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Specimen novae methodi curvarum quadraturas et rectificationes aliasque quantitates transcendentes inter se comparandi

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SPECIMEN
NOVAE METHODI
CVRVARVM QVADRATVRAS ET RECTIFICATIONES ALIASQVE QVANTITATES TRANSCENDENTES INTER SE COMPARANDI.

Auctore

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Quae nuper occaſione inuentorum Ill. Comitis Fagnani commentatus ſum de comparatione arcuum ellipſis hyperbolae et curuae lemniſcatae, multo latius mihi quidem patere ſtatim ſunt viſa. Cum enim methodis adhuc conſuetis eiusmodi tantum curuarum arcus inter ſe comparari poſſent, quarum rectificatio vel a quadratura circuli, vel a logarithmis penderet, quippe quae quantitates, etſi ſunt tranſcendentes, tamen ita iam in Analyſi prae ceteris ius quoddam ciuitatis ſunt adeptae, vt perinde atque algebraicae tractari queant: maxima certe attentione erat dignum, quod a Fagnano in hyperbola et ellipſi arcus ſint aſſignati, quorum differentia ſit algebraica; in lemniſcata autem eiusmodi arcus, qui adeo inter ſe ſint aequales, vel certam teneant rationem, propterea quod harum curuarum rectificatio neque ad quadraturam circuli, neque ad logarithmos reduci queat. Hinc certe Theoriae quantitatum transcendentium insigne lumen accenderetur, si modo via, qua Fagnanus est vsus, certam methodum suppeditaret in huiusmodi inuestigationibus vlterius progrediendi; sed quia tota substitutionibus precario factis et quasi casu fortuito adhibitis nititur, parum inde vtilitatis in Analysin redundat. Deinde iam notaui integrationes, quas operatio Fagnaniana complectitur, tantum esse particulares, neque idcirco methodum certam, a qua plura expectari liceat, suppeditare. Interim tamen ea amplissimum campum aperuisse est visa, in quo vlterius excolendo Geometrae vires suas summo cum fructu exerceant, ad insigne Analyseos incrementum.

Res autem huc redit, vt propositis duabus formulis integralibus $\int X dx$ et $\int Y dy$, non integrabilibus, vbi X sit functio quaepiam ipsius x, et Y ipsius y, eiusmodi relatio inter variabiles x et y definiatur, vt illae formulae vel inter se fiant aequales, vel datam rationem teneant, vel vt differentiam algebraice assignabilem obtineant. Quae inuestigatio cum latissime pateat, tum etiam insignes in se continet casus iam pridem non sine maximo Analyseos incremento euolutos; huc enim referenda sunt, quae de comparatione arcuum circularium, de lunulis quadrabilibus, de zonis cycloidalibus quadrabilibus; tum vero arcubus parabolicis, qui vel datam inter se teneant rationem vel differentiam algebraicam habeant, a geometris sunt tradita: quin etiam haec inuestigatio a Cel. *Ioh. Bernoulli* ad parabolas cubicales altiorisque ordinis est extensa, sed quia ratio, qua vsus est, nulla certa methodo nitebatur, vlteriori vsu fere penitus caruit. Huc quoque pertinet,

quod

quod multo ante iam acutiſſimus Hugenius in Horologio oſcillatorio expoſuerat, vbi propoſito ſphaeroide elliptico compreſſo, ſeu reuolutione circa axem minorem genito, inuenire docuit conoides hyperbolicum, ita vt ſumma vtriusque ſuperficiei circulo exhiberi poſſet; cum tamen neutra ſuperficies ſeorſim cum circulo comparari queat. Quae inuentio iam tum ſummis geometris maxime memorabilis viſa eſt; atque *Bernoullius* in litteris ad *Leibnizium* datis dolet, hanc inuentionem nulla certa methodo inniti, ex qua plura huius generis inuenta deriuare liceat; interim quia ſuperficies tam illius ſphaeroidis elliptici, quam conoidis hyperbolici a logarithmis pendet, reductio vtriusque iunctim ſumtae ad circulum, ſimili modo perfici poteſt, quo in parabola arcus algebraicam habentes differentiam aſſignari ſolent. Inprimis autem hoc loco non eſt praetereundum, *Tſchirnhauſium* quondam methodum a ſe inuentam iactaſſe, cuius beneficio curuarum quarumcunque non rectificabilium arcus ita inter ſe comparare poſſet, vt differentia fiat algebraica, ſed praeter quam quod methodum ſuam nunquam aperuerit, manifeſtum eſt, eum paralogismo quodam fuiſſe deceptum, vt ſaepius alias, cum certum ſit, rem ita generaliter omnino expediri non poſſe: neque ergo *Tſchirnhauſius* putandus eſt quicquam eorum habuiſſe, quae vel tum circa comparationem curuarum ſunt inuenta, vel adhuc forte elicientur.

Specimen igitur quoddam methodi huiusmodi quaeſtiones ſoluendi hic exhibere conſtitui, quod non obſcure maiores progreſſus in hac re promittere videtur;

tur; atque cum non ſolum difficillimum ſit, propoſitis in genere eiusmodi formulis integralibus, quaeſitam inter variabiles relationem eruere, ſed etiam hoc saepiſſime omnino ne fieri quidem poſſit; ordine inuerſo rem ita tentaui, vt aſſumta binarum variabilium relatione, inde ipſas formulas integrales inueſtigarem, quae per hanc relationem inter ſe comparari poſſent. Quae methodus, cum facillime procedat, ad multo ſublimiora perducere poſſe videtur, quae aliis methodis plane ſint imperuia: hac enim methodo non ſolum ea, quae habet Fagnanus, facili negotio, ac ſine taedioſo calculo, ſum aſſecutus, ſed etiam multo ampliora atque illuſtriora reddidi, vt quae ille nimis particulariter definiuerat, ego ſatis vniuerſaliter expediuerim: atque calculus, quo ſum vſus, ita comparatus eſt, vt, quoniam operationes prorſus ſingulares complectitur, viam ad multo ſublimiora ſternere videatur.

Tum vero quanquam variabilium mutua relatio per methodos conſuetas definiri poteſt, quoties integratio vtriusque formulae $\int X\,dx$ et $\int Y\,dy$, vel a quadratura circuli, vel a logarithmis pendet; tamen et hoc plerumque non ſine moleſto calculo perficitur, dum partes, vel arcus circulares, vel logarithmos continentes, ſe mutuo deſtruere debent: quemadmodum hoc in comparatione arcuum parabolicorum abunde perſpicitur. Per meam autem methodum hae difficultates cunctae penitus euaneſcunt, ac fere ſine vllo negotio iſtae comparationes, tam in circulo, quam in parabola, abſoluuntur: in quo ſine dubio non exigua vis huius methodi ſita eſſe cenſenda eſt, quod non ſolum multo facilius ea, quae

quae aliis methodis iam ſunt eruta, praebeat; ſed etiam ad eiusmodi inueſtigationes manuducat, in quibus aliae methodi nihil eſſent praeſtiturae. Quam ob rem hoc quidem loco iſtam methodum tantum ad eos caſus applicabo, qui etiam aliis methodis, ſed multo operoſius, expediri ſolent, quo cum principia, quibus innititur, hac occaſione expoſuero, deinceps facilius eius applicationem ad quaeſtiones ſublimiores ſuſcipere poſſim. Quoniam igitur mihi a relatione inter binas variabiles, quam pro lubitu conſtituo, ordiendum eſt, a ſimplicioribus incipiam, ac primo quidem ab eiusmodi, quae ad ſimiles formulas integrales perducant, ſeu in quibus X et Y ſimiles ſint prodituræ functiones ipſarum x et y. Formulae ergo integrales hinc natae ob ſimilitudinem quantitates tranſcendentes exhibebunt, ad eandem lineam curuam pertinentes, deinceps autem ad formulas quoque diſſimiles, quae ad diuerſas curuas pertineant, ſum progreſſurus.

RELATIO PRIMA
inter binas variabiles x et y.

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy.$$

1. Si hinc ſeorſim valores x et y extrahantur, reperietur:

$$y = \frac{-\delta x \pm \sqrt{((\delta\delta - \gamma\gamma)xx - \alpha\gamma)}}{\gamma}$$

$$x = \frac{-\delta y \pm \sqrt{((\delta\delta - \gamma\gamma)yy - \alpha\gamma)}}{\gamma}$$

vbi quouis caſu diſpiciendum eſt, vtrum ſignum quantitati radicali ſit praefigendum? Fieri enim poteſt, vt in vtraque formula, vel ſigna paria, vel diſparia, locum habeant,

beant, dum alterutrum arbitrio noſtro plane relinquitur: in quo iudicio inprimis natura variabilium x et y, vtrum affirmatiue, an negatiue accipiantur, ſpectari debet.

2. Ponantur breuitatis gratia membra irrationalia:

$$\pm\sqrt{((\delta\delta-\gamma\gamma)xx-\alpha\gamma)}=P, \text{ et } \pm\sqrt{((\delta\delta-\gamma\gamma)yy-\alpha\gamma)}=Q$$

vt ſit:

$$y=\frac{-\delta x+P}{\gamma}, \text{ et } x=\frac{-\delta y+Q}{\gamma}.$$

ſicque erit:

$$P=\gamma y+\delta x, \text{ et } Q=\gamma x+\delta y$$

vnde quouis caſu facile colligere licet, vtrum quantitates P et Q habiturae ſint valores affirmatiuos, an negatiuos.

3. Differentietur iam aequatio aſſumta, eritque:

$$dx(\gamma x+\delta y)+dy(\gamma y+\delta x)=0$$

atque ob $\gamma x+\delta y=Q$, et $\gamma y+\delta x=P$, habebitur haec aequatio:

$$Qdx+Pdy=0, \text{ ſiue } \frac{dx}{P}+\frac{dy}{Q}=0.$$

Reſtitutis ergo pro P et Q valoribus, huic aequationi integrali:

$$\int\frac{dx}{\sqrt{((\delta\delta-\gamma\gamma)xx-\alpha\gamma)}}+\int\frac{dy}{\sqrt{((\delta\delta-\gamma\gamma)yy-\alpha\gamma)}}=\text{Conſt.}$$

ſatisfacit relatio inter variabiles x et y aſſumta.

4. Euoluamus haec accuratius, et quo facilius applicatio fieri queat, ponamus:

$$-\alpha\gamma=Ap \quad \text{et} \quad \delta\delta-\gamma\gamma=Cp$$

vt ſit:

$$\int\frac{dx}{\sqrt{(A+Cxx)}}+\int\frac{dy}{\sqrt{(A+Cyy)}}=\text{Conſt.}$$

eritque

eritque $\alpha = -\frac{Ap}{\gamma}$ et $\delta = \sqrt{(Cp + \gamma\gamma)}$;
sicque quantitates p et γ arbitrio nostro relinquuntur.

5. Statuatur ergo $\gamma = A$, et $p = Akk$, ita vt k sit noua quantitas constans, a nostro arbitrio pendens; eritque

$$\alpha = -Akk, \ \gamma = A, \text{ et } \delta = \sqrt{A(A + Ckk)}$$

et aequatio canonica, nostrae aequationi integrali satisfaciens, erit:

$$0 = -Akk + A(xx + yy) + 2xy\sqrt{A(A + Ckk)}$$

seu $y = \frac{-x\sqrt{(A + Ckk)} + k\sqrt{(A + Cxx)}}{\sqrt{A}}$

et $x = \frac{-y\sqrt{(A + Ckk)} + k\sqrt{(A + Cyy)}}{\sqrt{A}}$.

6. Si $\sqrt{(A + Cyy)}$ negatiue capiatur, itemque $\sqrt{A}$, tum huius aequationis differentialis

$$\frac{dx}{\sqrt{(A + Cxx)}} = \frac{dy}{\sqrt{(A + Cyy)}}$$

integralis erit:

$$0 = -Akk + A(xx + yy) - 2xy\sqrt{A(A + Ckk)},$$

ideoque

vel $y = \frac{x\sqrt{(A + Ckk)} - k\sqrt{(A + Cxx)}}{\sqrt{A}}$

vel $x = \frac{y\sqrt{(A + Ckk)} + k\sqrt{(A + Cyy)}}{\sqrt{A}}$.

7. Quia ergo aequatio integralis constantem in se complectitur k, quae in differentiali non inest, indicio hoc est, integralem esse completam; sicque differentiali nulla alia satisfacit integralis, nisi quae in forma inuenta comprehendatur. Atque haec est integratio principalis, ad quam relatio inter x et y assumta perducit.

 8. Hinc

8. Hinc autem deriuari possunt innumerabiles aliae integrationes. Si enim sint X et Y eiusmodi functiones ipsarum x et y, vt vi relationis assumtae fit $X = Y$, eadem relatio satisfaciet quoque huic aequationi differentiali:

$$\frac{X\,dx}{\sqrt{(A+Cxx)}} = \frac{Y\,dy}{\sqrt{(A+Cyy)}}$$

Infinitis autem modis huiusmodi functiones aequales exhiberi possunt ex formulis pro x et y inuentis.

9. Quo autem haec inuestigatio latius pateat, et X et Y sint functiones similes, eas non assumo inter se aequales, eiusmodi autem pro iis valores indago, vt sit:

$$\frac{X\,dx}{\sqrt{(A+Cxx)}} - \frac{Y\,dy}{\sqrt{(A+Cyy)}} = dV$$

atque quantitas V prodeat algebraica, si scilicet relatio §. 6. tradita locum habeat.

10. Cum igitur sit $\frac{dy}{\sqrt{(A+Cyy)}} = \frac{dx}{\sqrt{(A+Cxx)}}$, erit

$$\frac{(X-Y)\,dx}{\sqrt{(A+Cxx)}} = dV$$

et ob $P = k\sqrt{A(A+Cxx)} = \gamma y + \delta x = Ay + x\sqrt{A(A+Ckk)}$ sumto per §. 6. $\sqrt{A}$ negatiuo, erit $\sqrt{(A+Cxx)} = \frac{x}{k}\sqrt{(A+Ckk)} - \frac{y}{k}\sqrt{A}$, vnde fiet:

$$\frac{(X-Y)k\,dx}{x\sqrt{(A+Ckk)} - y\sqrt{A}} = dV.$$

11. Cum sit porro ex aequatione differentiata

$$dx(Ax - y\sqrt{A(A+Ckk)}) = dy(x\sqrt{A(A+Ckk)} - Ay)$$

ponatur $xy = u$, erit $dy = \frac{du}{x} - \frac{y\,dx}{x}$, quo valore substituto fiet

$$dx\left(Ax - \frac{Ayy}{x}\right) = \frac{du}{x}(x\sqrt{A(A+Cxx)} - Ay)$$

seu

seu $\frac{dx}{x\sqrt{(A+Cxx)}-y\sqrt{A}}=\frac{du}{(xx-yy)\sqrt{A}}$;
sicque erit: $dV=\frac{k\,du}{\sqrt{A}}\cdot\frac{X-Y}{xx-yy}$.

12. Quoties ergo $\frac{X-Y}{xx+yy}$ eiusmodi functio ipsius u, quae ducta in du fiat integrabilis, toties valor quantitatis V algebraice exhiberi poterit: hoc autem euenit, quoties X et Y fuerint potestates parium exponentium ipsarum x et y, propterea cum sit ex aequatione assumta

$$xx+yy=kk+\tfrac{2u}{A}\sqrt{A(A+Ckk)}$$

13. Ponatur ergo $X=x^n$ et $Y=y^n$; erit posito $n=2$; $\frac{X-Y}{xx-yy}=1$; et $dV=\frac{k\,du}{\sqrt{A}}$ ideoque $V=\frac{ku}{\sqrt{A}}+\text{Const.}=\frac{kxy}{\sqrt{A}}+\text{Const.}$
Quam ob rem habebitur:

$$\int\frac{xx\,dx}{\sqrt{(A+Cxx)}}-\int\frac{yy\,dy}{\sqrt{(A+Cyy)}}=\text{Const.}+\frac{kxy}{\sqrt{A}}.$$

14. Sit iam $n=4$; eritque $\frac{X-Y}{xx-yy}=xx+yy$ $=kk+\frac{2u}{A}\sqrt{A(A+Ckk)}$
vnde $dV=\frac{k\,du}{A}(kk\sqrt{A}+2u\sqrt{(A+Ckk)})$
ergo $V=\frac{ku}{A}(kk\sqrt{A}+u\sqrt{(A+Ckk)})$.
Hoc igitur casu erit:

$$\int\frac{x^4\,dx}{\sqrt{(A+Cxx)}}-\int\frac{y^4\,dy}{\sqrt{(A+Cyy)}}=\text{Const.}+\frac{kxy}{A}(kk\sqrt{A}+xy\sqrt{(A+Ckk)})$$

similique modo vlterius progredi licet.

15. His igitur coniungendis si fuerit

$$xx+yy=kk+2xy\sqrt{(1+\tfrac{C}{A}kk)},\ \text{siue}$$
$$y=\frac{x\sqrt{(A+Ckk)}-k\sqrt{(A+Cxx)}}{\sqrt{A}}$$
$$x=\frac{y\sqrt{(A+Ckk)}+k\sqrt{(A+Cyy)}}{\sqrt{A}}$$

haec relatio inter x et y ſatisfaciet huic aequationi integrali:

$$\int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4)}{\sqrt{(A+Cxx)}} - \int \frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4)}{\sqrt{(A+Cyy)}} = \text{Conſt.}$$
$$+\frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{\sqrt{A}}(kk+xy\sqrt{(1+\tfrac{C}{A}kk)})$$

ſeu differentia iſtarum formularum integralium algebraice aſſignari poteſt.

RELATIO SECVNDA
inter binas variabiles x et y
$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy$

16. Quoniam, vti in praecedentibus deprehendimus, ambiguitas ſignorum radicalium ab arbitrio noſtro pendet, dummode eius ratio in concluſionibus finalibus debite habeatur, ſi ad differentiam binarum formularum integralium peruenire velimus, extrahendo radices, habebimus:

$$y=\frac{-\beta-\delta x-\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx)}}{\gamma}$$
$$x=\frac{-\beta-\delta y+\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy)}}{\gamma}.$$

17. Statuamus breuitatis gratia has formulas irrationales:

$$\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx)}=P$$
$$\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy)}=Q$$

eritque

$$-P=\beta+\gamma y+\delta x \quad\text{et}\quad Q=\beta+\gamma x+\delta y$$

vnde eliciuntur iſtae relationes

$$P+Q=(\gamma-\delta)(x-y)$$
$$\gamma P+\delta Q=\beta(\delta-\gamma)+(\delta\delta-\gamma\gamma)y$$
$$\delta P+\gamma Q=\beta(\gamma-\delta)-(\delta\delta-\gamma\gamma)x.$$

18. Ac-

18. Aequatio autem propofita differentiata dat:

$$dx(\beta+\gamma x+\delta y)+dy(\beta+\gamma y+\delta x)=0$$

fiue $Q\,dx-P\,dy=0$

vnde oritur:

$$\frac{dx}{P}=\frac{dy}{Q} \text{ feu } \int\frac{dx}{P}-\int\frac{dy}{Q}=\text{Conft.}$$

cui ergo aequationi integrali fatisfacit relatio propofita, indeque valores pro x et y extracti.

19. Vt hinc fimili modo alias integrationes obtineamus, fint iterum X et Y functiones fimiles ipfarum x et y; ac pofito:

$$\frac{X\,dy}{P}-\frac{Y\,dy}{Q}=dV$$

definiantur hae functiones ita, vt V prodeat quantitas algebraica, ficque habeatur:

$$\int\frac{X\,dx}{P}-\int\frac{Y\,dy}{Q}=V+\text{Conft.}$$

20. Cum igitur fit $\frac{dy}{Q}=\frac{dx}{P}$, erit $dV=\frac{(X-Y)dx}{P}$, feu $dV=\frac{-dx(X-Y)}{\beta+\gamma y+\delta x}$. Sit iterum $xy=u$, ideoque $dy=\frac{du}{x}-\frac{y\,dx}{x}$, erit pro aequatione differentiali

$$dx(\beta+\gamma x+\delta y)+\frac{du}{x}(\beta+\gamma y+\delta x)-\frac{y\,dx}{x}(\beta+\gamma y+\delta x)=0$$

feu $dx(\beta x-\beta y+\gamma xx-\gamma yy)+du(\beta+\gamma y+\delta x)=0$.

21. Valore hinc pro dx fubftituto habebitur:

$$dV=\frac{du(X-Y)}{(x-y)(\beta+\gamma(x+y))}$$

Ponatur autem vlterius $x+y=t$; erit $xx+yy=tt-2u$ et quia aequatio affumta in hanc formam abibit:

$$0=\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u$$

ex qua differentiando fit $dt(\beta+\gamma t)=(\gamma-\delta)du$, feu $\frac{du}{\beta+\gamma t}=\frac{dt}{\gamma-\delta}$.

22. Hinc

22. Hinc igitur simpliciori modo obtinetur

$$dV = \frac{dt(X - Y)}{(\gamma - \delta)(x - y)}$$

vnde patet, si X et Y fuerint potestates ipsarum x et y, tum fractionem $\frac{X-Y}{x-y}$ per t et u, ideoque et per solum t, ob $u = \frac{+\alpha + 2\beta t + \gamma tt}{2(\gamma - \delta)}$, commode exprimi posse.

23. Sit ergo $X = x^n$, et $Y = y^n$; ac ponatur primo $n = 1$, erit $\frac{X-Y}{x-y} = 1$, et $dV = \frac{dt}{\gamma - \delta}$; vnde fit $V = \frac{t}{\gamma - \delta}$. Quocirca pro hoc casu erit

$$\int \frac{x\,dx}{P} - \int \frac{y\,dy}{Q} = \text{Const.} + \frac{(x+y)}{\gamma - \delta}.$$

cui ergo aequationi integrali satisfit per relationem inter x et y assumtam.

24. Sit $n = 2$; eritque $\frac{X-Y}{x-y} = x + y = t$; vnde fit

$$dV = \frac{t\,dt}{\gamma - \delta} \text{ et } V = \frac{tt}{2(\gamma - \delta)} = \frac{(x+y)^2}{2(\gamma - \delta)}$$

Hoc ergo casu habebitur:

$$\int \frac{xx\,dx}{P} - \int \frac{yy\,dy}{Q} = \text{Const.} + \frac{(x+y)^2}{2(\gamma - \delta)}.$$

25. Si vlterius progredi lubeat, ponatur $n = 3$, eritque:

$$\frac{x^3 - y^3}{x - y} = xx + xy + yy = tt - u = \frac{(\gamma - 2\delta)tt - 2\beta t - \alpha}{2(\gamma - \delta)}$$

et $V = \frac{\frac{1}{3}(\gamma - 2\delta)t^3 - \beta tt - \alpha t}{2(\gamma - \delta)^2}$; sicque erit

$$\int \frac{x^3\,dx}{P} - \int \frac{y^3\,dy}{Q} = \text{Const.} + \frac{(\gamma - 2\delta)(x+y)^3 - 3\beta(x+y)^2 - 3\alpha(x+y)}{6(\gamma - \delta)^2}$$

26. His igitur formulis coniungendis, sequenti aequationi integrali

$\int dx$

$$\int \frac{dx(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx+\mathfrak{D}x^3)}{\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx)}} - \int \frac{dy(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}yy+\mathfrak{D}y^3)}{\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy)}}$$
$$= \text{Const.} + \frac{\mathfrak{B}(x+y)}{\gamma-\delta} + \frac{\mathfrak{B}(x+y)^2}{2(\gamma-\delta)} + \frac{\mathfrak{D}((\gamma-2\delta)(x+y)^3-3\beta(x+y)^2-3\alpha(x+y))}{6(\gamma-\delta)^2}$$

satisfacit relatio assumta

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy$$

indeque valores pro x et y initio eruti.

27. Quo applicatio ad casus particulares facilius fieri possit, ponamus $\beta\beta-\alpha\gamma = Ap$; $\beta(\delta-\gamma) = Bp$; et $\delta\delta-\gamma\gamma = Cp$; vt sit $P = \sqrt{p(A+2Bx+Cxx)}$ et $Q = \sqrt{p(A+2By+Cyy)}$ fiatque:

$$\gamma = A + Bk, \text{ et } \delta = \sqrt{A(A+2Bk+Ckk)}$$

erit $p = \frac{(AC-BB)kk}{C}$, et $\beta = \frac{B}{C}(\delta+\gamma)$ atque $\alpha = \frac{2BB}{CC}(\gamma+\delta) - \frac{(AC-BB)^2kk}{CC(A+Bk)}$.

RELATIO TERTIA

inter binas variabiles x et y

$$0 = \alpha + mxx + nyy + 2\delta xy$$

28. Extrahendo vtramque radicem habebitur

$$y = \frac{-\delta x + \sqrt{((\delta\delta-mn)xx-\alpha n)}}{n}$$
$$x = \frac{-\delta y - \sqrt{((\delta\delta-mn)yy-\alpha m)}}{m}$$

hinc posito:

$$P = \sqrt{(\delta\delta-mn)xx-\alpha n)} \text{ et } Q = \sqrt{((\delta\delta-mn)yy-\alpha m)}$$

erit $P = \delta x + ny$ et $-Q = \delta y + mx$.

29. Per differentiationem vero obtinemus;

$$dx(mx+\delta y) + dy(ny+\delta x) = 0$$

seu $-Q dx + P dy = 0$, ideoque $\frac{dy}{Q} = \frac{dx}{P}$

vnde

vnde aequatio assumta huic aequationi integrali

$$\int \frac{dy}{Q} = \int \frac{dx}{P} \text{ satisfacit.}$$

30. Sint iam X et Y functiones ipsarum x et y singulatim, ac ponatur

$$\int \frac{X\,dx}{P} - \int \frac{Y\,dy}{Q} = V$$

ita vt fiat V quantitas algebraica: eritque

$$\frac{(X-Y)\,dx}{P} = d\,V \; \frac{+(X-Y)\,dx}{\delta x + ny}.$$

31. Posito $xy = u$, vt sit $dy = \frac{du}{x} - \frac{y\,dx}{x}$, erit

$$dx(mxx - nyy) + du(ny + \delta x) = 0$$

vnde, cum fiat $\frac{dx}{\delta x + ny} = \frac{-du}{mxx - nyy}$, erit

$$dV = \frac{-du(X-Y)}{mxx - nyy},$$

hincque non difficulter casus integrabiles eliciuntur.

32. Sit enim primo $X = mxx$, et $Y = nyy$, erit

$$dV = -du, \text{ et } V = -u = -xy$$

Hinc relatio inter x et y assumta satisfacit huic aequationi integrali:

$$\int \frac{mxx\,dx}{P} - \int \frac{nyy\,dy}{Q} = \text{Const.} - xy.$$

33. Sit secundo $X = mmx^4$, et $Y = nny^4$, erit

$$dV = -du(mxx + nyy) = +du(\alpha + 2\delta u)$$

vnde fit $V = u(\alpha + \delta u) = xy(\alpha + \delta xy)$

Ergo huic aequationi integrali

$$\int \frac{mmx^4\,dx}{P} - \int \frac{nny^4\,dy}{Q} = \text{Const.} + xy(\alpha + \delta xy)$$

satisfacit relatio assumta inter x et y.

34. His

34. His igitur colligendis relatio inter x et y assumta satisfaciet huic aequationi integrali:

$$\int \frac{dx(\mathfrak{A}+\mathfrak{B}mxx+\mathfrak{C}m^2x^4)}{\sqrt{((\delta\delta-mn)xx-\alpha n)}} - \int \frac{dy(\mathfrak{A}+\mathfrak{B}myy+\mathfrak{C}m^2y^4)}{\sqrt{((\delta\delta-mn)yy-\alpha m)}}$$
$$= \text{Const.} - \mathfrak{B}xy + \mathfrak{C}xy(\alpha+\delta xy).$$

35. Ponamus ad faciliorem applicationem:

$$\delta\delta - mn = Cp; \quad \alpha n = -Ap \quad \text{et} \quad \alpha m = -Bp,$$

vt sit $P = \sqrt{p}(A + Cxx)$, et $Q = \sqrt{p}(B + Cyy)$, erit $\frac{m}{n} = \frac{B}{A}$. Sit ergo $m = B$, et $n = A$, erit $\alpha = -p$, et $\delta = \sqrt{(AB + Cp)}$. Sit ergo $p = Ckk$, vt sit $\alpha = -Ckk$, et aequatio relationem inter x et y definiens erit:

$$o = -Ckk + Bxx + Ayy + 2xy\sqrt{(AB + CCkk)}$$

36. Quam ob rem valores ipsius x et y hinc erunt:

$$y = \frac{-x\sqrt{(AB + CCkk)} + k\sqrt{C(A + Cxx)}}{A}$$
$$x = \frac{-y\sqrt{(AB + CCkk)} - k\sqrt{C(B + Cyy)}}{B}$$

existente:

$$P = k\sqrt{C}(A + Cxx) \quad \text{et} \quad Q = k\sqrt{C}(B + Cyy).$$

37. Hi igitur valores conueniunt huic aequationi integrali:

$$\int \frac{dx(\mathfrak{A}+\mathfrak{B}Bxx+\mathfrak{C}B^2x^4)}{\sqrt{(A+Cxx)}} - \int \frac{dy(\mathfrak{A}+\mathfrak{B}Ayy+\mathfrak{C}A^2y^4)}{\sqrt{(B+Cyy)}}$$
$$= \text{Const.} - \mathfrak{B}kxy\sqrt{C} + \mathfrak{C}kxy(-Ckk + xy\sqrt{(AB+C^2kk)})\sqrt{C}.$$

38. Ponatur $B = \frac{CE}{F}$, quae aequatio latius patere videatur, atque, constantibus mutatis, prodibit ista aequatio integralis:

$$\int \frac{dx(\mathfrak{A}+\frac{C}{A}\mathfrak{B}xx+\frac{CC}{AA}\mathfrak{C}x^4)\sqrt{C}}{\sqrt{(A+Cxx)}} - \int \frac{dy(\mathfrak{A}+\frac{F}{E}\mathfrak{B}yy+\frac{FF}{EE}\mathfrak{C}y^4)\sqrt{F}}{\sqrt{(E+Fyy)}}$$
$$= \text{Const.} - \frac{CF}{AE}\mathfrak{B}kxy - \frac{CCFF}{AAEE}\mathfrak{C}k^3xy + \frac{CCFF}{AAEE}\mathfrak{C}kxxyy\sqrt{(\tfrac{AE}{CF}+kk)}$$

 cui

cui ſatisfaciunt iſti valores:

$$\tfrac{Ay}{C} = k\sqrt{(\tfrac{A}{C}+xx)} - x\sqrt{(\tfrac{AE}{CF}+kk)}$$

$$\tfrac{Ex}{F} = -k\sqrt{(\tfrac{E}{F}+yy)} - y\sqrt{(\tfrac{AE}{CF}+kk)}$$

qui oriuntur ex hac aequatione:

$$kk = \tfrac{E}{F}xx + \tfrac{A}{C}yy + 2xy\sqrt{(\tfrac{AE}{CF}+kk)}.$$

39. Hae formulae ratione ſignorum vtcunque transmutari poſſunt. Primo enim in formulis integralibus nihil mutando, tam k, quam $\sqrt{(\tfrac{AE}{CF}+kk)}$ pro lubitu vel affirmatiue, vel negatiue, accipi poſſunt, dummodo eadem ſigni ratio vbique obſeruetur. Deinde etiam tam $\sqrt{C}$, quam $\sqrt{F}$, negatiue ſumi poteſt; illo autem caſu quoque $\sqrt{(\tfrac{A}{C}+xx)}$, quippe $\frac{\sqrt{(A+Cxx)}}{\sqrt{C}}$, hoc vero $\sqrt{(\tfrac{E}{F}+yy)}$ negatiue eſt accipiendum.

40. Denique patet, ſi C ſit quantitas poſitiua, tum quoque F quantitatem poſitiuam eſſe oportere, quia alioquin altera formula integralis fieret imaginaria. Sin autem C ſit quantitas negatiua, tum etiam F talis ſit neceſſe eſt; et quo hoc caſu imaginaria ſe deſtruant, pro kk quantitas negatiua accipienda erit; quo k et k^3 fiant quoque imaginariae.

41. Hoc ergo caſu ſequens habebitur aequatio integralis:

$$\int\frac{dx(\mathfrak{A}+\frac{C}{A}\mathfrak{B}xx+\frac{CC}{AA}\mathfrak{C}x^4)\sqrt{C}}{\sqrt{(A-Cxx)}} - \int\frac{dy(\mathfrak{A}+\frac{F}{E}\mathfrak{B}yy+\frac{FF}{EE}\mathfrak{C}y^4)\sqrt{F}}{\sqrt{(E-Fyy)}}$$

$$= \text{Conſt.} + \tfrac{CF}{AE}\mathfrak{B}kxy + \tfrac{CCFF}{AAEE}\mathfrak{C}k^3xy + \tfrac{CCFF}{AAEE}\mathfrak{C}kxxyy\sqrt{(\tfrac{AE}{CF}-kk)}$$

cui

cui ſatisfaciunt iſti valores:

$$\frac{Ay}{C} = x\sqrt{\left(\frac{AE}{CF} - kk\right)} - k\sqrt{\left(\frac{A}{C} - xx\right)}$$

$$\frac{Ex}{F} = y\sqrt{\left(\frac{AE}{CF} - kk\right)} + k\sqrt{\left(\frac{E}{F} - yy\right)}$$

ex hac aequatione oriundi:

$$kk = \frac{E}{F}xx + \frac{A}{C}yy - 2xy\sqrt{\left(\frac{AE}{CF} - kk\right)}.$$

42. Hae formulae etiam eas, quae ex hypotheſi prima ſunt erutae, in ſe complectuntur, ponendo ſcilicet $E = A$, et $F = C$, quin etiam formulae ſecundae hypotheſis his non latius patent. Si enim in relatione ſecundo loco aſſumta pro $x + \frac{\beta}{\gamma + \delta}$ et $y + \frac{\beta}{\gamma + \delta}$ ſcribatur x et y, aequatio omnino primae formae oritur, ſimilique modo ſi hanc relationem conſtituere velimus:

$$0 = \alpha + 2bx + 2\beta y + \gamma xx + cyy + 2\delta xy$$

ea facile ad relationem tertiam reduceretur, vnde eius euolutionem praetermitto.

43. Perſpicuum nunc eſt, ex his formulis infinitas comparationes inſtitui poſſe circa quantitates tranſcendentes, tam ratione ſpatiorum, quam arcuum, qui quidem vel a quadratura circuli pendent, vel a logarithmis. Etſi autem hae comparationes etiam vulgari calculo inſtitui poſſunt, tamen non inutile erit oſtendere, quemadmodum eaedem multo facilius ex his formulis deriuari queant; quod eo maius notatu dignum videtur, cum hic neque naturae circuli, neque logarithmorum, ratio peculiaris habeatur. Ex quo facilius intelligetur, quemadmodum haec methodus etiam pari

ſucceſſu ad eiusmodi formulas integrales ſe extendat, quae neque ad circuli, neque hyperbolae quadraturam reuocari poſſunt.

I.

De comparatione arcuum circularium.

44. Sit radius circuli, ſeu ſinus totus $=1$, ac poſito ſinu quocunque $=z$, ſit arcus ei reſpondens $=\Pi.z$, ſumto Π pro nota eius functionis, qua pendentia arcus a ſuo ſinu denotatur. Erit ergo, vti conſtat, $\Pi.z=\int\frac{dz}{\sqrt{(1-zz)}}$; atque vt formulas integrales §. 41. erutas huc transferamus, poni oportet: $A=E=C=F=1$; $\mathfrak{A}=1$, $\mathfrak{B}=0$, et $\mathfrak{C}=0$.

45. Ex his autem valoribus emerget haec aequatio integralia complectens:

$$\int\frac{dx}{\sqrt{(1-xx)}}-\int\frac{dy}{\sqrt{(1-yy)}}=\text{Conſt.}$$

cui ſatisfacere inuentae ſunt hae formulae:

$$y=x\sqrt{(1-kk)}-k\sqrt{(1-xx)}$$

$$x=y\sqrt{(1-kk)}+k\sqrt{(1-yy)}$$

quae oriuntur ex hac aequatione:

$$kk=xx+yy-2xy\sqrt{(1-kk)}.$$

46. Per has igitur determinationes ſatisfit huic aequationi:

$$\Pi.x-\Pi.y=\text{Conſt.}$$

in qua conſtans ita determinabitur: ponatur $y=0$ eritque $x=k$; ex quo caſu prodit $\Pi.k-\Pi.0=\text{Conſt.}$ ſeu

seu ob $\Pi.0 = 0$ erit Const. $= \Pi\, k$, seu arcui cuius sinus $= k$. Hinc generatim habebimus:

$$\Pi.x - \Pi.y = \Pi.k.$$

47. Hinc ergo statim arcuum tam additio, quam subtractio colligitur. Si enim duo habeantur arcus $\Pi.k$ et $\Pi.y$, quorum sinus sint k et y, et summae arcuum sinus ponatur $= x$, vt sit $\Pi\, x = \Pi.k + \Pi\, y$, erit $x = y\sqrt{(1-kk)} + k\sqrt{(1-yy)}$. Porro si maioris arcus sinus sit $= x$, minoris $= k$, sinusque differentiae ponatur $= y$, vt sit $\Pi.y = \Pi.x - \Pi.k$, erit:

$$y = x\sqrt{(1-kk)} - k\sqrt{(1-xx)};$$

vti ex elementis est manifestum.

48. Perspicuum etiam est, quemadmodum hinc arcuum multiplicationem deduci oporteat. Posito enim $y = k$, vt sit $x = 2k\sqrt{(1-kk)}$, erit $\Pi\, x = 2\,\Pi.k$. Ac si valor hic pro x inuentus loco y substituatur, in formula $x = y\sqrt{(1-kk)} + k\sqrt{(1-yy)}$, ob $\Pi.y = 2\,\Pi.k$, prodibit $\Pi.x = 3\,\Pi.k$.

49. In genere autem, si sit y sinus arcus nk, seu $\Pi.y = n\Pi.k$, et $\sqrt{(1-yy)}$ sit cosinus arcus nk, vti $\sqrt{(1-kk)}$ denotat cosinum arcus k, atque ponatur

$$x = y\sqrt{(1-kk)} + k\sqrt{(1-yy)} \text{ erit } \Pi.x = (n+1)\,\Pi.k.$$

Ex sinu ergo cuiusuis multipli arcus k reperietur sinus multipli vnitate altioris.

50. Quo autem haec facilius expediri queant, valorem quoque ipsius cosinus $\sqrt{(1-xx)}$ nosse conueniet; quem in finem cum ex formula prima sit:

$$k\sqrt{(1-xx)} = x\sqrt{(1-kk)} - y$$

 substi-

ſubſtituatur hic valor ipſius x ex altera formula erit $k\sqrt{(1-xx)}=y(1-kk)+k\sqrt{(1-kk)(1-yy)})-y$, ideoque

$$\sqrt{(1-xx)}=\sqrt{(1-kk)(1-yy)}-ky$$

ſimilique modo erit:

$$\sqrt{(1-yy)}=\sqrt{(1-kk)(1-xx)}+kx.$$

51. Inuentis ergo valoribus, tam pro x, quam pro $\sqrt{(1-xx)}$, multiplicetur ille per λ et productum ad hunc addatur, eritque

$$\sqrt{(1-xx)}+\lambda x=\sqrt{(1-kk)(1-yy)}-ky+\lambda y\sqrt{(1-kk)}+\lambda k\sqrt{(1-yy)}$$

ſeu $\sqrt{(1-xx)}+\lambda x=(\sqrt{(1-kk)}+\lambda k)\sqrt{(1-yy)}+y(\lambda\sqrt{(1-kk)}-k)$ Quo igitur hi factores ſimiles reddantur, neceſſe eſt, vt ſit $\lambda=\sqrt{-1}$, eritque:

$$\sqrt{(1-xx)}+x\sqrt{-1}=(\sqrt{(1-kk)}+k\sqrt{-1})(\sqrt{(1-yy)}+y\sqrt{-1}).$$

52. Hanc ergo formulam loco ſuperioris adhibendo, ſtatim patet, vt ſit $\Pi.x=2\,\Pi.k$, ob $y=k$, eſſe oportere

$$\sqrt{(1-xx)}+x\sqrt{-1}=(\sqrt{(1-kk)}+k\sqrt{-1})^2$$

Ac ſi hic valor pro x inuentus loco y ſubſtituatur, vt ſit

$\Pi.y=2\,\Pi.k$, prodibit:

$$\sqrt{(1-xx)}+x\sqrt{-1}=(\sqrt{(1-kk)}+k\sqrt{-1})^3 \text{ pro } \Pi.x=3\,\Pi.k$$

vnde in genere colligitur, vt ſit $\Pi.x=n\,\Pi.k$, debere eſſe:

$$\sqrt{(1-xx)}+x\sqrt{-1}=(\sqrt{(1-kk)}+k\sqrt{-1})^n.$$

53. Quia porro $\sqrt{-1}$ ob ſuam naturam tam negatiue, quam affirmatiue accipere licet, erit quoque pro eadem arcus multiplicatione: $\Pi.x=n\,\Pi.k$

$$\sqrt{(1-xx)}-x\sqrt{-1}=(\sqrt{(1-kk)}-k\sqrt{-1})^n$$

ideo-

ideoque vel

$$\sqrt{(1-xx)} = \frac{(\sqrt{(1-kk)}+k\sqrt{-1})^n+(\sqrt{(1-kk)}-k\sqrt{-1})^n}{2}$$

vel $x = \frac{(\sqrt{(1-kk)}+k\sqrt{-1})^n-(\sqrt{(1-kk)}-k\sqrt{-1})^n}{2\sqrt{-1}}$

quae formulae quoque valent pro valoribus fractis exponentis n.

II.

De Comparatione arcuum parabolicorum.

54. Sit AB axis et A vertex parabolae, quem tangat recta indefinita AV, super qua capiantur abscissae; posito ergo parabolae latere recto $=2$, sit abscissa quaeuis $AP = z$, erit applicata $Pp = \frac{1}{2}zz$, ex quo arcus parabolae huic abscissae respondens erit $Ap = \int dz \sqrt{(1+zz)}$: qui cum sit functio ipsius z, denotetur per $\Pi.z$, ita vt $\Pi.z$ significet arcum parabolae abscissae z conuenientem, seu sit Tab II. Fig. 1.

$$\Pi.z = \int dz \sqrt{(1+zz)}.$$

55 Irrationalitate in denominatorem translata erit: $\Pi.z = \int \frac{dz(1+zz)}{\sqrt{(1+zz)}}$. Ad hanc ergo formam vt formulae integrales §. 38. reuocentur, erit $A = E = 1$; $C = F = 1$, $\mathfrak{A} = 1$ et $\mathfrak{B} = 1$ atque $\mathfrak{C} = 0$. Vnde aequatio illa integralis in hanc abit formam:

$$\int \frac{dx(1+xx)}{\sqrt{(1+xx)}} - \int \frac{dy(1+yy)}{\sqrt{(1+yy)}} = \text{Const.} + kxy$$

cui satisfaciunt hi valores:

$$y = -k\sqrt{(1+xx)} + x\sqrt{(1+kk)} \quad \text{et} \quad x = k\sqrt{(1+yy)} + y\sqrt{(1+kk)}$$

sumtis tam k quam $\sqrt{(1+kk)}$ negatiuis.

56. Hac

56. Hac igitur inter x et y relatione subsistente pro arcubus parabolae erit:

$$\Pi . x - \Pi . y = \text{Const.} + kxy$$

ad quam constantem determinandam ponatur $y = 0$, et quia tum fit $x = k$, erit $\Pi . k = \text{Const.}$ Quocirca habebitur

$$\Pi . x - \Pi . y = \Pi . k + kxy.$$

57. Vt igitur haec aequatio locum habeat, relatio inter ternas abscissas k, x, et y eiusmodi erit:

$$x = k\surd(1+yy) + y\surd(1+kk), \text{ seu } y = x\surd(1+kk) - k\surd(1+xx)$$

vnde praeterea eruuntur istae determinationes:

$$\surd(1+xx) = \surd(1+kk)(1+yy) + ky \text{ et } \surd(1+yy) = \surd(1+kk)(1+xx) - kx$$

ex quibus porro elicitur:

$$x + \surd(1+xx) = (k + \surd(1+kk))(y + \surd(1+yy)).$$

58. Si manente cadem abscissa k, capiantur aliae duae abscissae q et p, vt sit

$$q = k\surd(1+pp) + p\surd(1+kk) \text{ et } p = q\surd(1+kk) - k\surd(1+qq)$$

seu $q + \surd(1+qq) = (k + \surd(1+kk))(p + \surd(1+pp))$

erit $\Pi . q - \Pi . p = \Pi . k + kpq$.

Ideoque hanc aequationem ab illa subtrahendo habebitur:

$$(\Pi . x - \Pi . y) - (\Pi\ q - \Pi\ p) = k(xy - pq).$$

59. Pro hoc igitur casu erit

$$\frac{x + \surd(1+xx)}{y + \surd(1+yy)} = \frac{q + \surd(1+qq)}{p + \surd(1+pp)}$$

vnde relatio inter p, q, x et y siue k obtinetur: Erit autem

$$k = x\surd(1+yy) - y\surd(1+xx) = q\surd(1+pp) - p\surd(1+qq) \text{ et}$$

$$\surd(1+kk) = \surd(1+xx)(1+yy) - xy = \surd(1+pp)(1+qq) - pq$$

60. Iam

60. Iam ob $\frac{1}{p+\surd(1+pp)} = \surd(1+pp)-p$ erit:
$$\surd(1+xx)+x = (\surd(1+yy)+y)(\surd(1+qq)+q)(\surd(1+pp)-p)$$
vnde reperitur:
$$x = y\surd(1+pp)(1+qq)+q\surd(1+pp)(1+yy)-p\surd(1+qq)(1+yy)-pqy$$
Quare erit:
$$(\Pi.x-\Pi.y)-(\Pi.q-\Pi.p) = (q\surd(1+pp)-p\surd(1+qq))(y\surd(1+pp)-p\surd(1+yy))(q\surd(1+yy)+y\surd(1+qq)).$$

Problema 1.

61. Dato arcu parabolae quocunque Ak, in vertice A terminato, ab alio quocunque puncto p arcum abſcindere pq, qui arcum illum Ak ſuperet quantitate algebraice aſſignabili. Tab. II. Fig. 1.

Solutio.

Poſita parabolae parametro $=2$, ſit k abſciſſa arcui Ak conueniens, abſciſſae autem punctis p et q reſpondentes ſint AP$=y$ et AQ$=x$; eritque arc.pq $=\Pi.x-\Pi.y$ et arc. A$k=\Pi.k$; cum igitur data ſit abſciſſa AP$=y$, ſi capiatur altera
$$AQ = x = y\surd(1+kk)+k\surd(1+yy)$$
erit $\Pi.x-\Pi.y=\Pi.k+kxy$, ideoque
$$\text{Arc.}\,pq = \text{Arc.}\,Ak + kxy.$$
Superabit ergo arcus pq, qui in dato puncto p terminatur, arcum Ak quantitate algebraice aſſignabili kxy.

Poterit etiam a puncto p antrorſum abſcindi arcus pq', qui pariter arcum Ak quantitate geometrica ſuperet; ad hoc ponatur $AP = x$ et $AQ^{\prime} = y$, sitque $y = x\sqrt{(1+kk)} - k\sqrt{(1+xx)}$; et cum sit Arc. $pq^{\prime} = \Pi.x - \Pi.y$, erit:

$$\text{Arc.} pq^{\prime} = \text{Arc. } Ak + kxy$$

Vtraque igitur solutio ita coniungetur, vt posita abscissa data $AP = p$; capiendum sit:

$AQ = p\sqrt{(1+kk)} + k\sqrt{(1+pp)}$, et $AQ^{\prime} = p\sqrt{(1+kk)} - k\sqrt{(1+pp)}$ quo facto erit:

$$\text{Arc.} pq = \text{Arc. } Ak + kp.AQ$$

$$\text{Arc.} pq^{\prime} = \text{Arc. } Ak + kp.AQ^{\prime}$$

sicque duplici modo problemati est satisfactum.

Coroll. 1.

62. Fieri autem nequit, vt excessus kxy, quo arcus pq arcum Ak superat, euanescat, deberet enim esse vel $x = 0$, vel $y = 0$. At casu $x = 0$ fieret $y = -k$, arcusque in ipso vertice A inciperet in altero ramo ipsi arcui Ak similis capiendus; altero autem casu, quo $y = 0$, fieret $x = k$, et arcus pq in arcum Ak abiret: vnde arcui Ak geometrice in parabola abscindi nequit alius arcus ipsi aequalis, qui ipsi non simul futurus sit similis.

Coroll. 2.

63. Vicissim ergo dato arcu quocunque pq in parabola, semper a vertice arcus abscindi poterit Ak, qui ab illo deficiat quantitate geometrica. Cum enim nunc datae sint abscissae $AP = y$ et $AQ = x$. erit $AK = k = x\sqrt{(1+yy)} - y\sqrt{(1+xx)}$, qua inuenta, erit Arc. pq - Arc. $Ak = kxy$.

Coroll.

Coroll. 3.

64. Quin etiam puncto p pro incognito habito, proposito arcu Ak, alius arcus pq assignari poterit, qui illum superet quantitate data, puta $= C$. Habebimus ergo has duas aequationes:

$$kxy = C \text{ et } xx + yy = kk + 2xy\sqrt{(1+kk)}$$

seu $xx + yy = kk + \frac{2C}{k}\sqrt{(1+kk)}$; ergo

$$x + y = \sqrt{(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)})}$$

$$x - y = \sqrt{(kk - \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)})}$$

Seu sint x et y binae radices huius aequationis quadraticae

$$zz - Pz + Q = 0; \text{ erit } Q = \frac{C}{k} \text{ et } P = \sqrt{(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)})}$$

vnde $z = \frac{1}{2}\sqrt{(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)})} \pm \frac{1}{2}\sqrt{(kk - \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)})}$.

Coroll. 4.

65. Quantacunque sit haec quantitas C, modo sit affirmatiua, semper prodeunt pro x et y valores reales, iique affirmatiui: At si sit $C = 0$, fiet $x = k$, et $y = 0$. Quin etiam poni potest C negatiuum, quo casu y reperitur quoque negatiuum, et arcus quaesitus vtrinque circa verticem A erit dispositus. Verum si sit $C = -D$, necesse est, vt sit $D < \frac{k^3}{2(1+\sqrt{(1+kk)})}$, seu $D < \frac{1}{2}k(\sqrt{(1+kk)} - 1)$; nam si D esset maius, vtraque abscissa fieret imaginaria.

Coroll. 5.

66. Casu autem $D = -C = \frac{1}{2}k(\sqrt{(1+kk)} - 1)$, erit $zz = \frac{D}{k}$; ideoque $x = +\sqrt{\frac{1}{2}(\sqrt{(1+kk)} - 1)}$

et $y = -\sqrt{\tfrac{1}{2}}(\sqrt{(1+kk)}-1)$; hocque casu orietur arcus vtrinque a vertice aeque extensus, cuius defectus ab arcu A k est minimus omnium, qui quidem geometrice construi possunt.

Problema 2.

Tab. II. Fig. 2. 67. Dato arcu parabolae quocunque ef, a dato eius puncto quocunque p alium abscindere arcum pq, ita vt arcuum ef et pq differentia geometrice possit assignari.

Solutio.

Posito parabolae latere recto $= 2$, tanget recta AV parabolam in vertice A, a quo capiantur abscissae, quae sint:

$$AE = e;\ AF = f;\ AP = p \text{ et } AQ = q$$

quarum tres priores e, f, p, sunt datae, haec vera q ita accipiatur, vt sit per §. 59.

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}$$

Tum vero sit $k = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$, scribendo e et f pro y et x, eritque $(\Pi.q - \Pi.p) - (\Pi.f - \Pi.e) = k(pq - ef)$.

Ideoque habebitur:

$$\text{Arc.}pq - \text{Arc.}ef = k(pq - ef).$$

Hinc etiam apparet, si punctum q fuerit datum, ex formula tradita simili modo punctum p antrorsum procedendo definiri posse, vt arcuum differentia prodeat geometrice assignabilis.

Coroll.

Coroll. 1.

68. Ex reductione §. 60. facta patet esse

$$pq - eff = (p\sqrt{(1+ee)} - e\sqrt{(1+pp)})(p\sqrt{(1+ff)} + f\sqrt{(1+pp)})$$

sicque, sumta abscissa q, ex aequatione $\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}$ erit :

$$\text{Arc.}\,pq - \text{Arc.}\,ef = (f\sqrt{(1+ee)} - e\sqrt{(1+ff)})(p\sqrt{(1+ee)} - e\sqrt{(1+pp)})(p\sqrt{(1+ff)} + f\sqrt{(1+pp)}).$$

Coroll. 2.

69. Si velimus punctum p ita accipere, vt arcuum differentia euanescat, seu fiat Arc. $pq =$ Arc ef, oportet esse

vel $p\sqrt{(1+ee)} - e\sqrt{(1+pp)} = 0$, vel $p\sqrt{(1+ff)} + f\sqrt{(1+pp)} = 0$

Priori casu fit $p = \pm e$; posteriori $p = \pm f$, vtroque autem casu arcus pq vel cum arcu ef congruit, vel eius fit similis in altero parabolae ramo assumtus; ita vt geometrice duo arcus aequales exhiberi nequeant, quae non simul sibi futuri sint similes.

Coroll. 3.

70. Cum sit $k = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$, erit $\sqrt{(1+kk)} = \sqrt{(1+ee)(1+ff)} - ef$; hinc $k\sqrt{(1+kk)}$
$= f\sqrt{(1+ff)} + 2eef\sqrt{(1+ff)} - 2eff\sqrt{(1+ee)}$
$- e\sqrt{(1+ff)}$

siue

$$k\sqrt{(1+kk)} = f\sqrt{(1+ff)} - e\sqrt{(1+ee)} - 2ef(f\sqrt{(1+ee)} - e\sqrt{(1+ff)})$$

ideoque $k\sqrt{(1+kk)} = f\sqrt{(1+ff)} - e\sqrt{(1+ee)} - 2efk.$

 Quo

Quo circa habebitur:

$$kef = \tfrac{1}{2}f\sqrt{(1+ff)} - \tfrac{1}{2}e\sqrt{(1+ee)} - \tfrac{1}{2}k\sqrt{(1+kk)}.$$

Coroll. 4.

71. Quia igitur k simili quoque modo pendet a p et q, erit etiam

$$kpq = \tfrac{1}{2}q\sqrt{(1+qq)} - \tfrac{1}{2}p\sqrt{(1+pp)} - \tfrac{1}{2}k\sqrt{(1+kk)}$$

Quare cum arcuum differentia sit $= kpq - kef$; si quatuor parabolae puncta e, f, p, q ita a se inuicem pendent, vt sit:

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}$$

erit

$$\text{Arc.}\,pq - \text{Arc.}\,ef = \tfrac{1}{2}q\sqrt{(1+qq)} - \tfrac{1}{2}p\sqrt{(1+pp)} - \tfrac{1}{2}f\sqrt{(1+ff)} + \tfrac{1}{2}e\sqrt{(1+ee)}$$

quae expressio, ob functiones quantitatum p, q, e, f a se inuicem separatas, est notatu digna.

Coroll. 5.

72. Relatio inter e, f, p, q etiam ita exprimi potest, vt sit

$$\sqrt{(1+qq)}+q = (\sqrt{(1+ee)}-e)(\sqrt{(1+ff)}+f)(\sqrt{(1+pp)}+p)$$

tum ob $\frac{1}{\sqrt{(1+qq)}+q} = \sqrt{(1+qq)} - q$ erit:

$$\sqrt{(1+qq)}-q = (\sqrt{(1+ee)}+e)(\sqrt{(1+ff)}-f)(\sqrt{(1+pp)}-p)$$

vnde datis e, f, et p, facile valor tam pro q, quam pro p, eruitur.

Coroll. 6.

73. Ex formula Coroll. 1. data apparet, arcum pq semper maiorem fore arcu ef, si punctum p a vertice

vertice parabolae A magis fuerit remotum, quam punctum e; contra autem arcum pq prodituram esse minorem. Ac si quidem sit $p=0$, erit Arc. ef − Arc. pq $=ef(f\sqrt{(1+ee)}-e\sqrt{(1+ff)})$; minimus autem omnium arcus pq euadet, si capiatur $p=-\sqrt{\tfrac{1}{2}}(\sqrt{(1+ee)(1+ff)}-ef-1)$ et $q=+\sqrt{\tfrac{1}{2}}(\sqrt{(1+ee)(1+ff)}-ef-1)$ tumque erit:

$$\text{Arc.}\,ef-\text{Arc}\;pq=\tfrac{1}{2}(e+f)(\sqrt{(1+ff)}-\sqrt{(1+ee)})$$

Arcusque pq vtrinque aeque circa verticem A erit dispositus.

Problema 3.

74. Dato arcu parabolae ef, a puncto dato p abscindere arcum pz, qui superet datum multiplum arcus ef quantitate geometrice assignabili. Tab. II. Fig. 3.

Solutio.

Posito parabolae latere recto $=2$, sint in verticis tangente abscissae datae $AE=e$, $AF=f$, et $AP=p$; tum capiantur abscissae $AQ=q$; $AR=r$; $AS=s$; $AT=t$; et vltima sit $AZ=z$; quae ita determinentur, vt sit:

Primo $\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}}=\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}$

eritque ex §. 71.

$$\text{Arc}\;pq-\text{Arc.}\,ef=\tfrac{1}{2}q\sqrt{(1+qq)}-\tfrac{1}{2}p\sqrt{(1+pp)}-\tfrac{1}{2}f\sqrt{(1+ff)}+\tfrac{1}{2}e\sqrt{(1+ee)}.$$

Deinde ex puncto q simili modo definiatur punctum r, vt sit:

$$\frac{r+\sqrt{(1+rr)}}{q+\sqrt{(1+qq)}}=\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}},\quad\text{seu}\quad\frac{r+\sqrt{(1+rr)}}{p+\sqrt{(1+pp)}}=\left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^2$$

eritque

eritque

$$\text{Arc.}\,qr - \text{Arc.}\,ef = \tfrac{1}{2}r\surd(1+rr) - \tfrac{1}{2}q\surd(1+qq) - \tfrac{1}{2}f\surd(1+ff) + \tfrac{1}{2}e\surd(1+ee)$$

qua aequatione ad illam addita prodibit:

$$\text{Arc.}\,pr - 2\,\text{Arc.}\,ef = \tfrac{1}{2}r\surd(1+rr) - \tfrac{1}{2}p\surd(1+pp) - \tfrac{2}{2}f\surd(1+ff) + \tfrac{2}{2}e\surd(1+ee)$$

Tertio ex puncto r capiatur punctum s, vt sit:

$$\frac{s+\surd(1+ss)}{r+\surd(1+rr)} = \frac{f+\surd(1+ff)}{e+\surd(1+ee)},\ \text{seu}\ \frac{s+\surd(1+ss)}{p+\surd(1+pp)} = \left(\frac{f+\surd(1+ff)}{e+\surd(1+ee)}\right)^3$$

eritque:

$$\text{Arc.}\,rs - \text{Arc.}\,ef = \tfrac{1}{2}s\surd(1+ss) - \tfrac{1}{2}r\surd(1+rr) - \tfrac{1}{2}f\surd(1+ff) + \tfrac{1}{2}e\surd(1+ee)$$

quae ad praecedentem addita praebet:

$$\text{Arc.}\,ps - 3\,\text{Arc.}\,ef = \tfrac{1}{2}s\surd(1+ss) - \tfrac{1}{2}p\surd(1+pp) - \tfrac{3}{2}f\surd(1+ff) + \tfrac{3}{2}e\surd(1+ee).$$

Atque hoc modo si vlterius progrediamur, sitque z punctum post n huiusmodi operationes inuentum, erit:

$$\frac{z+\surd(1+zz)}{p+\surd(1+pp)} = \left(\frac{f+\surd(1+ff)}{e+\surd(1+ee)}\right)^n$$

vnde immediate punctum z reperietur, ita vt sit:

$$\text{Arc.}\,pz - n\,\text{Arc.}\,ef = \tfrac{1}{2}z\surd(1+zz) - \tfrac{1}{2}p\surd(1+pp) - \tfrac{n}{2}f\surd(1+ff) + \tfrac{n}{2}e\surd(1+ee)$$

sicque arcus pz est inuentus a dato puncto p abscissus, qui arcum ef vicibus n sumtum superat quantitate geometrica.

Coroll. I.

75. Quodcunque ergo multiplum arcus ef proponatur, cuius multipli exponens sit numerus n, siue is sit integer, siue fractus, a dato puncto p semper abscindi

ſcindi poterit arcus pz, qui hoc multiplum excedat quantitate geometrice aſſignabili; erit enim:

$$\sqrt{(1+zz)}+z=(\sqrt{(1+pp)}+p)(\sqrt{(1+ff)}+f)^n(\sqrt{(1+ee)}-e)^n \text{ et}$$

$$\sqrt{(1+zz)}-z=(\sqrt{(1+pp)}-p)(\sqrt{(1+ff)}-f)^n(\sqrt{(1+ee)}+e)^n.$$

Coroll. 2.

76. Quodſi ergo ad abbreuiandum ponatur:

$$\sqrt{(1+ee)}+e=E;\ \sqrt{(1+ff)}+f=F;\ \sqrt{(1+pp)}+p=P$$

erit $\sqrt{(1+zz)}+z=\frac{PF^n}{E^n}$ et $\sqrt{(1+zz)}-z=\frac{E^n}{PF^n}$ vnde oritur:

$$\sqrt{(1+zz)}=\frac{P^2F^{2n}+E^{2n}}{2PE^nF^n} \text{ et } z=\frac{P^2F^{2n}-E^{2n}}{2PE^nF^n}.$$

Coroll. 3.

77. Hinc ergo fiet $\frac{1}{2}z\sqrt{(1+zz)}=\frac{P^4F^{4n}-E^{4n}}{8P^2E^{2n}F^{2n}}$

Quia tum ſimili modo eſt

$$\tfrac{1}{2}e\sqrt{(1+ee)}=\frac{E^4-1}{8EE};\ \tfrac{1}{2}f\sqrt{(1+ff)}=\frac{F^4-1}{8FF}$$

$$\text{et } \tfrac{1}{2}p\sqrt{(1+pp)}=\frac{P^4-1}{8PP}$$

erit:

$$\text{Arc.}\,pz-n\,\text{Arc.}\,ef=\frac{P^4F^{4n}-E^{4n}}{8P^2E^{2n}F^{2n}}-\frac{P^4+1}{8PP}-\frac{n(F^4-1)}{8FF}+\frac{n(E^4-1)}{8EE}$$

Coroll. 4.

78. Si huius expreſſionis partes binae in vnam congregentur, reperietur iſta differentia geometrica:

$$\text{Arc.}\,pz - n\,\text{Arc.}\,ef = \frac{(F^{2n}-E^{2n})(P^4F^{2n}+E^{2n})}{8P^2E^{2n}F^{2n}} - \frac{n(FF-EE)(EEFF+1)}{8EEFF}.$$

Coroll. 5.

79. Quemadmodum hic ex puncto dato p alterum punctum z determinauimus, ita viciſſim, ſi punctum z pro dato accipiatur, antrorſum progrediendo, ſimili modo punctum p ex eadem aequatione reperietur, ita vt Arc. pz ſuperet arcum ef, n vicibus ſumtum quantitate geometrice aſſignabili.

Problema 4.

80. Dato in parabola arcu quocunque ef, inuenire alium arcum pz, qui ſe habeat ad illum in data ratione $n:1$, ita vt ſit Arc. $pz = n$ Arc. ef.

Solutio.

Retentis iisdem denominationibus, quibus in probl. praecedenti eiusque Coroll. 2. vſi ſumus; quoniam fieri debet:

$$\text{Arc.}\,pz - n\,\text{Arc.}\,ef = 0$$

quantitas illa algebraica, cui haec arcuum differentia aequalis eſt inuenta, in nihilum abire debet. Habebimus ergo ex Coroll. 4. hanc aequationem:

$$F^{2n}P^4 + E^{2n} = \frac{nE^{2n-2}F^{2n-2}(FF-EE)(EEFF+1)}{F^{2n}-E^{2n}}P^2$$

Pona-

Ponamus breuitatis gratia $\frac{F}{E}=C$, eritque

$$C^{2n}P^4+1=\frac{nC^{2n-2}(CC-1)(CCE^4+1)}{(C^{2n}-1)EE}PP$$

vnde fit:

$$C^nP^2=\frac{nC^{n-2}(CC-1)(CCE^4+1)}{2(C^{2n}-1)EE}-\sqrt{\left(\frac{nnC^{n-4}(CC-1)^2(CCE^4+1)^2}{4(C^{2n}-1)^2E^4}-1\right)}$$

ideoque

$$P=\sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{2(C^{2n}-1)CCEE}-\sqrt{\left(\frac{nn(CC-1)^2(CCE^4+1)^2}{4(C^{2n}-1)^2C^4E^4}-\frac{1}{C^{2n}}\right)}\right)}$$

siue

$$P=\sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{4(C^{2n}-1)CCEE}+\frac{1}{2C^n}\right)}-\sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{4(C^{2n}-1)CCEE}-\frac{1}{2C^n}\right)}$$

Deinde si pari modo ponatur $\sqrt{(1+zz)}+z=Z$, erit $Z=C^nP$. Ex inuentis autem quantitatibus P et Z ita eliciuntur ipsae abscissae p et z, vt sit:

$$p=\frac{PP-1}{2P} \quad \text{et} \quad z=\frac{ZZ-1}{2Z}$$

Restituto autem pro C valore $\frac{F}{E}$, si ponamus:

$$\sqrt{\left(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})}+\frac{1}{2E^nF^n}\right)}=M$$

$$\sqrt{\left(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})}-\frac{1}{2E^nF^n}\right)}=N$$

reperietur:

$$P=E^n(M-N) \quad \text{et} \quad \frac{1}{P}=F^n(M+N)$$

$$Z=F^n(M-N) \quad \text{et} \quad \frac{1}{Z}=E^n(M+N)$$

vnde concluduntur ipsae abscissae

$$p=-\tfrac{1}{2}M(F^n-E^n)-\tfrac{1}{2}N(F^n+E^n)$$

$$z=+\tfrac{1}{2}M(F^n-E^n)-\tfrac{1}{2}N(F^n+E^n)$$

Coroll. 6.

86. Vt ergo arcus ef triplum exhiberi poſſit, is non in vertice A terminari poteſt, ſeu E debet eſſe maius quam 1, atque adeo limes dabitur, infra quem accipi nequeat. Ad quem limitem inueniendum, reſolvi oportet hanc aequationem

$$3E^3F^3 + 3EF = 2F^4 + 2EEFF + 2E^4.$$

In hunc finem ponatur $EF = S$, et $EE + FF = R$, erit:

$$3S^3 + 3S = 2RR - 2SS, \text{ ideoque } R = \sqrt{(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S)}$$

vnde fit:

$$F + E = \sqrt{(2S + \sqrt{(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S)})}$$
$$F - E = \sqrt{(-2S + \sqrt{(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S)})}$$

Et cum ſit $E > 1$, et $F > 1$, debet eſſe $R > 2$, et

$$3S^3 + 2SS + 3S > 8; \text{ ideoque } S > 1.$$

Coroll. 7.

87. Generatim ergo pro caſu $n = 3$ oportet ſit

$$3S^3 + 3S > 2RR - 2SS; \text{ ideoque } R < \sqrt{(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S)}$$

quare ſi α ſit numerus vnitate minor: reperitur

$$F + E = \sqrt{(2S + \alpha\sqrt{(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S)})}$$
$$F - E = \sqrt{(-2S + \alpha\sqrt{(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S)})}$$

Debet ergo eſſe $\alpha\alpha > \frac{8S}{3SS + 2S + 3}$ et $S > 1$.

Coroll. 8.

88. Ponamus $S = 2$; erit $\alpha\alpha > \frac{16}{19}$. Capiatur $\alpha = 1$, vt ſit $EF = 2$, et $EE + FF = \sqrt{19}$; erit

$$F+E = \sqrt{(\sqrt{19} + 4)}; \quad E = \tfrac{1}{2}\sqrt{(\sqrt{19} + 4)} - \tfrac{1}{2}\sqrt{(\sqrt{19} - 4)}$$
$$F-E = \sqrt{(\sqrt{19} - 4)}; \quad F = \tfrac{1}{2}\sqrt{(\sqrt{19} + 4)} + \tfrac{1}{2}\sqrt{(\sqrt{19} - 4)}$$

ergo

ergo $e = \frac{1}{8}\sqrt{(\sqrt{19}+4)} - \frac{3}{8}\sqrt{(\sqrt{19}-4)}$

et $f = \frac{1}{8}\sqrt{(\sqrt{19}+4)} + \frac{3}{8}\sqrt{(\sqrt{19}-4)}$

Porro reperitur :

$M = \frac{1}{2\sqrt{2}}$ et $N = 0$; vnde

$z = -p = \frac{1}{4\sqrt{2}}(2+\sqrt{19})\sqrt{(\sqrt{19}-4)}$

hic ergo arcus triplus vtrinque circa verticem aequaliter extenditur.

III.

De Comparatione ſuperficierum ſphaeroidis elliptici compreſſi et conoidis hyperbolici.

89. Sit igitur primum propoſitum ſphaeroides ellipticum genitum rotatione ellipſis BMA circa axem minorem AC. Ponatur ſemiaxis minor $CA = a$; et ſemiaxis maior $CB = a\sqrt{m}$, exiſtente m numero vnitate maiori. Sumta iam in axe minore a centro C abſciſſa $CP = x$, erit applicata $PM = \sqrt{m(aa - xx)}$, vnde elementum ellipticum $= dx\sqrt{\frac{aa + (m-1)xx}{aa - xx}}$. Tab. II. Fig. 4.

90. Poſita nunc ratione diametri ad peripheriam $= 1 : \pi$, erit portio ſuperficiei ſphaeroidicae, a reuolutione arcus AM genita, ſeu quae reſpondet abſciſſae $CP = x$, aequalis huic integrali $2\pi\int dx\sqrt{m(aa+(m-1)xx)}$. Indicetur hoc integrale, quod tanquam functio abſciſſae x ſpectetur, hoc modo

$$\int dx\sqrt{m(aa+(m-1)xx)} = \Pi . x.$$

91. Portio ergo, ſuperficiei ſphaeroidicae ellipticae abſciſſae $CP = x$ reſpondens, erit $= 2\pi . \Pi . x$: vbi functio

functio Πx, vti perspicuum est, a logarithmis, seu rectificatione parabolae pendet, eritque $\Pi x = 0$, si $x = 0$; sin autem ponatur $x = a$, tum $2\pi . \Pi a$ exhibebit semissem totius superficiei sphaeroidis.

92. Sit porro conoides hyperbolicum genitum reuolutione hyperbolae am circa suum axem cap, cuius centrum sit in c. Ponatur eius semiaxis transuersus $ca = c$, semiaxis autem coniugatus $= c\sqrt{n}$. Sumta ergo in axe a centro c abscissa quacunque $cp = y$, quae quidem sit $> c$, erit applicata $pm = \sqrt{n(yy - cc)}$, et elementum hyperbolicum $= \int dy \sqrt{\frac{(n+1)yy - cc}{yy - cc}}$.

93. Hinc erit portio superficiei conoidis istius hyperbolici, ex arcu am genita, seu abscissae $cp = y$ respondens $= 2\pi \int dy \sqrt{n((n+1)yy - cc)}$. Quod integrale cum spectari possit tanquam functio ipsius y, ita indicetur:

$$\int dy \sqrt{n((n+1)yy - cc)} = \Theta y$$

sitque $\Theta y = 0$, si capiatur $y = c$. Erit ergo superficies conoidis hyperbolici abscissae $cp = y$ respondens $= 2\pi . \Theta y$.

94. Comparentur hae binae formulae cum illis, quae supra §. 38. sunt expositae, et cum sit:

$$\Pi . x = \int \frac{dx(aa + (m-1)xx)\sqrt{m}}{\sqrt{(aa + (m-1)xx)}}$$

erit $A = aa$; $C = m - 1$; $\mathfrak{A}\sqrt{(m-1)} = aa\sqrt{m}$; et

$$\frac{m-1}{aa}\mathfrak{B}\sqrt{(m-1)} = (m-1)\sqrt{m};$$

vnde fit $\mathfrak{A} = \frac{aa\sqrt{m}}{\sqrt{(m-1)}}$ et $\mathfrak{B} = \frac{aa\sqrt{m}}{\sqrt{(m-1)}}$.

95. Deinde pro hyperbola cum sit

$$\Theta . y = \int \frac{dy(-cc + (n+1)yy)\sqrt{n}}{\sqrt{(-cc + (n+1)yy)}}$$

fiat

fiat $E=-cc$, et $F=n+1$; eritque ob $\mathfrak{C}=0$

$$-\int\frac{dy(\mathfrak{A}+\frac{F}{E}\mathfrak{B}yy)\sqrt{F}}{\sqrt{(E+Fyy)}}=\frac{aa\sqrt{m(n+1)}}{\sqrt{(m-1)}}\int\frac{dy(-1+\frac{(n+1)yy}{cc})}{\sqrt{(-cc+(n+1)yy)}}$$

ergo $$-\int\frac{dy(\mathfrak{A}+\frac{F}{E}\mathfrak{B}yy)\sqrt{F}}{\sqrt{(E+Fyy)}}=\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}.\Theta y$$

96. His ergo fubftitutionibus factis habebimus hanc aequationem:

$$\Pi\, x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\Theta y=\text{Conft.}+\frac{(n+1)\sqrt{m(m-1)}}{cc}kxy$$

cui fatisfacit haec relatio inter x et y:

$$\frac{aay}{m-1}=k\sqrt{(\frac{aa}{m-1}+xx)}-x\sqrt{(kk-\frac{aacc}{(m-1)(n+1)})}\text{ feu}$$

$$\frac{ccx}{n+1}=k\sqrt{(-\frac{cc}{n+1}+yy)}+y\sqrt{(kk-\frac{aacc}{(m-1)(n+1)})}$$

vbi $\sqrt{(kk-\frac{aacc}{(m-1)(n+1)})}$ negatiue accipi conueniet.

97. Vel ponatur $k=\frac{ae}{\sqrt{(m-1)}}$, et fi fuerit

$$y=\frac{e}{a}\sqrt{(aa+(m-1)xx)}+\frac{x\sqrt{(m-1)}}{a\sqrt{(n+1)}}\sqrt{((n+1)ee-ac)}$$

feu $$x=\frac{ae\sqrt{(n+1)}}{cc\sqrt{(m-1)}}\sqrt{((n+1)yy-cc)}-\frac{ay\sqrt{(n+1)}}{cc\sqrt{(m-1)}}\sqrt{((n+1)ee-ca)}$$

erit

$$\Pi.x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\Theta y=\text{Conft.}+\frac{(n+1)ae\sqrt{m)}}{cc}xy$$

98. Ad conftantem autem definiendam ponatur $x=0$, vt fit $\Pi.x=0$, eritque $y=e$, vnde prodit: Conft. $=\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1}}\Theta e$; ficque habebitur:

$$\Pi.x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta y-\Theta e)=\frac{(n+1)ae\sqrt{m}}{cc}xy$$

At fi in hyperbola capiatur abfciffa $cf=e$, erit fuperficies conoidis ex arcu em nata $=2\pi.(\Theta y-\Theta e)$.

 99. Quo-

99. Quoniam igitur y per x determinatur, erit quoque

$$\sqrt{((n+1(yy-cc)} = \frac{e}{a}x\sqrt{(m-1)(n+1)} + \frac{1}{a}\sqrt{(aa+(m-1)xx)((n+1)ee-cc)}$$

vnde fit:

$$y+\delta\sqrt{((n+1)yy-cc)} = \left(\frac{e}{a}+\frac{\delta}{a}\sqrt{((n+1)ee-cc)}\right)\sqrt{(aa+(m-1)xx)}$$
$$+x\left(\frac{\delta e}{a}\sqrt{(m-1)(n+1)}+\frac{\sqrt{(m-1)}}{a\sqrt{(n+1)}}\sqrt{((n+1)ee-cc)}\right)$$

fit $1:\delta\sqrt{(m-1)(n+1)} = \delta : \frac{\sqrt{(n-1)}}{\sqrt{(n+1)}}$ erit $\delta = \frac{1}{\sqrt{(n+1)}}$ hincque obtinetur:

$$\sqrt{((n+1)yy-cc)}+y\sqrt{(n+1)} =$$
$$\left(\frac{e\sqrt{(n+1)}}{a}+\frac{1}{a}\sqrt{((n+1)ee-cc)}\right)\left(\sqrt{(aa+(m-1)xx)}+x\sqrt{(m-1)}\right).$$

100. Datis ergo abſciſſis CP $=x$, et $cf=e$, abſciſſa $cp=y$ ita definiri debet, vt ſit

$$\frac{\sqrt{((n+1)yy-cc)}+y\sqrt{(n+1)}}{\sqrt{((n+1)ee-cc)}+e\sqrt{(n+1)}} = \sqrt{\left(1+\frac{(m-1)xx}{aa}\right.}+\frac{x}{a}\sqrt{(m-1)}$$

Deinde autem eſt

$$exy = \frac{ay\sqrt{((n+1)yy-cc)}}{2\sqrt{(m-1)(n+1)}} - \frac{ae\sqrt{((n+1)ee-cc)}}{2\sqrt{(m-1)(n+1)}} + \frac{ccx\sqrt{(aa+(m-1)xx)}}{2a(n+1)}.$$

Problema Hugenianum.

Dato ſphaeroide elliptico lato ABC, inuenire conoides hyperbolicum apm, ita vt circulus deſcribi poſſit geometrice, cuius area aequalis ſit futura vtrique ſuperficiei ſphaeroidicae et conoidicae iunctim ſumtae.

Solutio prima.

101. Manentibus pro vtroque corpore denominationibus, modo expoſitis, ſtatuatur $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = 1$, ſeu

ſeu $cc = \frac{aa\sqrt{m(n+1)}}{\sqrt{n(m-1)}}$, vnde ſemiaxis transuerſus hyperbolae c determinatur, numero n ſeu eius ſpecie arbitrio noſtro relicta: eritque ſtabilita ſuperiori relatione inter x et y

$$\text{II. } x + (\Theta y - \Theta e) = \frac{(n+1)ae\sqrt{m}}{cc}xy = \frac{exy\sqrt{n(m-1)(n+1)}}{a}.$$

102 Cum nunc ſit ſuperficies ſphaeroidis ex arcu BM nata, ſeu Sup. BM $= 2\pi . \Pi x$; et ſuperficies conoidis ex arcu em nata, ſeu Sup. $em = 2\pi(\Theta y - \Theta e)$; erit

$$\text{Sup. BM} + \text{Sup. } em = \frac{2\pi exy\sqrt{n(m-1)(n+1)}}{a}.$$

Vnde ſi hae duae ſuperficies iunctim ſumtae aequentur circulo, cuius radius $= r$, ob eius aream $= \pi rr$ erit

$$rr = \frac{2exy\sqrt{n(m-1)(n+1)}}{a}.$$

103. Hic iam continetur ſolutio problematis ſenſu multo latiori accepti. Caſu enim Hugeniano, quo integrum ſphaeroides aſſumitur, ſeu, quod eodem redit, eius ſemiſſis, erit $x = a$; tum vero punctum e in vertice a capi oportet, vnde fit $e = c$. Erit ergo hoc caſu:

$$y = c\sqrt{m} + \frac{c\sqrt{n(m-1)}}{\sqrt{(n+1)}} = cp,$$

fietque:

$$\text{Sup. BA} + \text{Sup. } am = 2\pi(n+1)aa\left(m + \frac{\sqrt{mn(m-1)}}{\sqrt{(n+1)}}\right).$$

104. Radio ergo circuli vtrique ſuperficiei ſimul aequalis poſito $= r$ erit, $rr = 2aa(m(n+1) + \sqrt{mn}$ $(m-1)(n+1))$

ſiue $r = a\sqrt{2(\sqrt{m(n+1)} + \sqrt{n(m-1)})\sqrt{m(n+1)}}$

Q 2 Atque

Atque erit $cp = y = \frac{c}{\sqrt{(n+1)}}(\sqrt{m(n+1)} + \sqrt{n(m-1)})$

tum vero accipi debet $c = a\sqrt[4]{\frac{m(n+1)}{n(m-1)}}$.

Quae eſt ſolutio ſimpliciſſima Problematis Hugeniani.

Solutio ſecunda.

105. Cum relatio inter x et y ſit ita comparata, vt ſit

$$\frac{\sqrt{((n+1)yy-cc)}+y\sqrt{(n+1)}}{\sqrt{((n+1)ee-cc)}+e\sqrt{(n+1)}} = \sqrt{(1+\frac{(m-1)xx}{aa})} + \frac{x}{a}\sqrt{(m-1)}$$

ſitque $\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta y - \Theta e) = \frac{\sqrt{m(n+1)}}{2cc}$.

$$\left(\frac{aay\sqrt{((n+1)yy-cc)}}{\sqrt{(m-1)}} - \frac{aae\sqrt{((n+1)ee-cc)}}{\sqrt{(m-1)}} + \frac{ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right)$$

Capiatur in conoide noua abſciſſa $cq = z$, et pro e iam ſumatur y, vt ſit

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{\sqrt{((n+1)yy-cc)}+y\sqrt{(n+1)}} = \sqrt{(1+\frac{(m-1)xx}{aa})} + \frac{x}{a}\sqrt{(m-1)}$$

erit pariter $\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta z - \Theta y) = \frac{\sqrt{m(n+1)}}{2cc} \times$

$$\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}} - \frac{aay\sqrt{((n+1)yy+cc)}}{\sqrt{(m-1)}} + \frac{ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right).$$

106. Addantur hae formulae inuicem, atque y prorſus eliminabitur; fiet enim

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{\sqrt{((n+1)ee-cc)}+e\sqrt{(n+1)}} = \left(\sqrt{(1+\frac{(m-1)xx}{aa})} + \frac{x}{a}\sqrt{(m-1)}\right)^2$$

eritque: $2\Pi . x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta z - \Theta e) = \frac{\sqrt{m(n+1)}}{2cc} \times$

$$\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}} - \frac{aae\sqrt{(n+1)ee-cc)}}{\sqrt{(m-1)}} + \frac{2ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right).$$

107. Statuatur iam $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = 2$, ſeu $cc = \frac{aa\sqrt{m(n+1)}}{2\sqrt{n(m-1)}}$ erit per $\frac{2\pi}{2}$ multiplicando

$$\text{Sup. BM} + \text{Sup } en = \frac{\pi\sqrt{m(n+1)}}{2cc} \times$$

$$\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}} - \frac{aae\sqrt{((n+1)ee-cc)}}{\sqrt{(m-1)}} + \frac{2ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right)$$

vnde

vnde facile radius circuli aequalis definitur.

108. Sit nunc pro casu Hugeniano $x = a$, et $e = c$, erit:

$$\frac{\sqrt{((n+1)zz - cc)} + z\sqrt{(n+1)}}{c(\sqrt{n} + \sqrt{(n+1)})} = (\sqrt{m} + \sqrt{(m-1)})^2$$

Hincque inuento z, existenteque $cc = \frac{aa\sqrt{m(n+1)}}{2\sqrt{n(m-1)}}$, erit

$$\text{Sup. BA} + \text{Sup. } an = \frac{\pi\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz - cc)}}{\sqrt{(m-1)}} - \frac{aacc\sqrt{n}}{\sqrt{(m-1)}} + \frac{2aacc\sqrt{m}}{\sqrt{(n+1)}}\right).$$

Solutio Generalis.

109. Si hac ratione continuo vlterius progrediamur, vt supra pro parabola est factum, reperietur, si abscissa $cq = z$ existente, $cf = e$ ita capiatur, vt sit

$$\frac{\sqrt{((n+1)zz - cc)} + z\sqrt{(n+1)}}{\sqrt{((n+1)ee - cc)} + e\sqrt{(n+1)}} = \left(\sqrt{\left(1 + \frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)}\right)^{\mu}$$

fore $\mu \Pi x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta z - \Theta e, = \frac{\sqrt{m(n+1)}}{2cc} \times$

$$\left(\frac{aaz\sqrt{((n+1)zz - cc)}}{\sqrt{(m-1)}} - \frac{aae\sqrt{((n+1)ee - cc)}}{\sqrt{(m-1)}} + \frac{\mu ccx\sqrt{(aa + (m-1)xx)}}{\sqrt{(n+1)}}\right)$$

$$= \frac{\mu}{2\pi}\text{Sup. BM} + \frac{aa\sqrt{m(n+1)}}{2\pi cc\sqrt{n(m-1)}} \cdot \text{Sup. } en.$$

110. Pro casu ergo Hugenii, posito $x = a$, et $e = c$, fiat $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = \mu$; et capiatur abscissa $cq = z$, ita vt sit:

$$\frac{\sqrt{((n+1)zz - cc)} + z\sqrt{(n+1)}}{c(\sqrt{n} + \sqrt{(n+1)})} = (\sqrt{m} + \sqrt{(m-1)})^{\mu}$$

eritque

$$\text{Sup. BA} + \text{Sup. } an = \frac{\pi\sqrt{m(n+1)}}{\mu cc}\left(\frac{aaz\sqrt{((n+1)zz - cc)}}{\sqrt{(m-1)}} - \frac{aacc\sqrt{n}}{\sqrt{(m-1)}} + \frac{\mu aacc\sqrt{m}}{\sqrt{(n+1)}}\right)$$

siue

$$\text{Sup. BA} + \text{Sup. } an = \pi\left(z\sqrt{n((n+1)zz - cc)} - ncc + \frac{\mu cc\sqrt{mn(m-1)}}{\sqrt{(n+1)}}\right)$$

$$= \pi(z\sqrt{n((n+1)zz - cc)} - ncc + maa).$$

 111. Quae-

111. Quaecunque ergo fuerit hyperbola, ex qua conoides nafcitur, dummodo fit $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}=\mu$ numerus rationalis, ab eo femper portio an abfcindi poterit, cuius fuperficies ad fuperficiem fphaeroidis BMA addita, per circulum exhiberi poteft, cuius radius r geometrice eft affignabilis: erit enim

$$r=\sqrt{(maa-ncc+z\sqrt{n((n+1)zz-cc)})}.$$

112. Quo autem facilius pateat, quomodo abfciffa $cq=z$ reperiri debeat, cum fit

$$\sqrt{\left(\frac{(n+1)zz}{cc}-1\right)}+\frac{z}{c}\sqrt{(n+1)}=(\sqrt{(n+1)}+\sqrt{n})(\sqrt{m}+\sqrt{(m-1)})^{\mu}$$

erit

$$\frac{z}{c}\sqrt{(n+1)}-\sqrt{\left(\frac{(n+1)zz}{cc}-1\right)}=(\sqrt{(n+1)}-\sqrt{n})(\sqrt{m}-\sqrt{(m-1)})^{\mu}$$

hinc facile tam z, quam $\sqrt{((n+1)zz-cc)}$ colligentur.

113. Hinc autem porro concluditur, fore

$$z\sqrt{n((n+1)zz-cc)}=\frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(\sqrt{(n+1)}+\sqrt{n})^2(\sqrt{m}+\sqrt{(m-1)})^{2\mu}$$
$$-\frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(\sqrt{(n+1)}-\sqrt{n})^2(\sqrt{m}-\sqrt{(m-1)})^{2\mu}$$

At fi ponatur breuitatis gratia $\sqrt{(m}+\sqrt{(m-1)}=M$, et $\sqrt{n}+\sqrt{(n+1)}=N$, erit $z=\frac{c}{2\sqrt{(n+1)}}(M^{\mu}N-M^{-\mu}N^{-1})$, et

$$r=\sqrt{(maa+\frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(M^{\mu}-M^{-\mu})(M^{\mu}N^2+M^{-\mu}N^{-2})}$$

ficque problema non difficulter conftruitur, dummodo exponens μ fuerit rationalis.

114. Haec igitur exempla fufficiant vfum, nouae methodi, quam adumbraui, oftendiffe; etfi enim haec eadem exempla methodo confueta iam fint foluta, tamen non folum ad calculos admodum intricatos deueniri

niri ſolet, ſed etiam integratione, qua formulae differentiales, vel ad quadraturam circuli, vel ad logarithmos reducantur, abſolute eſt opus. Huius igitur nouae methodi inſigne commodum in hoc conſiſtit, quod eius beneficio eadem problemata, tam ſine laborioſo calculo, quam ſine vlla integratione reſolui queant; quam ob cauſam inde merito multo maiora ac ſublimiora expectare licet, quae vim omnium conſuetarum methodorum penitus ſuperent.

DE-

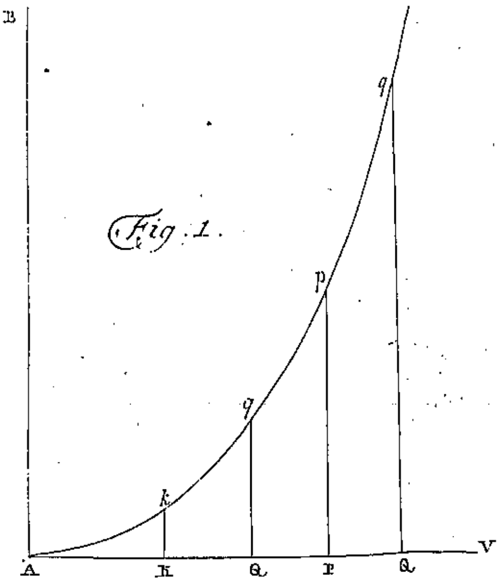

Fig. 1.

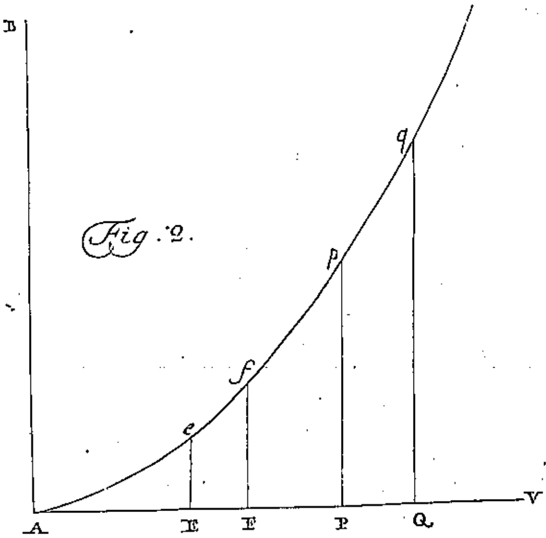

Fig. 2.

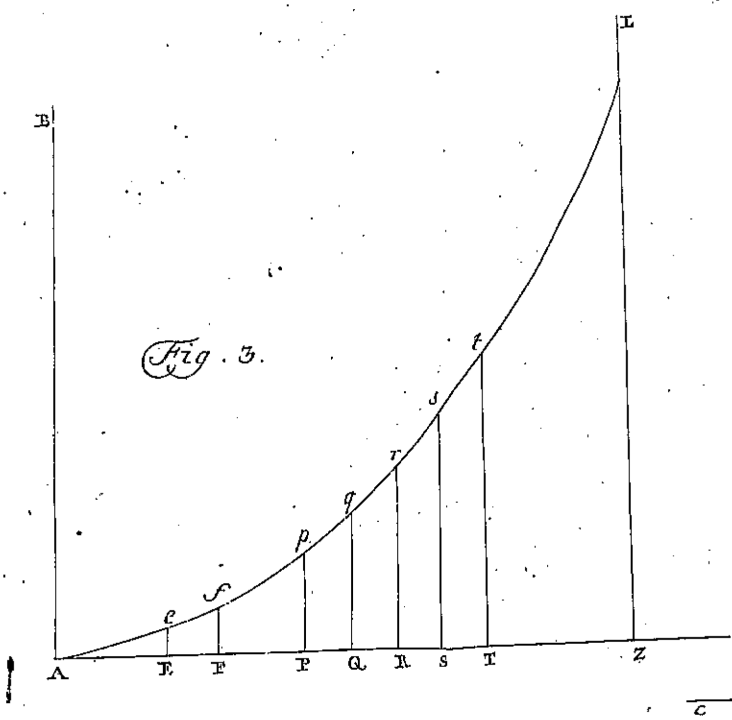

Fig. 3.

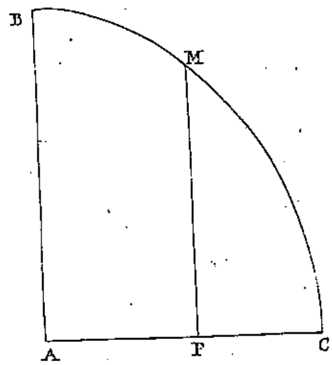

Fig. 4.