpha+2\beta x+\gamma xx)+\frac{2}{n+1}\cdot\frac{Ay^{n+1}}{(\alpha+2\beta x+\gamma xx)^{\frac{n+1}{2}}}+\gamma yydx^2=Cdx^2.$$

Superſunt autem caſus, quibus eſt vel $n=1$, vel $n=2$.

I. Sit $n=1$; et conditiones praecedentes poſtulant

$$LX+\left(\frac{ddL}{dx^2}\right)=0;\quad \frac{2}{n+1}K\left(\frac{dX}{dx}\right)+\frac{n+3}{n+1}X\left(\frac{dK}{dx}\right)+\frac{1}{2}\left(\frac{d^3K}{dx^3}\right)=0$$

ſeu $LXdx^2+ddL=0$ et $2KdX+4XdK+dx\left(\frac{d^3K}{dx^3}\right)=0$

hinc fit $2KKX+\int\frac{Kd^3K}{dx^2}=$ Conſt. ideoque

$$2KKXdx^2+KddK-\tfrac{1}{2}dK^2=Cdx^2$$

et $X=\dfrac{Edx^2+\frac{1}{2}dK^2-KddK}{2KK}$

pro priori conditione autem ponatur $L=0$. Quare erit

$Q=K$; $P=-y\left(\frac{dK}{dx}\right)$; atque huius aequationis

$$ddy+yXdx^2=0.$$

Exiſtente $X=\dfrac{Edx^2+\frac{1}{2}dK^2-KddK}{2KKdx^2}$, quaecunque functio ipſius x ſumatur pro K, erit integrale:

$$-ydxdy\left(\frac{dK}{dx}\right)+Kdy^2+yyKXdx^2+\tfrac{1}{2}yydx^2\left(\frac{ddK}{dx^2}\right)=Cdx^2$$

II. Sit $n=2$; et conditiones poſtulant:

$$2KdX+5XdK=0;\quad LX=\tfrac{1}{2}\left(\frac{d^3K}{dx^3}\right);\quad \left(\frac{ddL}{dx^2}\right)=0.$$

Prima

Prima dat $X = AK^{-\frac{5}{2}}$, qui in altera ſubſtitutus praebet

$$2ALK^{-\frac{5}{2}}dx^3 = d^3K;$$

verum, ob $\left(\frac{ddL}{dx^2}\right) = 0$, erit $L = \alpha + \beta x$, vnde, poſito $K = (\alpha + \beta x)^\mu$, erit $2A(\alpha + \beta x)^{1-\frac{5\mu}{2}} = \mu(\mu - 1)(\mu - 2)$
$(\alpha + \beta x)^{\mu - 3}\beta^3$
et $\mu = \frac{8}{7}$; hincque $2A = \frac{-48}{343}\beta^3$; et $X = \frac{A}{(\alpha+\beta x)^{\frac{20}{7}}} = \frac{-24\beta^3}{343(\alpha+\beta x)^{\frac{20}{7}}}$

Porro $Q = (\alpha + \beta x)^{\frac{8}{7}}$; $P = \alpha + \beta x - \frac{8}{7}\beta y(\alpha + \beta x)^{\frac{1}{7}}$
Conſequenter huius aequationis differentio-differentialis

$$ddy + y^2 X dx^2 = 0$$

exiſtente $X = \frac{-24\beta^3}{343(\alpha+\beta x)^{\frac{20}{7}}}$, integrale eſt

$$dx\,dy\left(\alpha + \beta x - \tfrac{8}{7}\beta y(\alpha+\beta x)^{\frac{1}{7}}\right) + dy^2(\alpha+\beta x)^{\frac{8}{7}} - \frac{16\beta^3 y^3 dx^2}{343(\alpha+\beta x)^{\frac{12}{7}}}$$
$$- \beta y dx^2 + \frac{4\beta^2 y^2 dx^2}{49(\alpha+\beta x)^{\frac{6}{7}}} = C dx^2$$

II. Si $n = 2$, adhuc caſus notari meretur, quo $L = \alpha$; et poſito

$K = x^\mu$, erit $2\alpha A x^{-\frac{5\mu}{2}} = \mu(\mu-1)(\mu-2)x^{\mu-3}$, vnde fit $\mu = \frac{6}{7}$ et $2\alpha A = \frac{6.1.8}{343}$; ideoque $\alpha = \frac{24}{343A}$. Quare erit

$$K = x^{\frac{6}{7}};\ L = \frac{24}{343A};\ X = \frac{A}{x^{\frac{15}{7}}};\ \text{ac porro}$$

$$Q = x^{\frac{6}{7}};\ P = \frac{24}{343A} - \frac{6y}{7x^{\frac{1}{7}}}.$$ Conſequenter huius aequationis:

$$ddy + \frac{Ay^2dx^2}{x^{\frac{15}{7}}} = 0$$

 inte-

integrale erit

$$\frac{24dxdy}{343A} - \frac{6ydxdy}{7x^{\frac{1}{7}}} + x^{\frac{6}{7}}dy^2 + \frac{2Ay^3dx^2}{3x^{\frac{9}{7}}} - \frac{yydx^2}{49x^{\frac{8}{7}}} = Cdx^2.$$

Solutio III.

Sumto multiplicatore $Pdx^2 + 2Qdxdy + 3Rdy^2$, prima integralis pars exiſtit $Pdx^2dy + Qdxdy^2 + Rdy^3$, et reliqua expreſſio integranda

$$y^nPXdx^4 + 2y^nQXdx^3dy + 3y^nRXdx^2dy^2$$
$$- dx^3dy\left(\frac{dP}{dx}\right) - dx^2dy^2\left(\frac{dP}{dy}\right)$$
$$- dx^2dy^2\left(\frac{dQ}{dx}\right) - dxdy^3\left(\frac{dQ}{dy}\right)$$
$$- dxdy^3\left(\frac{dR}{dx}\right) - dy^4\left(\frac{dR}{dy}\right)$$

vnde ſtatim, vt ante concludimus, $R = K$ functioni ipſius x: tum vero $Q = L - y\left(\frac{dK}{dx}\right)$, ergo $\left(\frac{dQ}{dx}\right) = \left(\frac{dL}{dx}\right) - y\frac{ddK}{dx^2}$. Deinde deſtructio terminorum per dy^2 affectorum praebet:

$$3y^nKX - \left(\frac{dP}{dy}\right) - \left(\frac{dL}{dx}\right) + y\left(\frac{ddK}{dx^2}\right) = 0, \text{ ex quo fit}$$
$$P = M - y\left(\frac{dL}{dx}\right) + \frac{1}{2}yy\left(\frac{ddK}{dx^2}\right) + \frac{3}{n+1}y^{n+1}KX.$$

Cum ergo ſit

$$\left(\frac{dP}{dx}\right) = \left(\frac{dM}{dx}\right) - y\left(\frac{ddL}{dx^2}\right) + \frac{1}{2}yy\left(\frac{d^3K}{dx^3}\right) + \frac{3}{n+1}y^{n+1}\left(\frac{d.KX}{dx}\right)$$

ob $2y^nQXdx^3dy = 2Xdx^3\left(y^nLdy - y^{n+1}dy\left(\frac{dK}{dx}\right)\right)$ termini per dy affecti praebent alteram integralis partem

$$dx^3\left\{\begin{array}{l}\frac{2}{n+1}LXy^{n+1} - \frac{2}{n+2}y^{n+2}X\left(\frac{dK}{dx}\right) - y\left(\frac{dM}{dx}\right) + \frac{1}{2}yy\left(\frac{ddL}{dx^2}\right)\\ - \frac{1}{6}y^3\left(\frac{d^3K}{dx^3}\right) - \frac{3}{(n+1)(n+2)}y^{n+2}\left(\frac{d.KX}{dx}\right) + N\end{array}\right\}$$

Iam

Iam vero, ob primum terminum $y^n \mathrm{PX}\, dx^4$, esse oportet:

$$0 = y^n \mathrm{MX} - y^{n+1}\mathrm{X}\left(\frac{d\mathrm{L}}{dx}\right) + \frac{1}{2} y^{n+2}\mathrm{X}\left(\frac{dd\mathrm{K}}{dx^2}\right) + \frac{3}{n+1} y^{2n+1}\mathrm{KXX}$$
$$- \frac{2}{n+1} y^{n+1}\left(\frac{d.\mathrm{LX}}{dx}\right) + \frac{2}{n+2} y^{n+2}\mathrm{X}\left(\frac{dd\mathrm{K}}{dx^2}\right) + \frac{2}{n+2} y^{n+2}\left(\frac{d\mathrm{X}}{dx}\right)\left(\frac{d\mathrm{K}}{dx}\right)$$
$$+ y\left(\frac{dd\mathrm{M}}{dx^2}\right) - \frac{1}{2} yy\left(\frac{d^3\mathrm{L}}{dx^3}\right) + \frac{1}{6} y^3\left(\frac{d^4\mathrm{K}}{dx^4}\right) + \frac{3}{(n+1)(n+2)} y^{n+2}\left(\frac{dd\,\mathrm{KK}}{dx^2}\right) - \frac{d\mathrm{N}}{dx}.$$

Hic autem, si n determinare nolimus, esse debet $\mathrm{L} = 0$, ideoque $\mathrm{R} = 0$, vnde hic casus ad praecedentem deduceretur. Consideremus ergo casus sequentes:

I. Sit $n = 1$; eritque $\mathrm{N} = 0$; $\mathrm{MX} + \left(\frac{dd\mathrm{M}}{dx^2}\right) = 0$; vnde ne X ad primam solutionem reuocetur, fieri debet $\mathrm{M} = 0$; tum vero habebitur:

$$-\mathrm{X}\left(\frac{d\mathrm{L}}{dx}\right) - \left(\frac{d.\mathrm{LX}}{dx}\right) - \frac{1}{2}\left(\frac{d^3\mathrm{L}}{dx^3}\right) = 0 \text{ et}$$
$$\frac{1}{2}\mathrm{X}\left(\frac{dd\mathrm{K}}{dx^2}\right) + \frac{3}{2}\mathrm{KXX} + \frac{2}{3}\mathrm{X}\left(\frac{dd\mathrm{K}}{dx^2}\right) + \frac{2}{3}\left(\frac{d\mathrm{X}}{dx}\right)\left(\frac{d\mathrm{K}}{dx}\right)$$
$$+ \frac{1}{6}\left(\frac{d^4\mathrm{K}}{dx^4}\right) + \frac{1}{2}\left(\frac{dd.\mathrm{KX}}{dx^2}\right) = 0.$$

Ac ne X ad modum casus praecedentis definiatur, quod erat $n = 1$, ponatur $\mathrm{L} = 0$; vnde X ex hac aequatione definiri debet:

$$\frac{3}{2}\mathrm{KXX}\,dx^4 + \frac{5}{3}\mathrm{X}\,dx^2\,dd\mathrm{K} + \frac{5}{3}dx^2\,d\mathrm{K}\,d\mathrm{X} + \frac{1}{2}\mathrm{K}\,dx^2\,dd\mathrm{X} + \frac{1}{6}d^4\mathrm{K} = 0$$

II. Sit $n = \frac{1}{2}$; eritque $2\mathrm{KXX} - \frac{1}{2}\left(\frac{d^3\mathrm{L}}{dx^3}\right) = 0$; $\mathrm{M} = 0$; $\mathrm{N} = 0$:

$$-\mathrm{X}\left(\frac{d\mathrm{L}}{dx}\right) - \frac{4}{3}\left(\frac{d.\mathrm{LX}}{dx}\right) = 0;\ \left(\frac{d^4\mathrm{K}}{dx^4}\right) = 0; \text{ et}$$
$$\frac{33}{20}\mathrm{X}\left(\frac{dd\mathrm{K}}{dx^2}\right) + 4\left(\frac{d\mathrm{X}}{dx}\right)\left(\frac{d\mathrm{K}}{dx}\right) + \frac{4}{5}\left(\frac{dd.\mathrm{KX}}{dx^2}\right) = 0$$

seu $\frac{21}{10}\mathrm{X}\,dd\mathrm{K} + \frac{12}{5}d\mathrm{K}\,d\mathrm{X} + \frac{4}{5}\mathrm{K}\,dd\mathrm{X} = 0$

sed huiusmodi casibus non immoror.

 Solutio

Solutio IV.

Tentetur etiam factor tertii ordinis

$$P\,dx^3 + 2\,Q\,dx^2\,dy + 3\,R\,dx\,dy^2 + 4\,S\,dy^3$$

vnde nascitur *integralis pars prima* :

$$P\,dx^3\,dy + Q\,dx^2\,dy^2 + R\,dx\,dy^3 + S\,dy^4$$

et reliqua expressio integranda erit :

$$y^n PX\,dx^5 + 2y^n QX\,dx^4 dy + 3y^n RX\,dx^3 dy^2 + 4y^n SX\,dx^2 dy^3$$
$$- dx^4 dy\left(\frac{dP}{dx}\right) - dx^3 dy^2\left(\frac{dP}{dy}\right)$$
$$- dx^3 dy^2\left(\frac{dQ}{dx}\right) - dx^2 dy^3\left(\frac{dQ}{dy}\right)$$
$$- dx^2 dy^3\left(\frac{dR}{dx}\right) - dx\,dy^4\left(\frac{dR}{dy}\right)$$
$$- dx\,dy^4\left(\frac{dS}{dx}\right) - dy^5\left(\frac{dS}{dy}\right).$$

Erit ergo $S = K$; $R = L - y\left(\frac{dK}{dx}\right)$; atque

$$4y^n KX\,dy - dQ - dy\left(\frac{dL}{dx}\right) + y\,dy\left(\frac{ddK}{dx^2}\right) = 0$$

Ne hic in calculos nimis molestos delabamur, ponamus $K = A$; $L = B$; vt sit $S = A$ et $R = B$; iam ob $\left(\frac{dL}{dx}\right) = 0$ et $\left(\frac{ddK}{dx^2}\right) = 0$, erit $Q = \frac{4A}{n+1} y^{n+1} X$

Tum vero habebimus :

$$3\,By^n X - \left(\frac{dP}{dy}\right) - \frac{4A}{n+1} y^{n+1}\left(\frac{dX}{dx}\right) = 0$$

ergo $P = \frac{3}{n+1} BXy^{n+1} - \frac{4A}{(n+1)(n+2)} y^{n+2}\left(\frac{dX}{dx}\right)$

et $\left(\frac{dP}{dx}\right) = \frac{3B}{n+1} y^{n+1}\left(\frac{dX}{dx}\right) - \frac{4A}{(n+1)(n+2)} y^{n+2}\left(\frac{ddX}{dx^2}\right).$

Hinc ergo nascitur *altera integralis pars* :

$$dx^4\left(\frac{4A}{(n+1)^2} XX y^{2n+2} - \frac{3B}{(n+1)(n+2)} y^{n+2}\left(\frac{dX}{dx}\right) + \frac{4A}{(n+1)(n+2)(n+3)} y^{n+3}\left(\frac{ddX}{dx^2}\right)\right)$$

esseque debet

$$0 = \frac{3B}{n+1} X^2 y^{2n+1} - \frac{4A}{(n+1)(n+2)} Xy^{2n+2}\left(\frac{dX}{dx}\right) - \frac{8A}{(n+1)^2} Xy^{2n+2}\left(\frac{dX}{dx}\right)$$
$$+ \frac{3B}{(n+1)(n+2)} y^{n+2}\left(\frac{ddX}{dx^2}\right) - \frac{4A}{(n+1)(n+2)(n+3)} y^{n+3}\left(\frac{d^3X}{dx^3}\right).$$

Cui

Cui aequationi vt satisfiat, ponatur $B = 0$; et $(\frac{d^3 X}{dx^3}) = 0$ seu

$X = \alpha + 2\beta x + \gamma xx$, fiatque $\frac{4A}{(n+1)(n+2)} + \frac{8A}{(n+1)^2} = 0$, siue $n = -\frac{5}{3}$.

vnde erit:

$$S = A;\ R = 0;\ Q = -6Ay^{-\frac{2}{3}}(\alpha + 2\beta x + \gamma xx) \text{ et}$$

$P = 36Ay^{\frac{1}{3}}(\beta + \gamma x)$. Quare haec aequatio differentio-differentialis:

$$ddy + y^{-\frac{5}{3}}dx^2(\alpha + 2\beta x + \gamma xx) = 0$$

fit integrabilis, si multiplicetur per

$$36y^{\frac{1}{3}}(\beta + \gamma x)dx^3 - 12y^{-\frac{2}{3}}(\alpha + 2\beta x + \gamma xx)dx^2 dy + 4dy^3$$

et integrale erit

$$36y^{\frac{1}{3}}(\beta + \gamma x)dx^3 dy - 6y^{-\frac{2}{3}}(\alpha + 2\beta x + \gamma xx)dx^2 dy^2 + dy^4$$
$$+ 9y^{-\frac{4}{3}}(\alpha + 2\beta x + \gamma xx)^2 dx^4 - 27\gamma y^{\frac{4}{3}}dx^4 = C dx^4$$

atque in hac solutione continetur exemplum quartum

Coroll. 1.

38. Quartum ergo exemplum supra allatum aequationem differentialem maxime memorabilem continet, propterea quod ea nonnisi per factorem tertii ordinis ad integrabilitatem perduci potest, vnde eius integratio multo minus ab aliis methodis expectari potest.

Coroll.

Coroll. 2.

39. Si viciſſim ergo ponamus $y = fz^{\frac{3}{2}}$; vt ſit $y^{\frac{1}{3}} = z^{\frac{1}{2}}\sqrt[3]{f}$ et $y^{\frac{5}{3}} = fz^{\frac{5}{2}}\sqrt[3]{ff}$; erit $dy = \frac{3}{2}fz^{\frac{1}{2}}dz$ et $ddy = \frac{3}{2}fz^{\frac{1}{2}}ddz + \frac{3}{4}fz^{-\frac{1}{2}}dz^2$

et aequatio propoſita:

$$\frac{3}{2}fz^{\frac{1}{2}}ddz + \frac{3}{4}fz^{-\frac{1}{2}}dz^2 + \frac{dx^2(\alpha + 2\beta x + \gamma xx)}{fz^{\frac{5}{2}}\sqrt[3]{ff}} = 0$$

ſit integrabilis, ſi multiplicetur per

$$36z^{\frac{1}{2}}(\beta + \gamma x)dx^3\sqrt[3]{f} - \frac{18(\alpha + 2\beta x + \gamma xx)dx^2dz}{z^{\frac{1}{2}}}\sqrt[3]{f} + \frac{27}{2}f^3z^{\frac{3}{2}}dz^2$$

et integrale erit:

$$54fz(\beta + \gamma x)dx^3dz\sqrt[3]{f} - \frac{27}{2}f(\alpha + 2\beta x + \gamma xx)dx^2dz^2\sqrt[3]{f} + \frac{81}{16}f^4zzdz^4$$
$$+ \frac{9(\alpha + 2\beta x + \gamma xx)^2dx^4}{fzz\sqrt[3]{f}} - 27\gamma fzzdx^4\sqrt[3]{f} = Cdx^4.$$

Coroll. 3.

40 Ponatur $ff\sqrt[3]{ff} = \frac{4}{9}$, vt habeatur haec aequatio:

$$2z^3ddz + zzdz^2 + dx^2(\alpha + 2\beta x + \gamma xx) = 0$$

haecque fiet integrabilis, ſi multiplicetur per:

$$\frac{2(\beta + \gamma x)dx^3}{zz} - \frac{(\alpha + 2\beta x + \gamma xx)dx^2dz}{z^3} + \frac{dz^3}{z}$$

eritque integrale:

$$4z(\beta + \gamma x)dx^3dz - (\alpha + 2\beta x + \gamma xx)dx^2dz^2 + \frac{1}{2}zzdz^4$$
$$+ \frac{(\alpha + 2\beta x + \gamma xx)^2dx^4}{2zz} - 2\gamma zzdx^4 = Cdx^4$$

quae

quae aequatio etiam hoc modo repraesentari potest:

$$((\alpha+2\beta x+\gamma xx)dx^2-zzdz^2)^2+8z^3(\beta+\gamma x)dx^3dz-4\gamma z^4=Ezzdx^4.$$

Coroll. 4.

41. Si sit $\alpha=0$; $\beta=0$; et $\gamma=a^2$, seu ista aequatio integranda proponatur:

$$2z^3ddz+zzdz^2+aaxxdx^2=0,$$

ea integrabilis reddetur per hunc multiplicatorem:

$$\frac{2aaxdx^3}{zz}-\frac{aaxxdx^2dz}{z^3}+\frac{dz^2}{z}$$

et aequatio integralis erit:

$$(aaxxdx^2-zzdz^2)^2+8aaxz^2dx^3dz-4aaz^4dx^4=Ezzdx^4$$

seu $(aaxxdx^2+zzdz^2)^2-4aa(zdx-xdz)^2zzdx^2=Ezzdx^4$.

Coroll. 5.

42. Posita ergo constante $E=0$, pro hoc casu gemina aequatio integralis particularis habebitur:

I. $aaxxdx^2+zzdz^2-2azdx(zdx-xdz)=0$

II. $aaxxdx^2+zzdz^2+2azdx(zdx-xdz)=0$

quarum illa resoluitur in $axdx+zdz=\pm zdx\sqrt{2a}$

haec vero in . . . $axdx-zdz=\pm zdx\sqrt{-2a}$.

Scholion.

43. Euolutio horum exemplorum ita est comparata, vt non parum vtilitatis in resolutione aequationum differentialium secundi gradus afferre videatur; cum enim haec exempla, si nonnullos casus faciliores excipiamus, ope methodorum adhuc vsitatarum expe-

diri nequeant, noua haec methodus, qua negotium per multiplicatores conficitur, non solum optimo cum successu adhibetur, sed etiam nullum est dubium, quin ea, si vberius excolatur, multo maiora commoda sit allatura. Pari autem quoque successu ad aequationes differentiales tertii et altiorum graduum extendi poterit, siquidem certum est, quacunque proposita aequatione differentiali cuiuscunque gradus, inter duas variabiles, semper dari eiusmodi quantitatem, per quam, si aequatio multiplicetur, reddatur integrabilis. Quod cum etiam verum sit in aequationibus differentialibus primi gradus, et harum resolutio per methodum tales factores inuestigandi non mediocriter promoueri poterit; vbi quidem totum negotium eo reducitur, vt quouis casu oblato idoneus multiplicator inueniatur; atque in aequationibus quidem differentialibus primi gradus hic factor semper erit functio ipsarum x et y tantum, verum ob hoc ipsum quod diuersitas ordinum locum non habet, eius inuestigatio multo difficilior videtur, imprimis quando iste factor est functio transcendens. Cum autem haec ratio integrandi naturae aequationum sit maxime consentanea, non sine eximio fructu studium in ea excolenda collocabitur.

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