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Demonstratio theorematis et solutio problematis in actis erud. Lipsiensibus propositorum

Leonhard Euler

DEMONSTRATIO THEOREMATIS ET SOLVTIO PROBLEMATIS IN ACTIS ERVD. LIPSIENSIBVS PROPOSITORVM.

Auctore

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Theorema istud et Problema versantur circa arcus ellipticos; illo semissis ellipseos quaeque ita secatur, vt partium differentia sit geometrice assignabilis, hoc vero constructio geometrica arcus postulatur, qui sit semissis quadrantis elliptici. Tam demonstratio Theorematis, quam solutio Problematis, sequuntur ex iis, quae iam aliquoties de comparatione linearum curuarum praelegi; et quoniam methodus, qua hoc argumentum pertractaui, non solum noua, sed etiam plurimum recondita videbatur, has propositiones ideo publicare constitueram, vt alii quoque vires suas in iis euoluendis exercerent, nouisque methodis, quibus forte eo pertingerent, fines Analyseos amplificarent. Cum autem nemo adhuc sit inuentus, qui hoc negotium cum successu susceperit, etiamsi vix dubitare liceat, quin plures id frustra tentauerint, merito mihi quidem inde concludere videor, praeter methodum, qua ego sum vsus, vix vllam aliam viam ad huiusmodi speculationes patere. Quia enim haec methodus perquam indirecte, et quasi per ambages procedit, neque verisimile

mile ſit, eam cuiquam, qui huiusmodi problemata ſit aggreſſurus, vnquam in mentem venire, mirum non eſt, has quaeſtiones ab aliis intactas eſſe relictas. Etſi igitur iam aliquot ſpecimina huius methodi ſingularis ediderim, tamen operae pretium fore arbitror, ſi eius explicationem magis illuſtrauero, atque ad enodationem Problematis ac Theorematis propoſiti, accuratius accommodauero, vt ea, ſaepius tractando, magis trita et familiaris reddatur. Cum enim eius ope ad maxime abſconditas proprietates ellipſis aliarumque curuarum, quaſi inopinato ſim deductus, nullum eſt dubium, quin in ea plurima alia profundiſſimae indaginis contineantur, quae non niſi poſt frequentiorem tractionem inde eruere liceat.

Lemma I.

1. Si binae variabiles x et y ita a ſe inuicem pendeant, vt ſit:

$$0 = \alpha + \beta(xx + yy) + 2\gamma xy + \delta xxyy$$

erit ſiue ſumma, ſiue differentia, harum formularum integralium

$$\int \frac{dy}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}} \pm \int \frac{dx}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)}}$$

aequalis quantitati conſtanti.

Demonſtratio.

Cum enim ſit $0 = \alpha + \beta\, xx + yy) + 2\gamma xy + \delta xxyy$, erit inde vtramque radicem extrahendo:

$$y = \frac{-\gamma x \pm \sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)}}{\beta + \delta xx}$$

$$x = \frac{-\gamma y \pm \sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}}{\beta + \delta yy}$$

 vnde

vnde fequitur fore:

$$\beta y + \gamma x + \delta xxy = \pm \sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)}$$
$$\beta x + \gamma y + \delta xyy = \pm \sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}$$

Quodfi vero aequatio propofita differentietur, orietur:

$$0 = \beta x dx + \beta y dy + \gamma y dx + \gamma x dy + \delta xyy dx + \delta xxy dy$$

feu $0 = dx(\beta x + \gamma y + \delta xyy) + dy(\beta y + \gamma x + \delta xxy)$

quae abit in hanc:

$$\frac{dy}{\beta x + \gamma y + \delta xyy} + \frac{dx}{\beta y + \gamma x + \delta xxy} = 0.$$

Subftituantur loco denominatorum formulae illae irrationales, vt prodeant duo membra differentialia, in quibus variabiles x et y fint a fe inuicem feparatae, ac fumendis integralibus obtinebitur:

$$\int \frac{dy}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}} \pm \int \frac{dx}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)}} = \text{Conft.}$$

Coroll. 1.

2. Summa harum formularum integralium erit conftans, fi in vtraque radicis extractione fignis radicalibus paria tribuantur figna; fin autem figna ftatuantur difparia, tum differentia formularum integralium erit conftans.

Coroll. 2.

3. Si ponamus:

$$-\alpha\beta = Ak;\quad \gamma\gamma - \alpha\delta - \beta\beta = Bk;\quad -\beta\delta = Ck,$$

vnde fiet:

$$\alpha = \frac{-Ak}{\beta};\quad \delta = \frac{-Ck}{\beta},\quad \text{et } \gamma = \frac{\sqrt{(ACkk + Bk\beta\beta + \beta^4)}}{\beta}$$

Quare fi relatio inter x et y hac aequatione exprimatur:

$$0 = -Ak + \beta\beta(xx + yy) + 2xy\sqrt{(ACkk + Bk\beta\beta + \beta^4)} - Ckxxyy$$

erit

erit

$$\int \frac{dy}{\sqrt{(A+Byy+Cy^4)}} \pm \int \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = \text{Conſt.}$$

Coroll. 3.

4. Subſtitutis autem loco α, δ, γ his valoribus, erit

$$y = \frac{-x\sqrt{(ACkk+Bk\beta\beta+\beta^4)} \pm \beta\sqrt{k(A+Bxx+Cx^4)}}{\beta\beta - Ckxx}$$

$$x = \frac{-y\sqrt{(ACkk+Bk\beta\beta+\beta^4} \pm \beta\sqrt{k(A+Byy+Cy^4)}}{\beta\beta - Ckyy)}$$

qui ergo ſunt valores illi aequationi integrali conuenientes, et quia in his formulis ineſt conſtans arbitraria $\frac{\beta\beta}{k}$, eae integrale completum exhibere ſunt cenſendae.

Coroll. 4.

5. Ad has formulas commodiores reddendas, quia poſito $x = 0$ fit $y = \pm\frac{\sqrt{Ak}}{\beta}$, ponatur $\frac{\sqrt{Ak}}{\beta} = f$; et prodibit:

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A - Cffxx}$$

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} \pm f\sqrt{A(A+Byy+y^4)}}{A - Cffyy}$$

quae ſunt radices huius aequationis:

$$0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy$$

Coroll. 5.

6. Si ergo relatio inter x et y hac aequatione exprimatur:

$$0 = -Aff + A(xx+yy) \pm 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy$$

tum erit:

$$\int \frac{dy}{\sqrt{(A+Byy+Cy^4)}} \pm \int \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = \text{Conſt.}$$

ſeu $\frac{dy}{\sqrt{(A+Byy+Cy^4)}} \pm \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = 0.$

 Coroll.

Coroll. 6.

7. Viciſſim ergo ſi habeatur haec aequatio differentialis:

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}}+\frac{dx}{\sqrt{(A+Bxx+Cx^4)}}=0$$

relatio inter x et y ita ſe habebit, vt ſit:

$$y=\frac{-x\sqrt{A(A+Bff+Cf^4)}+f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

ſeu $x=\frac{-y\sqrt{A(A+Bff+Cf^4)}+f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}$.

Coroll. 7.

8. Verum propoſita hac aequatione differentiali:

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}}-\frac{dx}{\sqrt{(A+Bxx+Cx^4)}}=0$$

aequatio integralis completa erit:

$$y=\frac{x\sqrt{A(A+Bff+Cf^4)}+f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx},$$

ſeu $x=\frac{y\sqrt{A(A+Bff+Cf^4)}-f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}$.

Scholion.

9. Retinebo determinationes huius poſtremi caſus, quibus efficitur, quod ſi relatio inter binas variabiles x et y fuerit

$$0=-Aff+A(xx+yy)-2xy\sqrt{A(A+Bff+Cf^4)}-Cffxxyy,$$

ſiue $y=\frac{x\sqrt{A(A+Bff+Cf^4)}+f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$

et $x=\frac{y\sqrt{A(A+Bff+Cf^4)}-f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}$

tum hanc aequationem differentialem locum habere:

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}}-\frac{dx}{\sqrt{(A+Bxx+Cx^4)}}=0,$$

ſeu ſumtis integralibus fore:

$$\int\frac{dy}{\sqrt{(A+Byy+Cy^4)}}-\int\frac{dx}{\sqrt{(A+Bxx+Cx^4)}}=\text{Conſt.}$$

Pro

Pro hoc ergo casu erit:

$$\sqrt{(A+Bxx+Cx^4)} = \frac{y(A-Cffxx)-x\sqrt{A(A+Bff+Cf^4)}}{f\sqrt{A}}$$

et $$\sqrt{(A+Byy+Cy^4)} = \frac{-x(A-Cffyy)+y\sqrt{A(A+Bff+Cf^4)}}{f\sqrt{A}}$$

sicque fiet:

$$\frac{f\,dy\sqrt{A}}{y\sqrt{A(A+Bff+Cf^4)}-x(A-Cffyy)} + \frac{f\,dx\sqrt{A}}{x\sqrt{A(A+Bff+Cf^4)}-y(A-Cffxx)} = 0.$$

Lemma 2.

10. Eadem manente relatione inter binas variabiles x et y, vt sit $0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy)$, seu

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)}+f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

et $$x = \frac{y\sqrt{A(A+Bff+Cf^4)}-f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}.$$

erit differentia harum formularum integralium

$$\int\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} - \int\frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}$$

geometrice assignabilis.

Demonstratio.

Ad hoc ostendendum ponamus hanc differentiam $= V$, vt sit:

$$\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} - \frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}} = dV$$

Quare cum sit $\frac{dy}{\sqrt{(A+Byy+Cy^4)}} = \frac{dx}{\sqrt{(A+Bxx+Cx^4)}}$, erit

$$dV = \frac{\mathfrak{B}(yy-xx)dx}{\sqrt{(A+Bxx+Cx^4)}} = \frac{\mathfrak{B}f(yy-xx)dx\sqrt{A}}{y(A-Cffxx)-x\sqrt{A(A+Bff+Cf^4)}}.$$

Ponamus iam $xy = u$, vt sit $y = \frac{u}{x}$; et

$$0 = -Aff + Axx + \frac{Auu}{xx} - 2u\sqrt{A(A+Bff+Cf^4)} - Cffuu$$

qua aequatione differentiata fit:

$$0 = Axdx - \frac{Auudx}{x^3} + \frac{Audu}{xx} - du\sqrt{A(A+Bff+Cf^4)} - Cffudu;$$

 vnde

vnde, ob $\frac{u}{x}=y$, per x multiplicando oritur:

$$\frac{dx}{y(A-Cffxx)-x\sqrt{A(A+Bff+Cf^4)}}=\frac{du}{A(yy-xx)}$$

quae multiplicata per $\mathfrak{B}f(yy-xx)\sqrt{A}$ praebet:

$$dV=\frac{\mathfrak{B}f\,du}{\sqrt{A}} \text{ et } V=\text{Const.}+\frac{\mathfrak{B}fxy}{\sqrt{A}}.$$

Quam ob rem pro formularum integralium differentia habebimus:

$$\int\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}}-\int\frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}=\text{Const.}+\frac{\mathfrak{B}fxy}{\sqrt{A}}$$

quae vtique est geometrice assignabilis.

Coroll. 1.

11. Propositis ergo duabus formulis integralibus similibus

$$\int\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} \text{ et } \int\frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}$$

eiusmodi relatio inter x et y exhiberi potest, vt harum formularum differentia fiat geometrice assignabilis.

Coroll. 2.

12. Hunc scilicet in finem talis relatio inter variabiles x et y statui debet, vt sit:

$$0=-Aff+A(xx+yy)-2xy\sqrt{A(A+Bff+Cf^4)}-Cffxxyy$$

cuius aequationis resolutio cum sit ambigua, capi debet:

$$y=\frac{x\sqrt{A(A+Bff+Cf^4)}+f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

$$\text{et}\quad x=\frac{y\sqrt{A(A+Bff+Cf^4)}-f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}.$$

Coroll. 3.

13. Quemadmodum hic y per x et f, atque x per y et f definitur, ita etiam ſimili modo f per x et y definiri poteſt. Erit enim

$$f = \frac{y\sqrt{A(A+Bxx+Cx^4)} - x\sqrt{A(A+Byy+Cy^4)}}{A - Cxxyy}$$

vnde patet, ſi ſit $x = 0$, fore $y = f$, ex quo caſu conſtans illa, in valorem ipſius V ingrediens, definiri debet.

Scholion.

14. Simili modo demonſtrari poteſt, etiam harum formularum integralium differentiam

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4+\mathfrak{D}y^6)}{\sqrt{(A+Byy+Cy^4)}} - \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4+\mathfrak{D}x^6)}{\sqrt{(A+Bxx+Cx^4)}} = V$$

eſſe geometrice aſſignabilem: Poſito enim $xy = u$ erit:

$$dV = \frac{f\,du}{(yy-xx)\sqrt{A}}(\mathfrak{B}(yy-xx)+\mathfrak{C}(y^4-x^4)+\mathfrak{D}(y^6-x^6)),$$ ideoque

$$dV = \frac{f\,du}{\sqrt{A}}(\mathfrak{B}+\mathfrak{C}(yy+xx)+\mathfrak{D}(y^4+xxyy+x^4))$$

At ex aequatione canonica habemus:

$$xx+yy = \frac{Aff+2u\sqrt{A(A+Bff+Cf^4)}+Cffuu}{A}$$

Ponamus breuitatis gratia $\sqrt{A(A+Bff+Cf^4)} = Fff$, vt ſit

$$xx+yy = \frac{ff}{A}(A+2Fu+Cuu),$$

eritque ob $y^4+xxyy+x^4 = (xx+yy)^2 - uu$

$$dV = \frac{f\,du}{\sqrt{A}}\left\{\begin{array}{l}\mathfrak{B}+\frac{\mathfrak{C}ff}{A}(A+2Fu+Cuu)\\ +\frac{\mathfrak{D}f^4}{AA}(A+2Fu+Cuu)^2-\mathfrak{D}uu\end{array}\right\}$$

ideoque integrando:

$$V = \frac{f}{\sqrt{A}}\left\{\begin{array}{l}\mathfrak{B}u+\frac{\mathfrak{C}ff}{A}(Au+Fuu+\frac{1}{3}Cu^3)-\frac{1}{3}\mathfrak{D}u^3\\ +\frac{\mathfrak{D}f^4}{AA}(AAu+2AFuu+\frac{2}{3}(AC+2FF)u^3+CFu^4+\frac{1}{5}CCu^5\end{array}\right\}$$

Verum

Verum pro praeſenti inſtituto, quo ellipſis nobis eſt propoſita, formulae in lemmate exhibitae ſufficiunt.

Lemma 3.

Tab. III. Fig. 1. 15. Si C ſit centrum ellipſeos, eiusque ſemiaxes $CA = a$, $CB = b$; atque ad verticem A ducatur tangens AD, in qua ſumatur portio indefinita $AZ = z$, et ex Z ad AD perpendicularis erigatur ZMV, erit arcus, huic abſciſſae $AZ = z$ reſpondens, $AM = \int \frac{dz}{b} \sqrt{\frac{b^4 - (b^2 - aa)zz}{bb - zz}}$.

Demonſtratio.

Ponatur $ZM = v$; et ipſe arcus $AM = s$; erit ex natura ellipſis:

$$VM = a - v = \frac{a}{b}\sqrt{(bb - zz)}, \text{ hincque}$$

$$v = a - \frac{a}{b}\sqrt{(bb - zz)} \text{ et } dv = \frac{azdz}{b\sqrt{(ab - zz)}}.$$

Quare cum ſit $ds = \sqrt{(dz^2 + dv^2)}$, erit

$$ds = dz\sqrt{\left(1 + \frac{aazz}{bb(bb - zz)}\right)} = \frac{dz}{b}\sqrt{\frac{b^4 - (bb - aa)zz}{bb - zz}}.$$

et integrando:

$$s = \text{Arc. } AM = \int \frac{dz}{b} \sqrt{\frac{b^4 - (bb - aa)zz}{bb - zz}}$$

integrali ita accepto, vt euaneſcat, poſito $z = 0$.

Coroll. 1.

16. Ad hanc formulam contrahendam ponamus hic et in ſequentibus perpetuo $\frac{bb - aa}{bb} = n$, vt ſit $a = b\sqrt{(1 - n)}$, eritque

Arcus abſciſſae $AZ = z$ reſpondens $AM = \int dz \sqrt{\frac{bb - nzz}{bb - zz}}$.

Seu

Seu cum sit $AM = \int \frac{dz(bb - nzz)}{\sqrt{(b^4 - (n+1)bbzz + nz^4)}}$, haec expressio ad nostram formam tractatam $\int \frac{dz(\mathfrak{A} + \mathfrak{B}zz)}{\sqrt{(A + Bzz + Cz^4)}}$ reducetur ponendo:

$$\mathfrak{A} = bb;\ \mathfrak{B} = -n;\ A = b^4;\ B = -(n+1)bb;\ C = n$$

ita vt sit $\sqrt{(A + Bzz + Cz^4)} = \sqrt{(bb - zz)(bb - nzz)}$.

Coroll. 2.

17. Cum ob $a = b\sqrt{(1-n)}$ sit $dv = \frac{zdz\sqrt{(1-n)}}{\sqrt{(bb-zz)}}$ et $ds = dz\sqrt{\frac{bb-nzz}{bb-zz}}$, erit anguli AMZ sinus $= \frac{dz}{ds} = \sqrt{\frac{bb-zz}{bb-nzz}}$; cosinus $= \frac{dv}{ds} = \frac{z\sqrt{(1-n)}}{\sqrt{(bb-nzz)}}$ et tangens $= \frac{dz}{dv} = \frac{\sqrt{(bb-zz)}}{z\sqrt{(1-n)}}$: quas formulas probe notasse iuuabit

$$\text{sinus } AMZ = \sqrt{\frac{bb-zz}{bb-nzz}}$$

$$\text{cosinus } AMZ = \frac{z\sqrt{(1-n)}}{\sqrt{(bb-nzz)}}$$

$$\text{tang } AMZ = \frac{\sqrt{(bb-zz)}}{z\sqrt{(1-n)}}.$$

Coroll. 3.

18. Designabo porro arcum AM, qui abscissae cuique $AZ = z$ respondet, hac expressione $\Pi : z$, vt sit $AM = \Pi : z = \int dz\sqrt{\frac{bb-nzz}{bb-zz}}$. Hinc si variae abscissae ponantur

$$AF = f;\ AP = p;\ AQ = q;\ AR = r;\ AD = AB = b$$

erunt arcus respondentes:

$$Af = \Pi : f;\ Ap = \Pi : p;\ Aq = \Pi : q;\ Ar = \Pi : r;\ AMB = \Pi : b.$$

Coroll. 4.

19. Hoc modo etiam arcus, qui non in puncto A terminantur, commode exprimi poterunt; sic enim erit:

arcus $fp = \Pi : p - \Pi : f$; arcus $pq = \Pi : q - \Pi : p$
arcus $qr = \Pi : r - \Pi : q$; arcus $pr = \Pi : r - \Pi : p$

item arcus $Bp = \Pi : b - \Pi : p$; arcus $Bq = \Pi : b - \Pi : q$ Denotat enim $\Pi : b$ arcum totius quadrantis AMB; ideoque $4\,\Pi : b$ totam ellipsis peripheriam.

Problema I.

Tab. III. Fig. I. 20. Proposito in ellipsi arcu Af in vertice A terminato, ab alio quouis puncto p arcum abscindere pq, qui ab illo arcu Af discrepet quantitate geometrice assignabili.

Solutio.

Positis abscissis, quae punctis f, p et q respondent, $AF = f$; $AP = p$; et $AQ = q$, ex datis f et p conuenienter determinari oportet q. Cum igitur pro lemmate secundo sit

$$\mathfrak{A} = bb;\ \mathfrak{B} = -n;\ A = b^4;\ B = -(n+1)bb,\ \text{et}\ C = n:$$

capiatur q ita, vt sit:

$$q = \frac{bbp\sqrt{(bb-ff)(bb-nff)} + bbf\sqrt{(bb-pp)(bb-npp)}}{b^4 - nffpp}$$

eritque per lemmatis conclusionem:

$$\int dq\sqrt{\frac{bb-nqq}{bb-qq}} - \int dp\sqrt{\frac{bb-npp}{bb-pp}} = \text{Const.} - \frac{nfpq}{bb}.$$

At est $\int dq\sqrt{\frac{bb-nqq}{bb-qq}} = \Pi : q$ et $\int dp\sqrt{\frac{bb-npp}{bb-pp}} = \Pi : p$, vnde

$$\Pi : q - \Pi : p = \text{Const.} - \frac{nfpq}{bb}$$

vbi tantum superest, vt constans debite definiatur. Verum quia posito $p = 0$, fit $q = f$, ad quem casum aequa-

aequatione translata fiet: $\Pi : f =$ Const. quo valore introducto habebimus:

$$\Pi : q - \Pi : p = \Pi : f - \frac{nfpq}{bb}$$

siue $\text{Arc} : pq = \text{Arc} : Af - \frac{nfpq}{bb}$.

Coroll. 1.

21. Quia vero eidem abscissae $AQ = q$, bina in ellipsi puncta q respondent, ad hoc punctum perfecte determinandum, etiam applicatae Qq magnitudo definiri debet: Est vero

$$Qq = a - \frac{a}{b}\sqrt{(bb - qq)} = (b - \sqrt{(bb - qq)})\sqrt{(1 - n)}, \text{ et}$$

$$\sqrt{(bb - qq)} = \frac{b^3\sqrt{(bb - ff)(bb - pp)} - bfp\sqrt{(bb - nff)(bb - npp)}}{b^4 - nffpp}$$

Tum etiam notari meretur

$$\sqrt{(bb - nqq)} = \frac{b^3\sqrt{(bb - nff)(bb - npp)} - nbf\sqrt{(bb - ff)(bb - pp)}}{b^4 - nffpp}$$

si igitur valor ipsius $\sqrt{(bb - qq)}$ fit negatiuus, punctum q in superiori ellipsis quadrante capi debet.

Coroll. 2.

22. Hic igitur primo relatio notari debet, quae inter tria puncta f, p et q intercedit, quae ita est comparata, vt ex binis datis tertium inueniri possit:

I. Si f et p sint data, erit

$$q = \frac{bbp\sqrt{(bb - ff)(bb - nff)} + bbf\sqrt{(bb - pp)(bb - npp)}}{b^4 - nffpp}$$

$$\sqrt{(bb - qq)} = \frac{b^3\sqrt{(bb - ff)(bb - pp)} - bfp\sqrt{(bb - nff)(bb - npp)}}{b^4 - nffpp}$$

$$\sqrt{(bb - nqq)} = \frac{b^3\sqrt{(bb - nff)(bb - npp)} - nbfp\sqrt{(bb - ff)(bb - pp)}}{b^4 - nffpp}$$

II. Si f et q sint data, erit:

$$p = \frac{bbq\sqrt{(bb-ff)(bb-nff)} - bbf\sqrt{(bb-qq)(bb-nqq)}}{b^4 - nffqq};$$

$$\sqrt{(bb-pp)} = \frac{b^3\sqrt{(bb-ff)(bb-qq)} + bfq\sqrt{(bb-nff)(bb-nqq)}}{b^4 - nffqq}$$

$$\sqrt{(bb-npp)} = \frac{b^3\sqrt{(bb-nff)(bb-nqq)} + nbfq\sqrt{(bb-ff)(bb-qq)}}{b^4 - nffqq}.$$

III. Si p et q sint data, erit:

$$f = \frac{bbq\sqrt{(bb-pp)(bb-npp)} - bbp\sqrt{(bb-qq)(bb-nqq)}}{b^4 - nppqq}$$

$$\sqrt{(bb-ff)} = \frac{b^3\sqrt{(bb-pp)(bb-qq)} + bpq\sqrt{(bb-npp)(bb-nqq)}}{b^4 - nppqq},$$

$$\sqrt{(bb-nff)} = \frac{b^3\sqrt{(bb-npp)(bb-nqq)} + nbpq\sqrt{(bb-pp)(bb-qq)}}{b^4 - nppqq}.$$

Hae autem formulae omnes ex hac nascuntur:

$$0 = -b^4ff + b^4pp + b^4qq - 2bbpq\sqrt{(bb-ff)(bb-nff)} - nffppqq$$

quae adeo ad hanc rationalem, in qua f, p, et q aequaliter insunt, reducitur:

$$0 = b^8(f^4+p^4+q^4) + 4(n+1)b^6ffppqq - 2b^8(ffpp+ffqq + ppqq) - 2nb^4ffppqq(ff+pp+qq) + nnf^4p^4q^4.$$

Coroll. 3.

23. Harum formularum igitur ope, si trium punctorum f, p et q data sint bina quaecunque, tertium inueniri poterit, vt arcuum Af et pq differentia geometrice fiat assignabilis: Erit enim

$$\text{Arc.}\, Af - \text{Arc.}\, pq = \text{Arc.}\, Ap - \text{Arc.}\, fq = \frac{nfpq}{bb}.$$

Coroll. 4.

24. Denotat autem b semiaxem ellipsis CB; et posito altero $CA = a$, fecimus $\frac{bb-aa}{bb} = n$: vnde si $n = 0$ ellipsis

ellipsis abit in circulum, et arcuum assignatorum differentia euanescit. Ellipsis autem abibit in parabolam, cuius semiparameter $= c$, si $bb = ac$, et $a = \infty$: Hoc ergo casu fiet $n = \frac{c - a}{c} = -\frac{a}{c}$, et $\frac{n}{bb} = -\frac{1}{cc}$: ideoque $n = -\frac{bb}{cc}$ et $\sqrt{(bb - ff)} = b$; $\sqrt{(bb - nff)} = b\sqrt{(1 - \frac{ff}{cc})}$: vnde formulae superiores ad parabolam transferri poterunt.

Coroll. 5.

25. Si easdem formulas ad hyperbolam accommodare velimus, semiaxem b ita imaginarium statui oportet, vt eius quadratum bb fiat quantitas negatiua. Seu, quod eodem redit, in nostris formulis vbique loco bb scribatur $-bb$, et semiaxis a capiatur negatiue, tum vero n erit numerus vnitate maior.

Problema 2.

26. In quadrante elliptico AB, dato puncto quocunque f, inuenire aliud punctum g, vt arcuum Af et Bg differentia sit geometrice assignabilis. Tab. III. Fig. 2.

Solutio.

Ex praecedente problemate hoc facile resoluitur; positis enim semiaxibus CA $= a$, CB $= b$ et $\frac{bb - aa}{bb} = n$, punctum q in praecedente problemate in B vsque promoueri oportet, vt fiat $q = b$; tum sint abscissae super tangente AD vel axe AB sumtae, punctis f et g respondentes, AF $=$ Cf $= f$ et AG $=$ C𝔊 $= g$, ita vt, quod ante erat p, nunc sit g, atque ex dato puncto f

determinatio puncti g per formulas (§. 22.) ita se habebit, ob $p = g$ et $q = b$.

$$g = \frac{b^3 \sqrt{(bb - ff)(bb - nff)}}{b^4 - nbbff} = b \sqrt{\frac{bb - ff}{bb - nff}}:$$

$$\sqrt{(bb - gg)} = \frac{bbf \sqrt{(bb - nff)(bb - nbb)}}{b^4 - nbbff} = \frac{bf \sqrt{(1 - n)}}{\sqrt{(bb - nff)}}$$

$$\sqrt{(bb - ngg)} - \frac{b^3 \sqrt{(bb - nff)(bb - nbb)}}{b^4 - nbbff} = \frac{bb \sqrt{(1 - n)}}{\sqrt{(bb - nff)}}.$$

Vnde si anguli, quos applicatae Ff et Gg cum curua faciunt, in computum ducantur, erit

$$g = b \sin AfF \text{ et } f = b \sin AgG.$$

Atque hinc sequitur ista constructio pro puncto g inueniendo: Ad punctum f ducatur tangens fT, donec axi CA producto occurrat in T, tum in ea, si opus est, producta capiatur $TV = CB = b$, et per V agatur recta GG axi CA parallela, eritque punctum g quaesitum, ita vt arcuum Af et Bg differentia sit geometrice assignabilis. Verum ex problemate praecedente, ob $p = g$ et $q = b$, erit haec differentia:

$$\text{Arc.} Af - \text{Arc.} Bg = \frac{nfg}{b} = nf \sqrt{\frac{bb - ff}{bb - nff}}.$$

Ad quam construendam notetur esse:

$$Ff = \frac{AF}{\sin AfF} = f \sqrt{\frac{bb - nff}{bb - ff}}$$

et ex natura ellipsis:

$$CT = \frac{ab}{\sqrt{(bb - ff)}} = \frac{bb \sqrt{(1 - n)}}{\sqrt{(bb - ff)}}.$$

Hinc si ex centro ellipsis C in tangentem Ff demittatur perpendiculum CS, ob ang. CTS = ang. AfF, eiusque sinum $= \sqrt{\frac{bb - ff}{bb - nff}}$ et cosinum $= \frac{f\sqrt{(1 - n)}}{\sqrt{(bb - nff)}}$, erit

$$TS = CT \cos CTS = \frac{bbf(1 - n)}{\sqrt{(bb - ff)(bb - nff)}} \text{ hincque}$$

$$Sf = Tf - Ts = \frac{bbf - nf^3 - bbf + nbbf}{\sqrt{(bb - ff)(bb - nff)}} = \frac{nf(bb - ff)}{\sqrt{(bb - ff)(bb - nff)}} = nf \sqrt{\frac{bb - ff}{bb - nff}}.$$

Portio

Portio igitur tangentis fS, inter perpendiculum CS et punctum contactus f contenta, praebebit differentiam arcuum Af et Bg, ita vt sit:

$$\text{Arc. } Af - \text{Arc. } Bg = \text{Arc. } Ag - \text{Arc } Bf = Sf.$$

Coroll. 1.

27. Haec differentia arcuum facilius inueniri potest, si in f ad tangentem ducatur normalis $f\mathfrak{S}$; tum enim ex natura ellipsis statim constat, esse $C\mathfrak{S} = f - \frac{aa}{bb}f = nf$. Quare cum CS ipsi $\mathfrak{S}f$ sit parallela, et angulus $BCS = CTS = TfF$, eiusque ergo sinus $= \sqrt{\frac{bb-ff}{bb-nff}}$, erit:

$$Sf = C\mathfrak{S} \sin BCS = nf\sqrt{\frac{bb-ff}{bb-nff}}.$$

Coroll. 2.

28. Simili modo ex puncto g definietur punctum f; si enim ad g ducatur tangens vsque ad axem CA, atque ab intersectione eius cum axe in ea capiatur portio alteri semiaxi CB aequalis, haec praecise in recta Ff terminabitur, ideoque punctum f monstrabit.

Coroll. 3.

29. Constructio ergo puncti g ex dato puncto f ita se habebit: Ad punctum f ducatur tangens, axi CA producto occurrens in T, in eaque a T abscindatur portio TV, semiaxi CB aequalis, et recta $G\mathfrak{G}$ axi CA parallela, per punctum V acta, in ellipsi punctum quaesitum g definiet. Tum enim, si ex centro ellipsis C in illam tangentem perpendiculum CS demittatur, erit

erit Arc. Af − Arc. Bg = Rectae Sf, hincque etiam Arc. Af − Recta fS = Arc. Bg.

Coroll. 4.

Tab. III. Fig. 3. 30. Casus notabilis est, quo bina puncta f et g in vnum colliquescunt, ita vt arcus quadrantis AfB in puncto f ita secari iubeatur, vt partium Af et Bf differentia fiat geometrice assignabilis. Hunc in finem ponatur in solutione $g = f$, vnde fit $f = b\sqrt{\frac{bb - f^2}{bb - nff}}$ hincque $2bbff - nf^4 = b^4$, et $\frac{bb}{ff} = 1 + \sqrt{(1-n)} = \frac{a+b}{b}$. Quare pro puncto hoc f capi debet abscissa $AF = f = b\sqrt{\frac{b}{a+b}}$: atque, ob $\sqrt{\frac{bb-ff}{bb-nff}} = \frac{f}{b}$, erit partium differentia $Af - Bf = \frac{nff}{b} = \frac{nbb}{a+b}$, quae cum sit $n = \frac{bb-aa}{bb}$, abit in $Af - Bf = b - a$, ita vt aequalis euadat differentiae semiaxium. Vnde puncto f hoc modo definito, vt sit $f = b\sqrt{\frac{b}{a+b}}$, erit etiam

$$AC + Af = BC + Bf$$

seu ducto radio Cf ambo trilinea ACf et BCf pari perimetro includuntur.

Coroll. 5.

31. Quia supra habuimus $CT = \frac{ab}{\sqrt{(bb-ff)}}$, erit pro praesenti casu $CT = \sqrt{(aa+ab)}$ ob $ff = \frac{b^3}{a+b}$; vnde sequens concinna puncti f constructio deducitur. Bisecto semiaxe BC in O, interuallo $OT = OC + AC$, definiatur in CA producta punctum T, vnde interuallo $Tf = BC$ punctum f in ellipsi designetur: eritque f punctum quaesitum, et recta Tf eius tangens.

Proble-

Problema 3.

32. Propoſita ſemiellipſi $AB\alpha$, in eaque ſumto quocunque puncto p, definire punctum q ita, vt arcus pBq differat a quadrante elliptico ApB quantitate geometrice aſſignabili. Tab. III. Fig. 4.

Solutio.

Poſitis, vt hactenus, ſemiaxibus $CA = a$, $CB = b$ et ad abbreuiandum $n = \frac{bb - aa}{bb}$, in ſolutione problematis primi promoueatur punctum f in B vsque, eritque vi eius arcuum AB et pq differentia geometrice aſſignabilis, vti requiritur. Demiſſis ergo ad tangentem AD ex p et q perpendiculis pP et qQ, ſint $AP = p$ et $AQ = q$, atque ob $f = b$ habebimus ex (22)

$$q = \frac{b\surd(bb - pp)(bb - npp)}{bb - npp} = b\surd\frac{bb - pp}{bb - npp}$$

$$\surd(bb - qq) = \frac{-p\surd(bb - nbb)(bb - npp)}{bb - npp} = \frac{-bp\surd(1 - n)}{\surd(bb - npp)}$$

cuius quantitatis ſignum $-$ indicat, vlteriorem interſectionem perpendiculi QK pro puncto q accipi oportere, ſecus atque in problemate praecedente. Cum igitur $\surd\frac{bb - pp}{bb - npp}$ exprimat ſinum anguli, quem applicata Pp cum curua facit, erit $q = b$ ſin ApP. Ad Qq, ſi opus eſt, productam, ex centro C dirigatur recta CK, ſemiaxi $CB = b$ aequalis, vt ſit $CK = b$, eritque $\frac{q}{b} = \frac{CQ}{CK} =$ ſin ApP, hincque ſin $CKQ =$ ſin ApP et $CKQ = ApP$. Ex quo patet rectam CK parallelam fore tangenti in puncto p. Quare iuncta Cp, eaque, vt ſemidiametro ſpectata, erit CL eius ſemidiameter coniugata, in qua proinde producta, ſi capiatur $CK = CB$,

 perpen-

perpendiculum KQ ad CB demissum in ellipsi definiet punctum q. Quo inuento ob $f=b$; et $q=b\sqrt{\frac{bb-pp}{bb-npp}}$ erit arcuum differentia:

$$\text{Arc. } AB-\text{Arc. } pq=\frac{nfpq}{bb}=np\sqrt{\frac{bb-pp}{bb-npp}}=np\,\text{sin}\,ApP.$$

Ducatur ad ellipsin in p normalis $p\mathfrak{N}$, erit $C\mathfrak{N}=np$, et producta $p\mathfrak{N}$ in N angulus $C\mathfrak{N}N=\text{ang. } ApP$: quare cum haec pN futura sit normalis in diametrum coniugatam CL, erit $CN=np\,\text{sin}\,ApP$; vnde demisso ex p in CL perpendiculo, interuallum CN aequabitur differentiae illorum arcuum, ita vt sit:

$$\text{Arc. } AB-\text{Arc. } pq=CN.$$

Coroll. 1.

33. Cum igitur punctum p pro lubitu assumi possit, infiniti arcus pq exhiberi possunt, qui a quadrante AB differunt quantitate geometrice assignabili. Quare etiam hi arcus inter se different quantitate geometrice assignabili.

Coroll. 2.

34. Ex dato ergo puncto p punctum q ita definitur: Ad ductam Cp iungatur semidiameter coniugata CL in K producenda, vt fiat CK aequalis semiaxi CB, ad quem ex K perpendiculum demittatur KQ, ellipsin secans in q, erit q punctum quaesitum. Atque demisso ex p in CL perpendiculo pN, erit $AB=pq=CN$.

Coroll.

Coroll. 3.

35. Quoties perpendiculum pN intra C et K cadit, arcus pq erit minor quadrante AB, contra autem, si ad alteram partem cadit, maior. Ita si prius punctum in π detur, et rectae $C\pi$ conueniat semidiameter coniugata CL, qua producta in K, vt sit $CK = CB$, et ex K ad CB, demisso perpendiculo KQ secante ellipsin in q, quia hic perpendiculum $\pi\nu$ in CL demissum ad alteram partem cadit, erit arcus πq — arcu $AB = C\nu$. Tab. III. Fig. 5.

Theorema demonstrandum.

36. Si ellipsis $AB\alpha\beta$ diametro quacunque $p\pi$ fuerit bisecta, ad eamque ducatur diameter coniugata $L\lambda$, cuius semissis CL producatur in K, vt fiat CK alteri semiaxi principali CB aequalis, ad quem ex K demittatur perpendiculum KQ, ellipsin secans in q, tum ellipsis semiperimeter $pBL\alpha\pi$ ita secabitur in q, vt partium $\pi\alpha q$ et pBq differentia sit geometrice assignabilis. Ductis enim ex p et π ad diametrum coniugatam $L\lambda$ normalibus pN et $\pi\nu$, interuallum $N\nu$ illi differentiae ita aequabitur, vt sit Arc. $\pi\alpha q$ — Arc. $pBq = N\nu$. Fig. 5.

Demonstratio.

Quia CL est semidiameter coniugata conueniens semidiametro Cp, ex constructione, qua punctum q est definitum, patet per §. 34. fore:

Arc. AB — Arc. $pq = CN$.

Deinde, quia CL eſt quoque ſemidiameter coniugata conueniens ſemidiametro $C\pi$, ex §. 35. patet eſſe

$$\text{Arc.}\,\pi q - \text{Arc.}\,AB = C\nu.$$

Addantur hae duae aequationes, ac reſultabit

$$\text{Arc.}\,\pi q - \text{Arc.}\,pq = CN + C\nu = N\nu.$$

Coroll.

37. Perinde eſt, vtri ſemiaxi principali ſemidiameter CL producta, eiusue portio, aequalis capiatur, dummodo ex eius termino ad eum ipſum axem perpendiculum demittatur. Ita in CL potuiſſet abſcindi portio Ck ſemiaxi minori $C\alpha$ aequalis; recta enim qkq, per k ad $C\alpha$ normaliter ducta, in ellipſi idem punctum q prodidiſſet.

Scholion.

38. En ergo demonſtrationem completam Theorematis in Actis Erud. Lipſ. propoſiti, quae ita eſt comparata, vt nullo modo ex vulgaribus ellipſis proprietatibus deriuari potuiſſet, neque etiam Analyſis infinitorum multum auxilii attulerit, niſi hoc ipſo modo, quo hic ſum vſus, in ſubſidium vocetur. Ex profundis quidem ſpeculationibus Ill. Comitis Fagnani hanc quoque demonſtrationem deducere liceret; verum inde vix via pateret, ad problema ibidem propoſitum reſoluendum, in cuius ergo gratiam ſequentia ſunt praemittenda.

Problema 4.

Tab. IV. Fig. 1. 39. Arcum ellipticum quemcunque Ag ad alterum axem principalem in A terminatum ita ſecare in

f, vt

f, vt partium Af et fg differentia ſit geometrice aſſignabilis.

Solutio.

Poſitis ſemiaxibus $CA = a$, $CB = b$, et breuitatis gratia $n = \frac{bb - aa}{bb}$, in verticis A tangente AD ſumantur abſciſſae, ac ponatur abſciſſa toti arcui Ag dato reſpondens $AG = g$, quaeſita autem, quae puncto f reſpondeat, ſit $AF = f$. Cum igitur differentia arcuum Af et fg debeat eſſe geometrice aſſignabilis, quaeſtio continetur in Probl. I. ſumendo ibi $p = f$, et ponendo $q = g$, vnde obtinebimus has formulas:

$$g = \frac{2bbf\sqrt{(bb - ff)(bb - nff)}}{b^4 - nf^4}$$

$$\sqrt{(bb - gg)} = \frac{b^3(bb - ff) - bff(bb - nff)}{b^4 - nf^4} = \frac{b(b^4 - 2bbff + nf^4)}{b^4 - nf^4}$$

$$\sqrt{(bb - ngg)} = \frac{b^3(bb - nff) - nbff(bb - ff)}{b^4 - nf^4} = \frac{b(b^4 - 2nbbff + nf^6)}{b^4 - nf^4}$$

Ex quibus combinatione oritur:

$$\sqrt{(bb - ngg)} - n\sqrt{(bb - gg)} = \frac{(1 - n)b(b^4 + nf^4)}{b^4 - nf^4}$$ hincque

$$\frac{nf^4}{b^4} = \frac{\sqrt{(bb - ngg)} - n\sqrt{(bb - gg)} - (1 - n)b}{\sqrt{(bb - ngg)} - n\sqrt{(bb - gg)} + (1 - n)b}$$

quae formula reducitur ad.

$$\frac{nnf^4}{b^4} = \frac{(\sqrt{(bb - ngg)} - n\sqrt{(bb - gg)} - (1 - n)b)^2}{2bb - (1 + n)gg - 2\sqrt{(bb - gg)(bb - ngg)}}.$$

vnde radice quadrata extracta fit:

$$\frac{nff}{bb} = \frac{\sqrt{(bb - ngg)} - n\sqrt{(bb - gg)} - (1 - n)b}{\sqrt{(bb - ngg)} - \sqrt{(bb - gg)}} = \frac{(b - \sqrt{(bb - gg)})(b - \sqrt{(bb - ngg)})}{gg}$$

ex qua porro elicimus:

$$\frac{bb - nff}{bb} = \frac{(1 - n)(b - \sqrt{(bb - gg)})}{\sqrt{(bb - ngg)} - \sqrt{(bb - gg)}} = \frac{(b - \sqrt{(bb - gg)})(\sqrt{(bb - ngg)} + \sqrt{(bb - gg)})}{gg}$$

$$\frac{n(bb - ff)}{bb} = \frac{(1 - n)(b - \sqrt{(bb - ngg)})}{\sqrt{(bb - ngg)} - \sqrt{(bb - gg)}} = \frac{(b - \sqrt{(bb - ngg)})(\sqrt{(bb - ngg)} + \sqrt{(bb - gg)})}{gg}$$

Punctum igitur quaesitum f ita determinabitur, vt sit:

$$f = \frac{b}{g\sqrt{n}}\sqrt{(b-\sqrt{(bb-gg)})(b-\sqrt{(bb-ngg)})}$$

$$\sqrt{(bb-ff)} = \frac{b}{g\sqrt{n}}\sqrt{(b-\sqrt{(bb-ngg)})}(\sqrt{(bb-gg)}+\sqrt{(bbn-gg)})$$

$$\sqrt{(bb-nff)} = \frac{b}{g}\sqrt{(b-\sqrt{(bb-gg)})}(\sqrt{(bb-gg)}+\sqrt{(bb-ngg)})$$

Verum hoc puncto f ita determinato, ob $p=f$ et $q=g$, partium inuentarum differentia erit

$$\text{Arc. A}f - \text{Arc.}fg = \frac{nffg}{bb} = \frac{(b-\sqrt{(bb-gg)})(b-\sqrt{(bb-ngg)})}{g}$$

Coroll. 1.

40. Casum huius problematis iam soluimus (§. 30), quo arcus secundus Ag toti quadranti AB assumitur aequalis. Si enim ponamus $g=b$, reperietur, vt ibi,

$$f = b\sqrt{\frac{1-\sqrt{(1-n)}}{n}} = b\sqrt{\frac{b(b-a)}{bb-aa}} = \frac{b\sqrt{b}}{\sqrt{(1+b)}}$$

et partium differentia prodit $= b - b\sqrt{(1-n)} = b-a$.

Coroll. 2.

41. Si arcus dati Ag alter terminus in superiori quadrante existat, eique eadem abscissa A$G=g$ respondeat, eaedem hae formulae valent, nisi quod valor radicalis $\sqrt{(bb-gg)}$ negatiue capi debeat, radicali $\sqrt{(bb-ngg)}$ non mutato.

Coroll. 3.

42. Ita si proponatur tota semiperipheria, erit $g=0$, et $\sqrt{(bb-gg)}=-b$, vnde pro hoc casu obtinebitur:

$$f = \frac{b}{g\sqrt{n}}\sqrt{2b(b-\sqrt{(bb-ngg)})} = b$$

scilicet

scilicet arcus Af abibit in quadrantem ellipsis. Sin autem integra ellipsis peripheria proponeretur, tum esset et $g = 0$ et $\sqrt{(bb-gg)} = +b$, sicque valor ipsius f prodiret euanescens, at pro $\sqrt{(bb-ff)}$ capi deberet $-b$.

Problema 5.

43 Proposito in ellipsi arcu Ag altero termino A, in axe principali terminato assignare arcum pq, qui sit praecise semissis arcus dati Ag.

Solutio.

Manentibus superioribus denominationibus, sint abscissae, punctis p et q respondentes, $AP = p$, et $AQ = q$, atque ex puncto p, quasi esset datum, quaeratur q, vt differentia arcuum Af et pq fiat geometrice assignabilis, tum enim quoque differentia arcuum fg et pq geometrice assignari poterit; siquidem secundum problema praecedens arcus datus Ag, pro quo est $AG = g$, ita sectus est in f, vt partium Af et fg differentia sit geometrice assignabilis. Hunc ergo in finem esse debet:

$$q = \frac{bbp\sqrt{(bb-ff)(bb-nff)} + bbf\sqrt{(bb-pp)(bb-npp)}}{b^4 - nffpp}$$

seu

$$0 = b^4(pp+qq-ff) - 2bbpq\sqrt{(bb-ff)(bb-nff)} - nffppqq$$

Quo facto erit

$$\text{Arc.}\,Af - \text{Arc.}\,pq = \frac{nfpq}{bb}; \text{ ideoque}$$

$$2\,\text{Arc.}\,Af - 2\,\text{Arc.}\,pq = \frac{2nfpq}{bb}$$

At ex problemate praecedente habemus:

$$\text{Arc.}\,Af - \text{Arc.}\,fg = \frac{nffg}{bb}$$

qua

qua aequatione ab illa ſubtracta relinquitur:

$$\text{Arc. A}g - 2\,\text{Arc.}\,pq = \frac{2nfpq}{bb} - \frac{nffg}{bb}$$

Quae differentia cum in nihilum abire debeat, habebimus:

$$2nfpq = nffg \quad \text{et} \quad 2pq = fg.$$

Pro pq ſubſtituatur iſte valor $\frac{1}{2}fg$, et obtinebimus

$$b^4(pp+qq) = b^4ff + bbfg\sqrt{(bb-ff)(bb-nff)} + \tfrac{1}{4}nf^4gg$$

exiſtente $g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4-nf^4}$, vel potius pro f introducatur valor ante inuentus:

$$f = \frac{b}{g\sqrt{n}}\sqrt{(b-\sqrt{(bb-gg)})(b-\sqrt{(bb-ngg)})}$$

vnde fit:

$$\sqrt{(bb-ff)(bb-nff)} = \frac{bb(\sqrt{(bb-gg)}+\sqrt{(bb-ngg)})}{gg\sqrt{n}}\sqrt{(b-\sqrt{(bb-gg)})(b-\sqrt{(bb-ngg)})}$$

Poſtea vero ambae abſciſſae p et q ex hac aequatione duplicata definiri poterunt:

$$pp \pm 2pp + qq = \frac{b^4ff \pm b^4fg + bbfg\sqrt{(bb-ff)(bb-nff)} + \frac{1}{4}nf^4gg}{b^4}$$

vel ſublata iſta irrationalitate ob $bbfg\sqrt{(bb-ff)(bb-nff)} = \frac{1}{2}gg(b^4-nf^4)$ habebimus:

$$p+q = \frac{\sqrt{(b^4ff + b^4fg + \frac{1}{2}b^4gg - \frac{1}{4}nf^4gg)}}{bb}$$

$$q-p = \frac{\sqrt{(b^4ff - b^4fg + \frac{1}{2}b^4gg - \frac{1}{4}nf^4gg)}}{bb}$$

vnde vtraque abſciſſa p et q ſeorſim facile aſſignatur.

Coroll. I.

44. Si quantitatem ſubſidiariam f penitus eliminemus, perueniemus ad has duas formulas:

$pp+qq$

$pp + qq = \frac{1}{4ngg}(b - \sqrt{(bb-gg)})(b - \sqrt{(bb-ngg)})$ in
$(5bb + 3b\sqrt{(bb-gg)} + 3b\sqrt{(bb-ngg)} + \sqrt{(bb-gg)(bb-ngg)})$
$2pq = \frac{b}{n\sqrt{}} \sqrt{(b - \sqrt{(bb-gg)}(b - \sqrt{(bb-ngg)})}$.

Coroll. 2.

45. Si arcus propositus Ag sit semiperipheriae aequalis, ideoque $g = 0$ et $\sqrt{(bb-gg)} = -b$, et $\sqrt{(bb-ngg)} = b - \frac{ngg}{2b}$, fiet pro hoc casu:

$$pp + qq = bb \text{ et } 2pq = bg = 0$$

ideoque $p = 0$ et $q = b$. Arcus scilicet pq abibit in quadrantem AB, vt natura rei postulat.

Problema soluendum.

46. In quadrante elliptico AB, arcum assignare pq, qui praecise sit semissis arcus quadrantis AB. Tab. IV. Fig. 2.

Solutio.

Ponantur ellipsis semiaxes $CA = a$, $CB = b$, sitque breuitatis gratia $\frac{bb-aa}{bb} = n$. Tum ad A ducatur tangens, in eamque ex punctis quaesitis p et q demissa concipiantur perpendicula pP et qQ, vocenturque $AP = p$ et $AQ = q$. Iam manifestum est, hoc problema esse casum praecedentis, quo punctum g in B assumitur, ita vt hoc sit $g = b$. Quo valore inducto formulae (§. 44.) praebebunt

$$pp + qq = \frac{1-\sqrt{(1-n)}}{4n}(5bb - 3bb\sqrt{(1-n)}) \text{ et}$$
$$2pq = bb\sqrt{\frac{1-\sqrt{(1-n)}}{n}}.$$

At ob $n = \frac{bb - aa}{bb}$ eſt $\surd(1 - n) = \frac{a}{b}$ et $\frac{1 - \surd(1-n)}{n} = \frac{b}{b+a}$ vnde fiet:

$$pp + qq = \frac{bb(5b + 3a)}{4(a+b)} \text{ et } 2pq = \frac{bb\surd b}{\surd(a+b)}.$$

hincque:

$$q + p = \tfrac{1}{2} b \surd \frac{5b + 3a + 4\surd b(a+b)}{a+b}$$

$$q - p = \tfrac{1}{2} b \surd \frac{5b + 3a - 4\surd b(a+b)}{a+b}$$

ideoque ipſae abſciſſae erunt:

$$AP = \tfrac{1}{4} b \surd \frac{5b + 3a + 4\surd b(a+b)}{a+b} - \tfrac{1}{4} b \surd \frac{5b + 3a - 4\surd b(a+b)}{a+b}$$

$$AQ = \tfrac{1}{4} b \surd \frac{5b + 3a + 4\surd b(a+b)}{a+b} + \tfrac{1}{4} b \surd \frac{5b + 3a - 4\surd b(a+b)}{a+b}$$

qui ambo valores geometrice per circinum et regulam conſtrui poſſunt.

Haecque eſt ſolutio adaequata problematis in Actis Erud. Lipſienſibus propoſiti.

Coroll. 1.

47. Si diſtantiae binorum punctorum p et q a centro ellipſis deſiderentur, notetur poſita $AP = p$ fore $Cp = \surd(aa + npp)$, atque hinc colligitur fore

$$Cp = \frac{\surd(5aa - 2ab + 5bb + (a-b)\surd(9aa + 14ab + 9bb))}{2\surd 2}$$

$$Cq = \frac{(5aa - 2ab + 5bb + (b-a)\surd(9aa + 14ab + 9bb))}{2\surd 2}.$$

Coroll. 2.

48. Ambae abſciſſae p et q etiam hoc modo ad conſtructionem fortaſſe aptius exprimi poſſunt, vt ſit:

$$AP = p = \frac{b\surd(5b + 3a - \surd(9aa + 14ab + 9bb))}{2\surd 2(a+b)}$$

$$AQ = q = \frac{b\surd(5b + 3a + \surd(9aa + 14ab + 9bb))}{2\surd 2(a+b)}.$$

Coroll.

Coroll. 3.

49. Si ad puncta p et q tangentes ducantur ad occursum axis CA, magnitudo harum tangentium commode exprimitur. Reperietur enim

$$Tp = \frac{\sqrt{(9aa + 14ab + 9bb)} - 3a - b}{4}$$

pro puncto autem q erit eadem tangens $= \frac{\sqrt{(9aa + 14ab + 9bb)} + 3a + b}{4}$.

Coroll. 4.

50. Concipiatur tangens Tp ad alterum vsque axem CB continuata, et concursus littera Θ notari, eritque permutatis literis a et b:

$$\Theta p = \frac{\sqrt{(9aa + 14ab + 9bb)} + a + 3b}{4}$$

ideoque $\Theta p - Ap = a + b$.

Coroll. 5.

51. Solutio igitur huius problematis ad hanc quaestionem mere geometricam reducitur:

In quadrante elliptico AB *duo eiusmodi puncta* p *et* q *assignare, ita vt ad ea ductis tangentibus* $Tp\Theta$, $tq\theta$ *quoad axibus productis occurrant, sit pro vtroque* Tab. IV. Fig. 3.

$$\Theta p - Tp = CA + CB \text{ et } tq - \theta q = CA + CB$$

seu vt differentia partium vtriusque tangentis aequalis sit semisummae axium principalium.

Hoc problemate constructo, puncta p et q simul ita sunt comparata, vt arcus interceptus pq ad totum quadrantem AB rationem teneat subduplam.

Scholion.

52. Demonstrato nunc Theoremate, solutoque Problemate, quae in Actis Erud. Lipſ. extant propoſita, antequam huic inueſtigationi finem imponam, problema adhuc multo difficilius pertractabo, quo in ellipſi arcus aſſignari iubetur, qui totius perimetri ellipſeos ſit triens. Quoniam enim facillime arcus aſſignatur, qui totius perimetri ſit ſemiſſis, vel quadrans, vel ope problematis praecedentis etiam octans, haud parum notatu dignus videtur caſus, quo triens poſtulatur, cuius ſolutio, etiamſi ob ſummam facilitatem, qua res de ſemiſſi et quadrante expeditur, non admodum difficilis videatur, tamen ad inueſtigationes perquam prolixas et operoſas deducitur, quas ſuperare tentabo.

Problema 7.

Tab. IV. Fig. 1. 53. Datum ellipſis arcum Ab, ad alterum axem principalem in A terminatum, ita ſecare in duobus punctis f et g, vt trium partium Af, fg et gb binae quaeuis quantitate geometrice aſſignabili diſcrepent.

Solutio.

Ex punctis f, g, b ad rectam AD, quae ellipſin in A tangit, demiſſis perpendiculis vocentur abſciſſae:

$$AF = f;\ AG = g;\ \text{et } AH = b$$

quarum haec $AH = b$ datur, illas vero duas f et g determinari oportet. Cum autem arcuum Af et fg differentia geometrica eſſe debeat, erit ex praecedentibus:

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4-nf^4}$$

$$\text{et } Af - fg = \frac{nffg}{bb}.$$

Deinde

Deinde quia arcuum Af et gh differentia debet esse geometrica, erit per formulas superiores:

$$g = \frac{bbh\sqrt{(bb-ff)(bb-nff)} - bbf\sqrt{(bb-hh)(bb-nhh)}}{b^4 - nffhh}$$

et $Af - gh = \frac{nfgh}{bb}$.

Tum igitur quoque tertia differentia erit

$$fg - gh = \frac{nfg}{bb}(h-f).$$

Quodsi iam ambo hi valores ipsius g inter se aequentur, obtinebitur aequatio inter f et h, per quam propterea abscissa f determinabitur, qua inuenta porro abscissa g innotescit.

Coroll. 1.

54. Aequatis autem duobus valoribus ipsius g, eruetur:

$$(b^4h - nf^4h - 2b^4f + 2nf^3hh)\sqrt{(bb-ff)(bb-nff)}$$
$$= (b^4f - nf^5)\sqrt{(bb-hh)(bb-nhh)}$$

quae, sumtis vtrinque quadratis, ad duodecimum gradum ascendit.

Coroll. 2.

55. Si sit $h = b$, seu arcus Ah in B terminetur, habebitur ista aequatio resoluenda:

$$b^5 - nbf^4 - 2b^4f + 2nbbf^3 = 0$$

seu $nf^4 - 2nbf^3 + 2b^3f - b^4 = 0$.

Problema 8.

56. In ellipsi arcum pq assignare, qui sit tertia pars totius perimetri ellipsis. Tab. IV. Fig. 4.

 Solutio.

Solutio.

Positis semiaxibus $CA = a$, $CB = b$, et breuitatis ergo $n = \frac{bb - aa}{bb}$, diuidatur primo tota peripheria ellipsis ita in punctis f et g, vt partium ABf, $f\alpha g$, $g\beta A$ differentiae sint geometrice assignabiles. Statuantur his punctis f et g abscissae respondentes $AF = f$ et $AG = -g$ quatenus haec in plagam oppositam cadit. Problema igitur praecedens ad hunc casum accommodabitur, si ob punctum h in A incidens ponatur $h = 0$ et $\sqrt{(bb - hh)} = + b$, quo facto habebimus:

$$g = \frac{2bbf\sqrt{(bb - ff)}(bb - nff)}{b^4 - nf^4} \text{ et } g = -f$$

sicque erit $AG = AF = f$: et ternae partes ellipsis ita different, vt sit:

$$f\alpha g - ABf = \frac{nf^3}{bb} \text{ et } ABf - A\beta g = 0.$$

Cum autem sit $g = -f$ erit:

$$2bbf\sqrt{(bb - ff)}(bb - nff) = -(b^4 - nf^4)f$$

vnde quadratis sumtis elicitur:

$$nnf^8 - 6nb^4f^4 + 4(n+1)b^6ff - 3b^8 = 0.$$

Ad hanc aequationem resoluendam fingantur eius factores:

$$(nf^4 + Pff + Q)(nf^4 - Pff + R) = 0$$

esseque oportet:

$$-6nb^4 = n(Q+R) - PP; \quad 4(n+1)b^6 = P(R-Q); \quad -3b^8 = QR$$

ex quibus fit:

$$R + Q = \frac{PP - 6nb^4}{n}; \quad R - Q = \frac{4(n+1)b^6}{P}$$

vnde

vnde valores ipſarum Q et R in poſtrema aequatione ſubſtituta praebent:

$$P^6 - 12nb^4P^4 + 48nnb^8P^2 = 16nn(n+1)^2b^{12}$$

vbi commode euenit, vt ſubtrahendo vtrinque $64n^3b^{12}$ cubus relinquatur, cuius radice extracta fiet:

$$PP - 4nb^4 = 2b^4\sqrt[3]{2nn(1-n)^2}$$

et $P = bb\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})}$

Quo valore ſubſtituto, reperietur:

$$R + Q = \frac{-2b^4(n - \sqrt[3]{2nn(1-n)^2})}{n}$$

$$R - Q = \frac{2b^4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4})}}{n}$$

Deinde vero ipſa reſolutio ſuppeditat:

$$ff = \frac{-P \pm \sqrt{(PP - 4nQ)}}{2n} \text{ et } ff = \frac{+P \pm \sqrt{(PP - 4nR)}}{2n}$$

vnde, ſubſtitutis valoribus inuentis, obtinebitur:

$$\frac{2nff}{bb} = -\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})} \pm \sqrt{(8n - 2\sqrt[3]{2nn(1-n)^2}} \\ + 4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4}))}$$

$$\frac{2nff}{bb} = +\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})} \pm \sqrt{(8n - 2\sqrt[3]{2nn(1-n)^2}} \\ - 4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4}))}$$

ex his autem quaternis valoribus alii locum habere nequeunt, niſi qui ff praebeant poſitiuum et minus quam bb.

Inuento iam valore idoneo pro f, pro punctis quaeſitis p et q ponantur abſciſſae $AP = p$ et $AQ = q$, ac ſtatuatur:

$$0 = b^4(pp + qq - ff) - 2bbpq\sqrt{(bb - ff)(bb - nff)} - nffppqq$$

eritque

eritque $Af - pq = \frac{nfpq}{bb}$; hincque

$$3Af - 3pq = \frac{3nfpq}{bb}.$$ Supra autem habebamus

$$fg - Af = \frac{nf^2}{bb}$$

$$Ag - Af = 0$$

quae tres aequationes additae dant:

$$Af + fg + gA - 3pq = \frac{3nfpq + nf^2}{bb}.$$

Quare vt arcus pq praecise sit triens totius peripheriae, necesse est, vt sit $3pq = ff$, seu $pq = -\frac{1}{3}ff$, vnde fit:

$$pp + qq = ff - \frac{2ff}{3bb}\sqrt{(bb-ff)(bb-nff)} + \frac{nf^6}{9b^4}$$

hincque porro:

$$qq \pm 2pq + pp = ff + \frac{2}{3}ff - \frac{2ff}{3bb}\sqrt{(bb-ff)(bb-nff)} + \frac{nf^6}{9b^4}$$

Fiet ergo:

$$q - p = \frac{f}{3bb}\sqrt{(15b^4 + nf^4 - 6bb\sqrt{(bb-ff)(bb-nff)})}$$

$$q + p = \frac{f}{3bb}\sqrt{(3b^4 + nf^4 - 6bb\sqrt{(bb-ff)(bb-nff)})}.$$

Quia rectangulum $pq = -\frac{1}{3}ff$ est negatiuum, patet binarum abscissarum p et q alteram esse positiuam, alteram negatiuam. Cum autem singulis abscissis bina curvae puncta respondeant, vtrum conueniat ex valoribus $\sqrt{(bb-pp)}$ et $\sqrt{(bb-qq)}$ siue sint positiui, siue negatiui, dignoscitur. Eorum autem signa ita comparata esse oportet, vt satisfiat huic formulae.

$$\sqrt{(bb-qq)} = \frac{b^3\sqrt{(bb-ff)(bb-pp)} - bfp\sqrt{(bb-nff)(bb-npp)}}{b^4 - nffpp}.$$

Casus $n = \frac{1}{2}$

57. Prae ceteris hic casus $n = \frac{1}{2}$, seu $bb = 2aa$, est notatu dignus, quod hoc solo radicale cubicum rationale

rationale euadit. Erit ſcilicet $\sqrt[3]{2nn(1-n)^2} = \frac{1}{2}$, et $P = bb\sqrt{3}$; vnde $R + Q = 0$ et $R - Q = 2b^4\sqrt{3}$; ideoque $Q = -b^4\sqrt{3}$, et $R = +b^4\sqrt{3}$. Cum iam ſit

$$ff = -P \pm \sqrt{(PP - 2Q)} \quad \text{et} \quad ff = +P \pm \sqrt{(PP - 2R)}$$

erit

$$\frac{ff}{bb} = -\sqrt{3} \pm (3 + 2\sqrt{3}) \quad \text{et} \quad \frac{ff}{bb} = +\sqrt{3} \pm \sqrt{(3 - 2\sqrt{3})}$$

Horum quatuor valorum bini poſteriores ſunt imaginarii, priorum vero ſolus poſitiuus locum habet, ita vt ſit:

$$ff = bb(-\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}), \text{ quia hinc } ff < bb.$$

Cum porro punctum f ſupra axem ellipſis CB exiſtat, erit

$$\sqrt{(bb - ff)} = -b\sqrt{(1 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})})} \quad \text{et}$$

$$\sqrt{(bb - nff)} = \frac{b}{\sqrt{2}}\sqrt{(2 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})})} \quad \text{vnde}$$

$$\sqrt{(bb - ff)(bb - nff)} = \frac{-bb}{\sqrt{2}}\sqrt{(8 + 5\sqrt{3} - (3 + 2\sqrt{3})\sqrt{(3 + 2\sqrt{3})})}$$

ſiue

$$\sqrt{(bb - ff)(bb - nff)} = -\tfrac{1}{2}bb(\sqrt{(9 + 6\sqrt{3})} - 2 - \sqrt{3}).$$

Cum nunc ſit $ff = bb(\sqrt{(3 + 2\sqrt{3})} - \sqrt{3})$, erit

$$2pq = -\tfrac{2}{3}bb(\sqrt{(3 + 2\sqrt{3})} - \sqrt{3}) \quad \text{et}$$

$$pp + qq = +\tfrac{2}{3}bb(3 - \tfrac{1}{3}\sqrt{(9 + 6\sqrt{3})})$$

ex quibus ſit

$$(q + p)^2 = \tfrac{2}{3}bb(+3 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})} - \tfrac{1}{3}\sqrt{(9 + 6\sqrt{3})})$$

$$(q - p)^2 = \tfrac{2}{3}bb(+3 - \sqrt{3} + \sqrt{(3 + 2\sqrt{3})} - \tfrac{1}{3}\sqrt{(9 + 6\sqrt{3})})$$

et radicibus extractis

$$q + p = \tfrac{1}{3}b\sqrt{(3 + \sqrt{3})(6 - 2\sqrt{(3 + 2\sqrt{3})})}$$

$$q - p = \tfrac{1}{3}b\sqrt{(3 - \sqrt{3})(6 + 2\sqrt{(3 + 2\sqrt{3})})}$$

 Hinc

Hinc in fractionibus decimalibus erit

$$ff = 0,8104090\,bb; \qquad f = 0,9002272\,b$$

$$\sqrt{(bb-ff)} = -0,4354205\,b; \quad \sqrt{(bb-nff)} = +0,7712300\,b$$

$$2pq = -0,5402727\,bb; \quad (q+p)^2 = 0,4811342\,bb$$

$$pp+qq = +1,0214069\,bb; \quad (q-p)^2 = 1,5616796\,bb$$

$$q+p = 0,6936383\,b; \qquad p = 0,9716548\,b$$

$$p-q = 1,2496712\,b; \qquad q = -0,2780165\,b$$

quos valores pro p et q figura propemodum refert: atque ex formula $\sqrt{(bb-pp)}$ et $\sqrt{(bb-qq)}$ inuoluente intelligitur, punctum p infra axem βB, punctum q vero supra eum capi debere.

DE

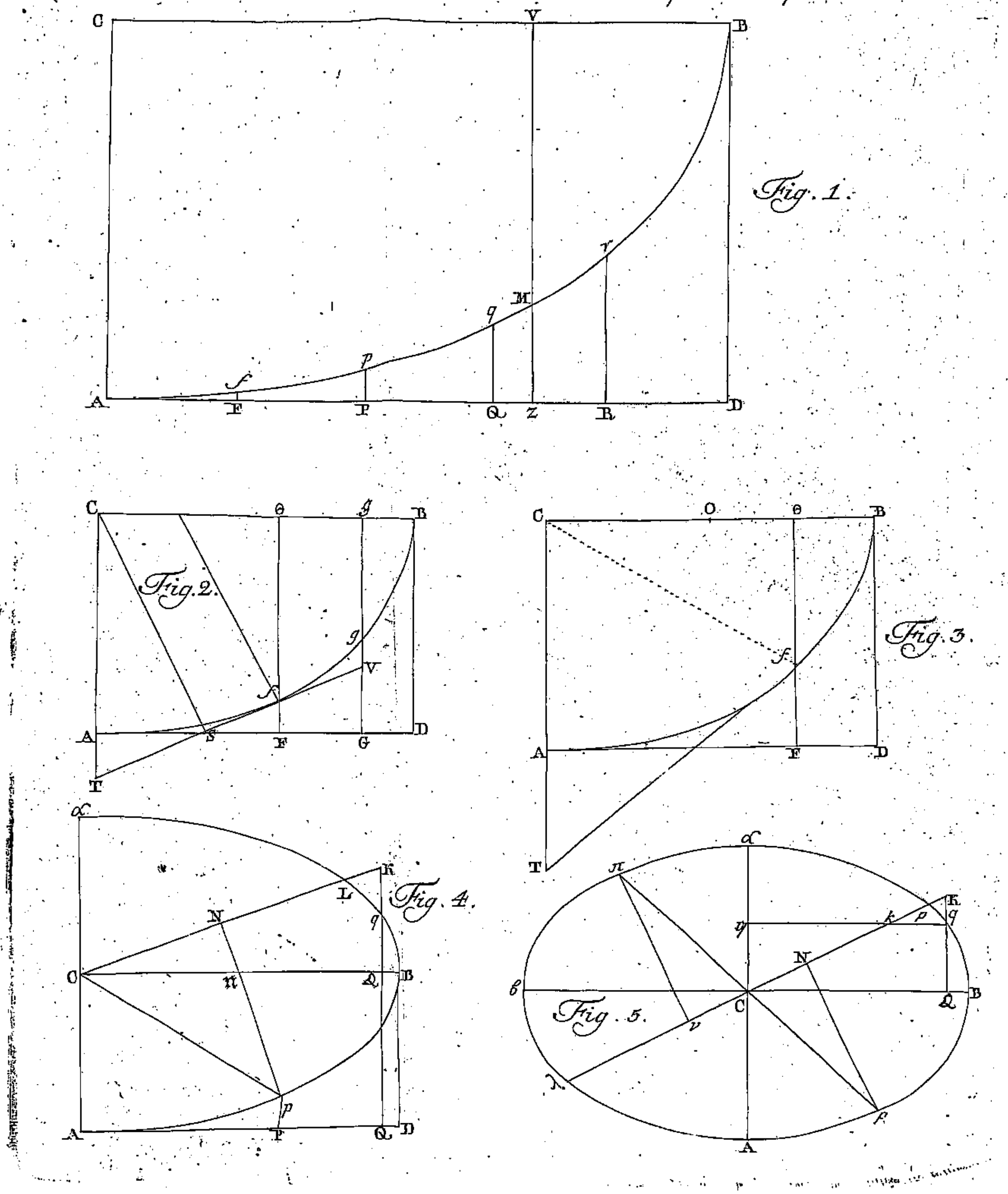
Fig. 1.
C
V
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Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.

Fig. 1.

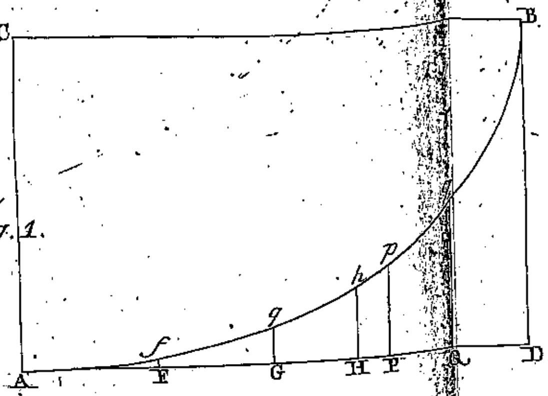

Fig. 2.

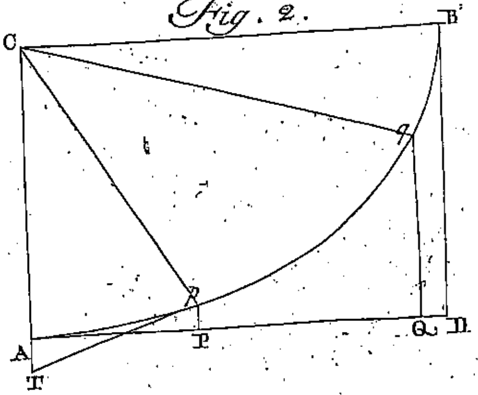

Fig. 3.

Fig. 4.

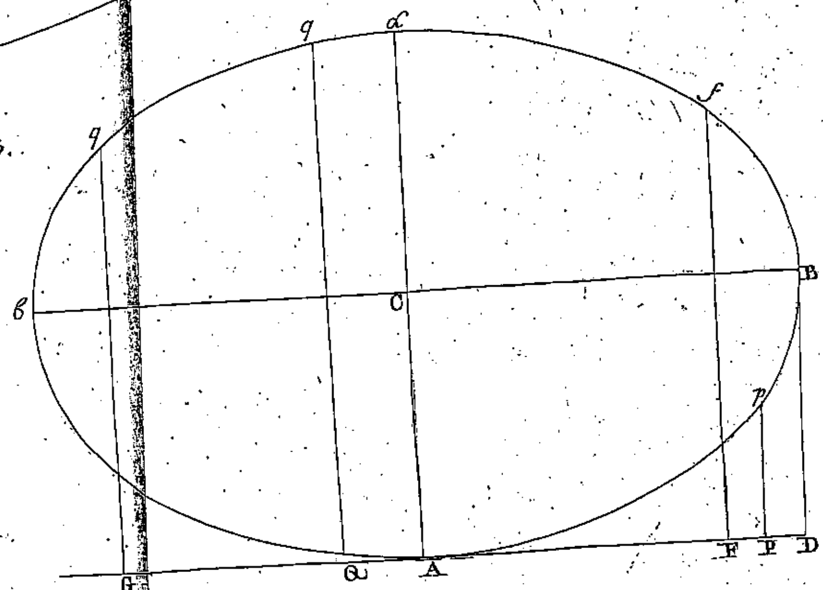

Fig. 5