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De expressione integralium per factores

Leonhard Euler

DE EXPRESSIONE INTEGRALIVM PER FACTORES.

Auctore

L. EVLERO.

In integralibus ad ſeries infinitas reuocandis Geometrae adhuc plurimum fuere occupati, cum ad naturam ſerierum accuratius perſpiciendam, tum ob ſummum vſum, quem ſeries praeſtant ad integralium valores proxime cognoſcendos. Iam vero oſtendi in Tomo Comment. Petrop. XI. ob easdem rationes reductionem integralium ad producta ex infinitis factoribus conſtantia non minus eſſe dignam, quae omni cura excolatur, ibique plurima iam huius reductionis dedi ſpecimina, quae in Analyſi haud contemnendum vſum afferre videntur; etiamſi ipſa pertractatio nondum ſatis fuerit polita atque in ordinem digeſta. Quam ob rem hoc argumentum denuo ita reſumere eſt viſum, vt primo fundamenta, quibus innititur, luculentius exponerem, tum vero plures caſus, qui inprimis memorabiles videntur, accuratius euoluerem.

Ante omnia autem notari conuenit, hunc modum, integralia per factores exprimendi, non in genere ita tradi poſſe, vt ad omnes quantitatis variabilis valores aeque pateat, ad quod inſtitutum ſeries infinitae potiſſimum ſunt accommodatae, ſed factores tum ſolum

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commode in vſum vocari poſſunt, quando is integralis tantum valor inueſtigatur, cum variabili valor quidem determinatus tribuitur. Neque vero hunc valorem pro lubitu aſſumere licet, ſed potius ita comparatum eſſe oportet, vt iam in formula differentiali ſingulari gaudeat proprietate, dum eam, vel ad nihilum, vel ad infinitum, redigit.

Huiusmodi autem caſus iam prae ceteris notatu inprimis digni, atque in applicatione ad praxin potiſſimum quaeri ſolent, ita vt plerumque quaeſtio verſari ſoleat in valore integralium pro huiusmodi quodam caſu inueniendo. Ita ſi de circuli quadratura agitur, vel huius formulae $\int \frac{dx}{\sqrt{(1-xx)}}$ valor deſideratur, caſu, quo $x = 1$, vel huius formulae $\int \frac{dx}{1+xx}$, caſu, quo $x = \infty$: ibi autem hoc caſu differentiale ipſum euadit infinitum, hic vero euaneſcit.

Quo igitur rem generalius complectar, duplicis generis formulas integrales hic euoluam: quae ſint $\int x^{m-1} dx (1-x^n)^k$ et $\int \frac{x^{m-1} dx}{(1+x^n)^k}$, quarum vtramque ita integrari aſſumo, vt euaneſcat poſito $x = 0$. Tum vero prioris integralis $\int x^{m-1} dx (1-x^n)^k$ eum tantum valorem determinare in animo eſt, quem accipit ſi ponatur $x = 1$: poſterioris vero integralis $\int \frac{x^{m-1} dx}{(1+x^n)^k}$ illum valorem, quem caſu $x = \infty$ ſortitur, tantum inveſtigabo. Euidens autem eſt hos integralium caſus prae reliquis tali eminenti praerogatiua gaudere, vt inprimis euolui mereantur.

Quanquam hic elegantiae consulens coefficientes omisi, tamen perspicuum est, has formulas aeque late patere, ac si tales coefficientes essent adiecti. Formula namque huiusmodi $\int \gamma y^{m-1} dy (\alpha - \beta y^n)^k$, posito $\frac{\beta y^n}{\alpha} = x^n$ manifesto ad allatam $\int x^{m-1} dx (1 - x^n)^k$, reducitur, neque propterea latius patere est censenda, ac simili reductione haec formula $\int \frac{\gamma y^{m-1} dy}{(\alpha + \beta y^n)^k}$ in altera $\int \frac{x^{m-1} dx}{(1 + x^n)^k}$ continetur, vnde omnino superfluum esset, loco formularum nostrarum simpliciori specie expressarum, has magis complicatas adhibere velle.

Verum etiam altera formularum sumtarum in altera continetur, ita vt sufficiat alterutram tantummodo, quam sum traditurus, tractasse.

Si enim ponatur $x = \frac{y}{(1 + y^n)^{\frac{1}{n}}}$, erit $1 - x^n = \frac{1}{1 + y^n}$: $x^m = \frac{y^m}{(1 + y^n)^{\frac{m}{n}}}$ et $\frac{dx}{x} = \frac{dy}{y(1 + y^n)}$: quibus valoribus substitutis obtinebitur

$$\int x^{m-1} dx (1 - x^n)^k = \int \frac{y^{m-1} dy}{(1 + y^n)^{k+1+\frac{m}{n}}},$$

his integralibus ita sumtis, vt euanescant posito $x = o$ et $y = o$; quae conditio hic semper subintelligi debet. Cum igitur posito $y = \infty$, fiat $x = 1$, habebimus sequens Theorema.

Theorema I.

I. *Valor formulae integralis* $\int x^{m-1}dx(1-x^n)^k$ *casu* $x=1$, *aequalis est valori huius formulae integrali* $\int\frac{y^{m-1}dy}{(1+y^n)^{k+1+\frac{m}{n}}}$ *casu* $y=\infty$.

Cuius aequalitatis ratio est, quod illa forma actu transmutatur in hanc, si ponatur $x=\frac{y}{(1+y^n)^{\frac{1}{n}}}$

Sequens Theorema, quod per similem reductionem oritur, non parum quoque vtilitatis habebit, quod ideo cum sua demonstratione apponam.

Theorema 2.

2. Valor huius formulae integralis $\int x^{m-1}dx(1-x^n)^k$ casu $x=1$, aequalis est valori huius formulae integralis $\int y^{nk+n-1}dy(1-y^n)^{\frac{m-n}{n}}$ etiam casu $y=1$.

Demonstratio.

Ponatur $x=(1-y^n)^{\frac{1}{n}}$, vt sit $1-x^n=y^n$; $x^m=(1-y^n)^{\frac{m}{n}}$ et $\frac{dx}{x}=\frac{-y^{n-1}dy}{1-y^n}$, quibus valoribus substitutis habebitur

$$x^{m-1}dx(1-x^n)^k=-y^{nk+n-1}dy(1-y^n)^{\frac{m-n}{n}}$$

Sit $Y=\int y^{nk+n-1}dy(1-y^n)^{\frac{m-n}{n}}$, integrali ita sumto, vt euanescat posito $y=0$; tum posito $y=1$, abeat Y in A. Iam cum illas formulas ita integrari oporteat, vt euanescant posito $x=0$, quo casu fit $y=1$, erit:

$$\int x^{m-1}dx(1-x^n)^k=A-Y.$$

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Ponatur nunc $x = 1$, quo casu fit $y = o$ ideoque et $Y = o$, et formula nostra integralis fiet $= A$: seu integrale $\int x^{m-1} dx (1 - x^n)^k$ casu $x = 1$ aequale erit integrali $\int y^{nk+n-1} dy (1 - y^n)^{\frac{m-n}{n}}$ casu $y = 1$. Q. E. D.

Coroll. 1.

3. Cum igitur hae tres formulae:

$$\text{I. } \int x^{m-1} dx (1-x^n)^k; \text{ II. } \int \frac{y^{m-1} dy}{(1+y^n)^{k+1+\frac{m}{n}}}; \text{ III. } \int z^{nk+n-1} dz$$

$(1 - z^n)^{\frac{m-n}{n}}$ ita a se inuicem pendeant, vt prima transeat in secundam posito $x = \frac{y}{(1+y^n)^{\frac{1}{n}}}$, at vero posito $x = (1 - z^n)^{\frac{1}{n}}$ ea abeat in tertiam negatiue sumtam, manifestum est, quoties vna harum fuerit absolute integrabilis, toties et binas reliquas fore absolute integrabiles.

Coroll. 2.

4. Prima autem absolute est integrabilis, vti per se est perspicuum, si sit k numerus integer affirmatiuus: quicunque numerus pro m statuatur. Excipiuntur tamen casus, quibus m aequatur cuipiam numero huius progressionis: $o; -n; -2n; -3n; \ldots\ldots -kn$; his enim casibus pars integralis pendebit a logarithmis. Casus ergo hi excipiendi huc redeunt, vt integratio absoluta succedat, existente k numero integro affirmatiuo, nisi $-\frac{m}{n}$ sit numerus integer affirmatiuus, vel minor, quam k, vel ipsi k aequalis. Vel nisi $k + \frac{m}{n}$ sit numerus integer affirmatiuus non maior quam k.

Coroll.

Coroll. 3.

5. Simili modo forma secunda erit integrabilis, si $-k-1-\frac{m}{n}$ fuerit numerus integer affirmatiuus, puta i; casus autem excipiuntur, quibus $-\frac{m}{n}$ pariter est numerus integer affirmatiuus non maior quam i. Vel si denotet ω numerum quemcunque affirmatiuum integrum ex hac serie:

$$0, 1, 2, \ldots\ldots\ldots i,$$

casus excipiuntur, quibus $-\frac{m}{n}=\omega$.

Coroll. 4.

6. Tertia autem formula absolute erit integrabilis, si $\frac{m-n}{n}$ fuerit numerus integer affirmatiuus, puta i; excipiuntur autem casus, quibus $-k-1=\omega$, denotante ω numerum quemcunque integrum affirmatiuum, non maiorem, quam i.

Coroll. 5.

7. His ergo notatis formula $\int x^{m-1}dx(1-x^n)^k$ absolute erit integrabilis, casibus sequentibus, in quibus i numerum affirmatiuum integrum quemcunque denotat, ω autem quemlibet numerum integrum affirmatiuum ipso i non maiorem.

I. Si $k=i$ neque tamen $-\frac{m}{n}=\omega$.

II. Si $k-1-\frac{m}{n}=i$ neque tamen $-\frac{m}{n}=\omega$ (vel $-k-1=\omega$)

III. Si $\frac{m-1}{n}=i$ neque tamen $-k-1=\omega$.

Coroll. 6.

8. Manifestum autem est, hos eosdem integrabilitatis casus locum esse habituros in formula hac latius paten-

patente $\int x^{m-1}dx(a+bx^n)^k$, pro quo demonſtratio pari modo adornatur. Atque ex his tribus conditionibus caſus integrabilitatis omnium huiusmodi formularum diiudicari ſolent.

Quanquam haec ad meum inſtitutum non pertinent, tamen quia tam facile ex binis Theorematibus praemiſſis fluunt, non incongruum eſt viſum, ea his adiicere. Nunc igitur ad verum fundamentum dicendorum progredior, quod reductione integralium ad alias formas nititur. Quam quo diſtinctius exponam, hanc formam algebraicam contemplor: $x^\alpha(1-x^n)^\gamma = P$, qua differentiata obtineo:

$$dP = \alpha x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}$$

quae adhuc aliis modis in duo membra diſpeſci poteſt, veluti:

$$dP = \alpha x^{\alpha-1}dx(1-x^n)^{\gamma-1} - (\alpha+\gamma n)x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}$$

Tum vero, ſi in membro poſteriori pro x^n ſcribatur $1-(1-x^n)$, prior forma dabit

$$dP = (\alpha+\gamma n)x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n x^{\alpha-1}dx(1-x^n)^{\gamma-1}$$

poſterior vero eodem redit. Vnde integrando obtinemus

$$P = \alpha\int x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}$$

$$P = \alpha\int x^{\alpha-1}dx(1-x^n)^{\gamma-1} - (\alpha+\gamma n)\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}$$

$$P = (\alpha+\gamma n)\int x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$$

Quae integralia cum euaneſcere debeant poſito $x=0$, neceſſe eſt, vt eodem caſu $Px^\alpha(1-x^n)^\gamma$ euaneſcat, quod quidem ſemper fit, ſi α ſit numerus poſitiuus quicunque. Iam ſi γ quoque fuerit numerus poſitiuus, euidens eſt poſito $x=1$, et hoc caſu fieri $P=0$: vnde ſequentia elicimus Theoremata:

Theorema. 3.

9. *Si* α et γ fuerint numeri poſitiui, ac poſt integrationem ponatur $x=1$, habebuntur ſequentes formularum integralium aequalitates.

I. $\alpha\int x^{\alpha-1}dx(1-x^n)^\gamma=\gamma n\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}$

II. $\alpha\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}=(\alpha+\gamma n)\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}$

III. $(\alpha+\gamma n)\int x^{\alpha-1}dx(1-x^n)^\gamma=\gamma n\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$.

Demonſtratio.

Cum enim poſt integrationem ponatur $x=1$, pro hoc caſu in ſuperioribus formulis fit $P=0$, indeque aperte ſequuntur aequationes hic propoſitae. Q. E. D.

Coroll. 1.

10. Harum trium aequationum quaelibet iam in duabus reliquis continetur, vnde eae in hac forma comprehendentur:

$$\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}=\frac{\alpha}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^\gamma=\frac{\alpha}{\alpha+\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$$

ſeu ſequentes tres formulae integrales inter ſe aequabuntur:

$$\frac{1}{\alpha}\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}=\frac{1}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^\gamma=\frac{1}{\alpha+\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$$

ſi quidem α et γ fuerint numeri poſitiui.

Coroll. 2.

11. Cum ſit per Theor. 2. $\int x^{m-1}dx(1-x^n)^k$ $=\int x^{nk+n-1}dx(1-x^n)^{\frac{m-n}{n}}$ poſito itidem $x=1$, aequalitas

litas habebitur inter sex sequentes formulas integrales:

I. $\frac{1}{\alpha}\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}$; II. $\frac{1}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma}$; III. $\frac{1}{\alpha+\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$

IV. $\frac{1}{\alpha}\int x^{n\gamma-1}dx(1-x^n)^{\frac{\alpha}{n}}$; V. $\frac{1}{\gamma n}\int x^{n\gamma+n-1}dx(1-x^n)^{\frac{\alpha-n}{n}}$; VI. $\frac{1}{\alpha+\gamma n}\int x^{n\gamma-1}dx(1-x^n)^{\frac{\alpha-n}{n}}$

dummodo exponentes α et γ fuerint affirmatiui.

Coroll. 3.

12. Si α fuerit numerus infinitus,

erit $\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$

atque ob eandem rationem,

erit $\int x^{\alpha+2n-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$

vnde generatim colligitur fore $\int x^{\alpha+\mu-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$

dummodo μ fuerit numerus finitus existente α infinito.

Coroll. 4.

13. Pari modo si γ fuerit numerus infinitus, erit

$$\int x^{\alpha-1}dx(1-x^n)^{\gamma} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$$

eodemque modo erit $\int x^{\alpha-1}dx(1-x^n)^{\gamma+1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma}$, vnde generatim colligitur fore:

$$\int x^{\alpha-1}dx(1-x^n)^{\gamma\pm\mu} = \int x^{\alpha-1}dx(1-x^n)^{\gamma}$$

siquidem μ sit numerus finitus existente γ infinito.

Problema 1.

14. Si m et k sint numeri positiui, atque i denotet numerum integrum affirmatiuum quemcunque, definire rationem formulae $\int x^{m-1}dx(1-x^n)^{k-1}$ ad formulam $\int x^{m-1}dx(1-x^n)^{k+i}$ casu $x=1$.

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Solutio.

Cùm sit $\int x^{\alpha-1}dx(1-x^n)^{\gamma-1} = \frac{\alpha+\gamma n}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma}$, erit ponendo m et k pro α et γ:

$$\int x^{m-1}dx(1-x^n)^{k-1} = \frac{m+kn}{kn}\int x^{m-1}dx(1-x^n)^{k}$$

si nunc manente $\alpha = m$ ponatur $\gamma = k+1$, erit γ multo magis numerus affirmatiuus, cum k sit talis; ideoque pari modo habebitur

$$\int x^{m-1}dx(1-x^n)^{k} = \frac{m+(k+1)n}{(k+1)n}\int x^{m-1}dx(1-x^n)^{k+1}$$

ac pari modo progrediendo, erit

$$\int x^{m-1}dx(1-x^n)^{k+1} = \frac{m+(k+2)n}{(k+2)n}\int x^{m-1}dx(1-x^n)^{k+2}.$$

Hinc ergo ingenere concluditur fore, denotante i numerum integrum quemcunque:

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m-1}dx(1-x^n)^{k+i}} = \frac{m+kn}{kn}\cdot\frac{m+kn+n}{kn+n}\cdot\frac{m+kn+2n}{kn+2n}\cdot\frac{m+kn+3n}{kn+3n}\cdots\cdots\frac{m+kn+in}{k+in}$$

Q. E. I.

Coroll. 1.

15. Cùm sit $\int x^{m-1}dx(1-x^n)^{k-1} = \int x^{nk-1}dx(1-x^n)^{\frac{m-n}{n}}$ ideoque etiam:

$\int x^{m-1}dx(1-x^n)^{k+i} = \int x^{kn+in+n-1}dx(1-x^n)^{\frac{m-n}{n}}$, erit quoque:

$$\frac{\int x^{kn-1}dx(1-x^n)^{\frac{m-n}{n}}}{\int x^{kn+in+n-1}dx(2\ x^n)^{\frac{m-1}{n}}} = \frac{m+kn}{kn}\cdot\frac{m+kn+n}{kn+n}\cdot\frac{m+kn+2n}{kn+2nn}\cdots\cdots\frac{m+kn+in}{kn+in}$$

Coroll. 2.

16. Si hic ponatur $kn = \mu$; et $\frac{m}{n} = x$ seu $m = xn$, ita vt iam μ et x sint numeri affirmatiui, habebitur haec reductio:

$\int x$

$$\frac{\int x^{\mu-1}dx(1-x^n)^{\varkappa-1}}{\int x^{\mu+in+n-1}dx(1-x^n)^{\varkappa-1}}=\frac{\mu+\varkappa n}{\mu}\cdot\frac{\mu+\varkappa n+n}{\mu+n}\cdot\frac{\mu+\varkappa n+2n}{\mu+2n}\cdots\frac{\mu+\varkappa n+in}{\mu+in}$$

ſcriptis autem pro μ et $\varkappa$ litteris m et k, erit:

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m+in+n-1}dx(1-x^n)^{k-1}}=\frac{m+kn}{m}\cdot\frac{m+kn+n}{m+n}\cdot\frac{m+kn+2n}{m+2n}\cdots\frac{m+kn+in}{m+in}.$$

Coroll. 3.

17. Si haec expreſſio per expreſſionem in problemate inuentam diuidatur, prodibit:

$$\frac{\int x^{m-1}dx(1-x^n)^{k+i}}{\int x^{m+in+n-1}dx(1-x^n)^{k-1}}=\frac{kn}{m}\cdot\frac{kn+n}{m+n}\cdot\frac{kn+2n}{m+2n}\cdots\frac{kn+in}{m+in}.$$

in quibus factoribus tam numeratores quam denominatores in arithmetica progreſſione progrediuntur, cuius differentia eſt $=n$.

Problema 2.

18. Valorem formulae $\int x^{m-1}dx(1-x^n)^{k-1}$, quem accipit caſu $x=1$, per factores infinitos exprimere, ſiquidem exponentes m et k ſint poſitiui.

Solutio.

Statuatur in forma praecedentis problematis numerus i infinitus; et habebitur:

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m-1}dx(1-x^n)^{k+i}}=\frac{m+kn}{kn}\cdot\frac{m+kn+n}{kn+n}\cdot\frac{m+kn+2n}{kn+2n}\cdot\frac{m+kn+3n}{kn+3n}\cdot\text{etc. in infinitum.}$$

Iam manente i eodem numero infinito, loco k alius ſumatur numerus finitus $\varkappa$ quicunque; et habebitur ſimili modo:

$$\frac{\int x^{m-1}dx(1-x^n)^{\kappa-1}}{\int x^{m-1}dx(1-x^n)^{\kappa+i}}=\frac{m+xn}{xn}\cdot\frac{m+xn+n}{xn+n}\cdot\frac{m+xn+2n}{xn+2n}\cdot\frac{m+xn+3n}{xn+3n}\text{ etc.}$$

vbi numerus factorum aequalis eſt numero factorum praecedentis expreſſionis, vtrinque ſcilicet infinitus $=i+1$. At ob i infinitum, eſt vti §. 13. notauimus $\int x^{m-1}dx(1-x^n)^{k+i}=\int x^{m-1}dx(1-x^n)^{\kappa+i}$ quare priori forma per poſteriorem diuiſa orietur:

$$\frac{\int x^{m-1}dx(1-x^n)^{\kappa-1}}{\int x^{m-1}dx(1-x^n)^{\kappa-1}}=\frac{x(m+kn)}{k(m+xn)}\cdot\frac{(x+1)(m+kn+n)}{(k+1)(m+xn+n)}\cdot\frac{(x+2)(m+kn+2n)}{(k+2)(m+xn+2n)}\text{ etc.}$$

ſtatuatur iam $x=1$, eritque $\int x^{m-1}dx(1-x^n)^{\kappa-1}=\frac{x^m}{n2}=\frac{1}{m}$ poſito $\kappa=1$, vnde fiet

$$\int x^{m-1}dx(1-x^n)^{k-1}=\frac{1}{m}\cdot\frac{1(m+kn)}{k(m+n)}\cdot\frac{2(m+kn+n)}{(k+1)(m+2n)}\cdot\frac{3(m+kn+2n)}{(k+2)(m+3n)}\cdot\frac{4(m+kn+3n)}{(k+3)(m+4n)}\text{ etc.}$$

Q. E. I.

Aliter.

Tractetur ſimili modo forma §. 16. inuenta, ſtatuendo i numerum infinitum, eritque:

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m+in-1}dx(1-x^n)^{k-1}}=\frac{m+kn}{m}\cdot\frac{m+kn+n}{m+n}\cdot\frac{m+kn+2n}{m+2n}\cdot\frac{m+kn+3n}{m+3n}\text{ etc.}$$

Iam poſito pro m alio numero finito, μ erit pari modo

$$\frac{\int x^{\mu-1}dx(1-x^n)^{k-1}}{\int x^{\mu+in-1}dx(1-x^n)^{k-1}}=\frac{\mu+kn}{\mu}\cdot\frac{\mu+kn+n}{\mu+n}\cdot\frac{\mu+kn+2n}{\mu+2n}\cdot\frac{\mu+kn+3n}{\mu+3n}\text{ etc.}$$

Cum autem ſit ob i numerum infinitum:

$$\int x^{m+in-1}dx(1-x^n)^{k-1}=\int x^{\mu+in-1}dx(1-x^n)^{k-1}=\int x^{in}dx(1-x^n)^{k-1}$$

euaneſcentibus quantitatibus finitis prae infinitis; et quia vtrinque idem factorum numerus habetur, formam priorem per poſteriorem diuidendo orietur:

$\int x^{m-}$

$$\frac{\int x^{m-1}dx(1-x)^{k-1}}{\int x^{\mu-1}dx(1-x)^{k-1}}=\frac{\mu(m+kn)}{m(\mu+kn)}\cdot\frac{(\mu+n)(m+nk+n)}{(m+n)(\mu+kn+n)}\cdot\frac{(\mu+2n)(\mu+kn+2n)}{(m+2n)(\mu+kn+2u)}\text{ etc.}$$

ſtatuatur iam $\mu=n$, fiet $\int x^{n-1}dx(1-x^n)^{k-1}=\frac{1-(1-x^n)^k}{nk}$ integratione ita peracta, vt euaneſcat poſito $x=0$. Poſito nunc $x=1$, iſte valor abit in $\frac{1}{nk}$, vnde obtinebitur:

$$\int x^{m-1}dx(1-x^n)^{k-1}=\frac{1}{nk}\cdot\frac{1(m+kn)}{m(1+k)}\cdot\frac{2(m+kn+n)}{(m+n)(2+k)}\cdot\frac{3(m+kn+2n)}{(m+2n)(3+k)}\text{ etc.}$$

En ergo aliud productum ex infinitis factoribus conſtans, priori non admodum diſſimile, eique adeo aequale, quo valor quaeſitus formulae integralis propoſitae exprimitur. Q. E. I.

Coroll. 1.

19. Has autem duas formas in infinitum excurentes inter ſe eſſe aequales, per ſe perſpicuum eſt: poſteriori enim per priorem diuiſa, ob ſingulorum membrorum numeratores aequales, prodit:

$$1=\frac{m}{nk}\cdot\frac{k(m+n)}{m(k+1)}\cdot\frac{(k+1)(m+2n)}{(m+n)(k+2)}\cdot\frac{(k+2)(m+3n)}{(m+2n)(k+3)}\text{ etc.}$$

At duo factores primi dant $\frac{m+n}{n(k+1)}$; tres $\frac{m+2n}{n(k+2)}$; quatuor $\frac{m+3n}{n(k+3)}$ et infiniti dant $\frac{m+in}{n(k+i)}=\frac{in+m}{in+kn}=1$.

Coroll. 2.

20. Huiusmodi formae factorum infinitorum innumerabiles formari poſſunt, quarum valor $=1$. Cum enim ſit

$$\frac{p}{p+q}\cdot\frac{p+q}{p+2q}\cdot\frac{p+2q}{p+3q}\cdot\frac{p+3q}{p+4q}\cdot\ldots=\frac{p}{p+iq}=\frac{p}{iq}$$

$$\frac{r+s}{r}\cdot\frac{r+2s}{r+s}\cdot\frac{r+3s}{r+2s}\cdot\frac{r+4s}{r+3s}\cdot\ldots=\frac{r+is}{r}=\frac{is}{r}$$

multiplicando has duas formas habebimus

$$1=\frac{qr}{ps}\cdot\frac{p(r+s)}{r(p+q)}\cdot\frac{(p+q)(r+2s)}{(r+s)(p+2q)}\cdot\frac{(p+2q)(r+3s)}{(r+2s)(p+3q)}\cdot \text{etc.}$$

Coroll. 3.

21. Si ergo valor formulae integralis inuentus per hanc expressionem $=1$ multiplicetur, prodibit expressio latius patens eidem aequalis, scilicet:

$$\int x^{m-1}dx(1-x^n)^{k-1}=\frac{qr}{knps}\cdot\frac{1(m+kn)p(r+s)}{m(k+1)r(p+q)}\cdot\frac{2(m+kn+n)(p+q)(r+2s)}{(m+n)(k+2)(r+s)(p+2q)}\cdot\frac{3(m+kn+2n)(p+2q)(r+3s)}{(m+2n)(k+3)(r+2s)(p+3q)}\cdot \text{etc.}$$

vbi pro p, q, r, s numeros quoscunque assumere licet. Pluribus modis ergo ita accipi possunt, vt quilibet factor ad formam simpliciorem redigatur.

Coroll. 4.

22. Sit $p=m$, et $q=n$, eritque:

$$\int x^{m-1}dx(1-x^n)^{k-1}=\frac{r}{mks}\cdot\frac{1(m+kn)(r+s)}{(m+n)(k+1)r}\cdot\frac{2(m+kn+n)(r+2s)}{(m+2n)(k+2)(r+s)}\cdot\frac{3(m+kn+2n)(r+3s)}{(m+3n)(k+3)(r+2s)}\cdot \text{etc.}$$

si porro ponatur $r=k$, et $s=1$, erit,

$$\int x^{m-1}dx(1-x^n)^{k-1}=\frac{1}{m}\cdot\frac{1(m+kn)}{(m+n)k}\cdot\frac{2(m+kn+n)}{(m+2n)(k+1)}\cdot\frac{3(m+kn+2n)}{(m+3n)(k+2)}\cdot \text{etc.}$$

quae est expressio primum inuenta. Sin autem sit $r=m+kn$, et $s=n$, erit,

$$\int x^{m-1}dx(1-x^n)^{k-1}=\frac{m+kn}{mkn}\cdot\frac{1(m+kn+n)}{(m+n)(k+1)}\cdot\frac{2(m+kn+2n)}{(m+2n)(k+2)}\cdot\frac{3(m+kn+3n)}{(m+3n)(k+3)}\cdot \text{etc.}$$

Coroll. 5.

23. Si ponatur $p=k+1$, et $q=1$, erit:

$$\int x^{m-1}dx(1-x^n)^{k-1}=\frac{r}{k(k+1)ns}\cdot\frac{1(m+kn)(r+s)}{mr(k+2)}\cdot\frac{2(m+kn+n)(r+2s)}{(m+n)(r+s)(k+3)}\cdot\frac{3(m+kn+2n)(r+3s)}{(m+2n)(r+2s)(k+4)}$$

sit

ſit porro $r = 1$, et $s = 1$, erit:

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{1}{k(k+1)n} \cdot \frac{2(m+kn)}{m(k+2)} \cdot \frac{3(3m+kn+n)}{(m+n)(k+3)} \frac{4(m+kn+2n)}{(m+2n)(k+4)} \text{ etc.}$$

ſin auten ponatur $r = m + kn$, et $s = n$, erit:

$$\int x^{m-1} dx (1-x^n)^{k-1} = \frac{m+kn}{k(k+1)nn} \cdot \frac{1(m+kn+n)}{m(k+2)} \cdot \frac{2(m+kn+2n)}{(m+n)(k+3)} \frac{3(m+kn+3n)}{(m+2n)(k+4)} \text{ etc.}$$

Coroll. 6.

24. Si manente exponente k reliquos exponentes m et n mutemus, habebimus

$$\int x^{\mu-1} dx (1-x^\nu)^{k-1} = \frac{1}{\mu} \cdot \frac{1(\mu+k\nu)}{(\mu+\nu)k} \cdot \frac{2(\mu+k\nu+\nu)}{(\mu+2\nu)(k+1)} \cdot \frac{3(\mu+k\nu+2\nu)}{(\mu+3\nu)(k+2)} \text{ etc.}$$

dummodo μ, ν et k ſint numeri affirmatiui. Diuiſa ergo illa forma per hanc obtinebimus:

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{\mu-1} dx (1-x^\nu)^{k-1}} = \frac{\mu}{m} \cdot \frac{(\mu+\nu)(m+kn)}{(m+n)(\mu+k\nu)} \cdot \frac{(\mu+2\nu)(m+kn+n)}{(m+2n)(\mu+k\nu+\nu)} \cdot \frac{(\mu+2\nu)(m+kn+2n)}{(m+3n)(\mu+k\nu+2\nu)} \text{ etc.}$$

Coroll. 7.

25. Sin autem etiam in altera forma k in $\varkappa$ mutetur, habebitur:

$$\frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{m-1} dx (1-x^n)^{\varkappa-1}} = \frac{\mu}{m} \cdot \frac{\varkappa(\mu+\nu)(m+kn)}{k(m+n)(\mu+\varkappa\nu)} \cdot \frac{(\varkappa+1)(\mu+2\nu)(m+kn+n)}{(k+1)(m+2n)(\mu+\varkappa\nu+\nu)} \cdot \frac{(\varkappa+2)(\mu+3\nu)(m+kn+2n)}{(k+2)(m+3n)(\mu+\varkappa\nu+2\nu)} \text{ etc.}$$

poſito poſt integrationem $x = 1$, et exiſtentibus omnibus exponentibus m, n, k ac $\mu, \nu, \varkappa$ affirmatiuis.

Scholion.

26. His conuerſionibus formularum integralium in factores infinitos expoſitis, videamus viciſſim quomodo propoſita huiusmodi expreſſio infinita per factores procedens, ad integrationes formularum caſu, quo $x = 1$, reduci

reduci debeat. Hic autem ante omnia ſpectari debent membra, quae illud productum infinitum conſtituunt, ex quot illa factoribus ſint compoſita: quae membra primum ita comparata eſſe debent, vt infiniteſima in vnitatem abeant. In hunc finem erunt fracta, et ex certo, tam numeratorum, quam denominatorum, numero conſtabunt, et vtrique per ſingula membra ſecundum progreſſionem arithmeticam procedent, ita vt in illis eadem habeatur differentia; etiamſi enim variae partes diuerſas obtineant differentias, eae tamen facile ad eandem reducentur. Cum igitur nihil obſtet, quo minus haec differentia vnitati aequalis conſtituatur, pro diuerſo factorum cuiusque membri numero, ſequentes huiusmodi productorum infinitorum ordines habebimus:

$$\frac{a}{b}\cdot\frac{a+1}{b+1}\cdot\frac{a+2}{b+2}\cdot\frac{a+3}{b+3}\cdot\frac{a+4}{b+4}\cdot\frac{a+5}{b+5}\cdot \text{ etc.}$$

$$\frac{ac}{be}\cdot\frac{(a+1)(c+1)}{(b+1)(e+1)}\cdot\frac{(a+2)(c+2)}{(b+2)(e+2)}\cdot\frac{(a+3)(c+3)}{(b+3)(e+3)}\cdot \text{ etc.}$$

$$\frac{acf}{beg}\cdot\frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)}\cdot\frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)}\cdot \text{ etc.}$$

$$\frac{acfh}{begk}\cdot\frac{(a+1)(c+1)(f+1)(h+1)}{(b+1)(e+1)(g+1)(k+1)}\cdot\frac{(a+2)(c+2)(f+2)(h+2)}{(b+2)(e+2)(g+2)(k+2)}\cdot \text{ etc.}$$

Quomodo ergo cuiusque horum productorum valor per formulas integrales exprimendus ſit, videamus.

Pro-

Problema 3.

27. Per formulas integrales definire valorem huius producti infiniti ex membris simplicibus constantis:

$$P = \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdot \frac{a+3}{b+3} \cdot \frac{a+4}{b+4} \cdot \frac{a+5}{b+5} \cdot \text{ etc.}$$

Solutio.

Denotante i numerum infinitum vidimus esse

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m}dx(1-x^n)^{k-1}} = \frac{m+kn}{m} \cdot \frac{m+kn+n}{m+n} \cdot \frac{m+kn+2n}{m+2n} \cdot \text{ etc.}$$

quae forma ad propositam reducetur ponendo $n=1$; $m+k=a$, et $m=b$, vnde fit $k=a-b$. Cum ergo k debeat esse numerus affirmatiuus, si fuerit $a>b$, erit

$$P = \frac{\int x^{b-1}dx(1-x)^{a-b-1}}{\int x^i dx(1-x)^{a-b-1}} = \frac{\int x^{a-b-1}dx(1-x)^{b-1}}{\int x^{a-b-1}dx(1-x)^i}$$

sin autem sit $b>a$, erit inuerse:

$$P = \frac{\int x^i dx(1-x)^{b-a-1}}{\int x^{a-1}dx(1-x)^{b-a-1}} = \frac{\int x^{b-a-1}dx(1-x)^i}{\int x^{b-a-1}dx(1-x)^{a-1}}. \text{ Q. E. I.}$$

Corollarium.

28. Manifestum autem est, si sit $a>b$, valorem P fore infinitum, sin autem sit $b>a$, fore $P=0$. Casu autem $a=b$ fit $P=1$: qui casus cum ad vtrumque expositorum aeque pertineat, euidens est, esse $\int\frac{x^{a-1}dx}{1-x} = \int\frac{x^i dx}{1-x}$ quae integralia casu $x=1$ vtique fiunt ita infinita, vt rationem aequalitatis obtineant. Est autem in genere $\int\frac{x^{a-1}dx}{1-x} = \int\frac{x^{b-1}dx}{1-x}$.

Problema 4.

29. Per formulas integrales definire valorem huius producti infiniti ex membris duplicatis constantis:

$$P=\frac{ac}{be}\cdot\frac{(a+1)(c+1)}{(b+1)(e+1)}\cdot\frac{(a+2)(c+2)}{(b+2)(e+2)}\cdot\frac{(a+3)(c+3)}{(b+3)(e+3)}.\text{ etc.}$$

Solutio.

Cum sit per §. 24. denotantibus m, n, k, μ, ν numeros positiuos

$$\frac{(\mu+\nu)(m+kn)}{(m+n)(\mu+k\nu)}\cdot\frac{(\mu+2\nu)(m+kn+n)}{(m+2n)(\mu+k\nu+\nu)}\text{ etc.}=\frac{m}{\mu}\cdot\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{k-1}}$$

ponatur $n=1$; $\nu=1$; $\mu+1=a$; $m+k=c$, $m+1=b$; $\mu+k=e$; erit $\mu=a-1$; $m=b-1$; et $k=c-b+1=e-a+1$. Quare, quo haec forma ad propositam possit reuocari, necesse est, vt sit $c-b=e-a$; nisi enim haec conditio locum habeat, valor producti propositi P esset vel infinitus, vel euanescens. Quod incommodum ne locum habeat, sit $c-b=e-a$, seu $a+c=b+e$; atque dum sint $a-1$; $b-1$; et $c-b$, vel $e-a$, numeri affirmatiui, erit:

$$P=\frac{b-1}{a-1}\cdot\frac{\int x^{b-2}dx(1-x)^{c-b}}{\int x^{a-2}dx(1-x)^{e-a}}.$$

Vel consideretur haec forma:

$$\frac{\mu(m+kn-n)}{m(\mu+k\nu-\nu)}\cdot\frac{(\mu+\nu)(m+kn)}{(m+n)(\mu+k\nu)}\text{. etc.}=\frac{m+kn-n}{\mu+k\nu-\nu}\cdot\frac{\int x^{m-1}dx(1-x^n)^{k-2}}{\int x^{\mu-1}dx(1-x^n)^{k-2}}$$

quae ex illa luculenter nascitur, ac ponatur $n=1$; $\nu=1$, $\mu=a$; $m=b$; $c=m+k-1$; et $e=\mu+k-1$; erit-

eritque $k-1=c-b=e-a$; iterum ergo esse debet $a+c=b+e$. Nunc ergo, dummodo sint a, b, et $c-b+1$, vel $e-a+1$, numeri positiui, erit

$$P=\frac{c}{e}\cdot\frac{\int x^{b-1}dx(1-x)^{c-b}}{\int x^{a-1}dx(1-x)^{e-a}}.$$

Quoties ergo fuerit $a+c=b+e$, valor quaesitus P est finitus: ac per has formulas integrales casu $x=1$ innotescit. Q. E. I.

Coroll. 1.

30. Cum sit $a+c=b+e$, si sit $c>b$, erit quoque $e>a$, et a et b in primo membro $\frac{ac}{be}$ denotant factores minores numeratoris et denominatoris. Requiritur autem tantum, vt $c-b+1$ sit numerus positiuus. Quare si etiam $c-e+1$ sit numerus positiuus, alio insuper modo valor quaesitus P exprimi poterit: scilicet permutandis b et e hoc modo:

$$P=\frac{c}{b}\cdot\frac{\int x^{e-1}dx(1-x)^{c-e}}{\int x^{a-1}dx(1-x)^{b-a}}.$$

Coroll. 2.

31. Atque quaelibet harum formularum locum habebit:

$$P=\frac{c}{b}\cdot\frac{\int x^{e-1}dx(1-x)^{c-e}}{\int x^{a-1}dx(1-x)^{b-a}}=\frac{c}{e}\cdot\frac{\int x^{b-1}dx(1-x)^{c-b}}{\int x^{a-1}dx(1-x)^{e-a}}=\frac{a}{b}\cdot\frac{\int x^{e-1}dx(1-x)^{a-c}}{\int x^{c-1}dx(1-x)^{b-c}}=\frac{a}{e}\cdot\frac{\int x^{b-1}dx(1-x)^{a-b}}{\int x^{c-1}dx(1-x)^{e-c}}$$

Quarum prima locum habet, si $c-e+1=b-a+1$ sit >0, secunda si $c-b+1=e-a+1>1$, tertia si $a-e+1=b-c+1>0$, et quarta, si $a-b+1=e-c+1>0$.

 Coroll.

Coroll. 3.

32. Prima forma et quarta simul valebunt, si differentia inter a et b sit vnitate minor, ideoque et inter c et e. Atque omnes quatuor simul locum habebunt, si insuper differentia inter a et e fuerit vnitate minor.

Coroll. 4.

33. Si ergo ponatur $a = p + m$; $b = p + n$; $c = p - m$ et $e = p - n$, vt sit $a + c = b + e = 2p$, fueritque $m + n < 1$, erit

$$P = \frac{p-m}{p+n} \cdot \frac{\int x^{p-n-1} dx (1-x)^{n-m}}{\int x^{p+m-1} dx (1-x)^{n-m}} = \frac{p+m}{p+n} \cdot \frac{\int x^{p-n-1} dx (1-x)^{n+m}}{\int x^{p-m-1} dx (1-x)^{n+m}}$$

$$P = \frac{p-m}{p-n} \cdot \frac{\int x^{p+n-1} dx (1-x)^{-n-m}}{\int x^{p+m-1} dx (1-x)^{-n-m}} = \frac{p+m}{p-n} \cdot \frac{\int x^{p+n-1} dx (1-x)^{m-n}}{\int x^{p-m-1} dx (1-x)^{m-n}}.$$

Atque hae quatuor formulae inter se erunt aequales.

Problema 5.

34. Per formulas integrales exprimere valorem huius producti infiniti ex membris triplicatis constantes.

$$P = \frac{acf}{beg} \cdot \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \cdot \text{etc.}$$

Solutio.

Cum inuenerimus §. 25.

$$\frac{m(\mu+\nu)(m+kn)}{k(m+n)(\mu+k\nu)} \cdot \frac{(k+1)(\mu+2\nu)(m+kn+n)}{(k+1)(m+2n)(\mu+k\nu+\nu)} \cdot \text{etc.} = \frac{m}{\mu} \frac{\int x^{m-1} dx (1-x^n)^{k-1}}{\int x^{\mu-1} dx (1-x^\nu)^{k-1}}$$

erit

erit membrum anterius adhuc adiiciendo:

$$\frac{(\varkappa-1)\mu(m+kn-n)}{(k-1)m(\mu+\varkappa\nu-\nu)}\cdot\frac{\varkappa(\mu+\nu)(m+kn)}{k(m+n)(\mu+\varkappa\nu)}\cdot \text{etc.} = \frac{(\varkappa-1)(m+kn-n)}{(k-1)(\mu+\varkappa\nu+\nu)}\cdot\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{\varkappa-1}}$$

quae forma quo ad propositam reducatur, statuatur:

$\varkappa-1=a$; $k-1=b$; $\mu=c$; $m=e$; $n=1$; $\nu=1$;

ac $m+k-1=e+b+f$; $\mu+\varkappa-1=c+a=g$.

Cum ergo haec reductio non succedat, nisi ista conditione sit $f=b+e$ et $g=a+c$; vt habeatur hoc productum infinitum:

$$P=\frac{ac(b+e)}{be(a+c)}\cdot\frac{(a+1)(c+1)(b+e+1)}{(b+1)(e+1)(a+c+1)}\cdot\frac{(a+2)(c+2)(b+e+2)}{(b+2)(e+2)(a+c+2)}\text{etc.}$$

Quare cum hoc casu sit $m=e$; $k=b+1$; $\mu=c$ et $\varkappa=a+1$, existentibus $n=\nu=1$, erit

$$P=\frac{a(b+e)}{b(a+c)}\cdot\frac{\int x^{e-1}dx(1-x)^b}{\int x^{c-1}dx(1-x)^a}$$

dummodo sint c, e, $b+1$ et $a+1$ numeri positiui. Q. E. I.

Coroll. I.

35. Cum per §. 9. sit $\int x^{\alpha-1}dx(1-x)^{\gamma-1}=\frac{\alpha+\gamma}{\alpha}\int x^\alpha dx(1-x)^{\gamma-1}$, erit

$\int x^{e-1}dx(1-x)^b=\frac{b+e+1}{e}\int x^e dx(1-x)^b$, ideoque

$$P=\frac{ac(b+e)(b+e+1)}{be(a+c)(a+c+1)}\cdot\frac{\int x^e dx(1-x)^b}{\int x^c dx(1-x)^a}$$

Et cum sit $\int x^{\alpha-1}dx(1-x)^\gamma=\frac{\gamma}{\alpha+\gamma}\int x^{\alpha-1}dx(1-x)^{\gamma-1}$, erit $\int x^{e-1}dx(1-x)^b=\frac{b}{b+e}\int x^{e-1}dx(1-x)^{b-1}$, habebitur quoque

$$P=\frac{\int x^{e-1}dx(1-x)^{b-1}}{\int x^{c-1}dx(1-x)^{a-1}}$$

Coroll.

Coroll. 2.

36. Formula haec autem locum habet, dummodo a, b, c et e ſint numeri affirmatiui: et quia iam a et c item b et e inter ſe permutari poſſunt, erit quoque $P = \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{c-1}}$: quae conuerſio autem ex Th. 2. per ſe eſt manifeſta.

Scholion 1.

37. Problema ergo propoſitum non in genere eſt ſolutum, ſed tantum caſu, quo $f = b + e$; et $g = a + c$; ſicque ſolutio noſtra duplici limitatione reſtringitur. Vnica vero tantum reſtrictione eſt opus, ne valor ipſius P vel fiat infinitus, vel euaneſcens, qua requiritur, vt ſit $a + c + f = b + e + g$. Quo autem problema pro hac vnica limitatione ſoluatur, neceſſe eſt plures formulas integrales in computum ducere, quod hoc modo praeſtari poterit. Poſito igitur $a + c + f = b + e + g$, cum ſit

$$P = \frac{acf}{beg} \cdot \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \cdot \text{etc.}$$

ſtatuatur $P = QR$, ſitque

$$Q = \frac{(p+q)(p-q)}{(p+r)(p-r)} \cdot \frac{(p+q+1)(p-q+1)}{(p+r+1)(p-r+1)} \cdot \text{etc.} \frac{p+q}{p+r} \cdot \frac{\int x^{p-r-1} dx (1-x)^{q+r}}{\int x^{p-q-1} dx (1-x)^{q+r}}$$

$$\text{et } R = \frac{\alpha\gamma(\beta+\varepsilon)}{\beta\varepsilon(\alpha+\gamma)} \cdot \frac{(\alpha+1)(\gamma+1)(\beta+\varepsilon+1)}{(\beta+\varepsilon)(\varepsilon+1)(\alpha+\gamma+1)} \text{ etc.} = \frac{\int x^{\beta-1} dx (1-x)^{\varepsilon-1}}{\int x^{\alpha-1} dx (1-x)^{\gamma-1}}$$

Fiat iam primum membrum producti QR aequale primo membro formae propoſitae P, ſcilicet

$\frac{\alpha\gamma}{\beta\varepsilon}$

$\frac{\alpha\gamma(\beta+\varepsilon)(p+q)(p-q)}{\beta\varepsilon(\alpha+\gamma)(p+r)(p-r)} = \frac{acf}{beg}$, quod pluribus modis fieri poteſt. Primum enim illud pluribus modis ad tres factores poteſt reduci, ponatur ſcilicet $\beta+\varepsilon = p+r$ et $\alpha+\gamma = p+q$; vt habeatur $q = \alpha+\gamma-p$; et $r = \beta+\varepsilon-p$; eritque:

$$\frac{\alpha\gamma(2p-\alpha-\gamma)}{\beta\varepsilon(2p-\beta-\varepsilon)} = \frac{acf}{beg}.$$

Quod ſi ergo ſtatuatur $\alpha = a$; $\beta = b$; $\gamma = c$; $\varepsilon = e$; et $2p = a+c+f = b+e+g$; erit $q = a+c-p$; et $r = b+e-p$. Sicque nulla alia reſtrictio hic inuoluitur, niſi vt ſit $a+c+f = b+e+g = 2p$. Pro hoc ergo caſu erit producti infiniti propoſiti valor

$$P = \frac{a+c}{b+e} \cdot \frac{\int x^{2p-b-e-1}dx(1-x)^{a+b+c+e-2p}}{\int x^{2p-a-c-1}dx(1-x)^{a+b+c+e-2p}} \cdot \frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}}$$

vbi iam tam litteras a et c, quam b et e pro lubitu inter ſe permutari licet.

Alio modo ſtatuatur $\gamma = p+r$ et $\varepsilon = p-q$, vt ſit

$$\frac{\alpha(\beta+\varepsilon)(p+q)}{\beta(\alpha+\gamma)(p-r)} = \frac{acf}{beg}.$$

Iam ſit $\alpha = a$; $\beta = b$; $\varepsilon = c-b$; $\gamma = e-a$; erit $q = p-c+b$ et $r = e-a-p$; hincque $f = 2p-c+b$ et $g = 2p-e+a$. Sin autem ponatur ſumma $a+c+f = b+e+g = s$; erit $a+b+2p = s$, et $2p = s-a-b$; ſicque $p+q = s-a-c = f$; $p-q = c-b$; $p+r = e-a$; $p-r = s-b-e = g$; et $q+r = b+e-a-c$. Atque hinc oritur:

$$P = \frac{s-a-c}{e-a} \cdot \frac{\int x^{s-b-e-1}dx(1-x)^{b+e-a-c}}{\int x^{c-b-1}dx(1-x)^{b+e-a-c}} \cdot \frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}}$$

Vbi iterum tam litteras a et c, quam b et e inter ſe permutare licet. Vel erit etiam ob plures valores ipſius Q

$$P = \frac{c-b}{e-a} \cdot \frac{\int x^{g-1} dx (1-x)^{c+e-s}}{\int x^{f-1} dx (1-x)^{c+e-s}} \cdot \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

At formula prius inuenta, ponendo s pro $2p$, abit in hanc,

$$P = \frac{a+c}{b+e} \cdot \frac{\int x^{g-1} dx (1-x)^{b+e-f}}{\int x^{f-1} dx (1-x)^{b+e-f}} \cdot \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{c-1}}$$

Scholion 2.

38. Quodſi iam omnes iſtae permutationes adhibeantur, quae pro formula Q obtinent, atque formula propoſita fuerit:

$$P = \frac{acf}{beg} \cdot \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \cdot \text{ etc.}$$

fueritque $a+c+f = b+e+g$, reperientur ſequentes valores pro valore P, ſcilicet:

$$P = \frac{f}{g} \cdot \frac{\int x^{e-a-1} dx (1-x)^{a+f-e}}{\int x^{c-b-1} dx (1-x)^{a+f-e}} \cdot \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{f}{e-a} \cdot \frac{\int x^{g-1} dx (1-x)^{f-g}}{\int x^{c-b-1} dx (1-x)^{f-g}} \cdot \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{c-b}{g} \cdot \frac{\int x^{e-a-1} dx (1-x)^{g-f}}{\int x^{f-1} dx (1-x)^{g-f}} \cdot \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{c-b}{e-a} \cdot \frac{\int x^{g-1} dx (1-x)^{e-a-f}}{\int x^{f-1} dx (1-x)^{e-a-f}} \cdot \frac{\int x^{b-1} dx (1-x)^{c-b-1}}{\int x^{a-1} dx (1-x)^{e-a-1}}$$

$$P = \frac{f}{g} \cdot \frac{\int x^{b+e-1} dx (1-x)^{f-b-e}}{\int x^{a+c-1} dx (1-x)^{f-b-e}} \cdot \frac{\int x^{b-1} dx (1-x)^{e-1}}{\int x^{a-1} dx (1-x)^{c-1}}$$

P =

$$P=\frac{f}{b+e}\cdot\frac{\int x^{g-1}dx(1-x)^{f-g}}{\int x^{a+c-1}dx(1-x)^{f-g}}\cdot\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}}$$

$$P=\frac{a+c}{g}\cdot\frac{\int x^{b+e-1}dx(1-x)^{g-f}}{\int x^{f-1}dx(1-x)^{g-f}}\cdot\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}}$$

$$P=\frac{a+c}{b+e}\cdot\frac{\int x^{g-1}dx(1-x)^{b+e-f}}{\int x^{f-1}dx(1-x)^{b+e-f}}\cdot\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}}$$

Porro autem hic tam ternas litteras a, c, f, quam has b, e, g, pro lubitu inter ſe permutare licet, ex quo maxima copia formularum, quae omnes eidem valori P ſunt aequales, enaſcetur.

Scholion 3.

39. Hinc etiam pro producto ſimpliciori

$$P=\frac{ac}{be}\cdot\frac{(a+1)(c+1)}{(b+1)(e+1)}\cdot\frac{(a+2)(c+2)}{(b+2)(e+2)}\cdot \text{ etc.}$$

ſi fuerit $a+c=b+e$, praeter valores ſupra inuentos plures alii exhiberi poterunt. Primum enim quia $a+c=b+e$ valor in problemate 5 inuentus huc pertinet: $P=\frac{\int x^{e-1}dx(1-x)^{b-1}}{\int x^{c-1}dx(1-x)^{a-1}}$. Deinde ſi in ſerie §. praec. vna litterarum a, c, f vni ex b, e, g aequalis ſtatuatur, vel haec eadem expreſſio, vel aliae obtinebuntur, quae cum praecedentibus erunt:

$$P=\frac{\int x^{e-1}dx(1-x)^{a-e-1}}{\int x^{c-1}dx(1-x)^{b-c-1}};\quad P=\frac{\int x^{b-1}dx(1-x)^{a-b-1}}{\int x^{c-1}dx(1-x)^{e-c-1}}$$

$$P=\frac{\int x^{e-1}dx(1-x)^{c-e-1}}{\int x^{a-1}dx(1-x)^{b-a-1}};\quad P=\frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}}$$

$$P=\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}};\quad P=\frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}}$$

vbi eſt $e-a=c-b$ et $c-e=b-a$.

In ſequentibus eſt n numerus arbitrarius:

$$P=\frac{\int x^{e-n-1}dx(1-x)^{n+a-e-1}}{\int x^{c-n-1}dx(1-x)^{n+b-c-1}}\cdot\frac{\int x^{n-1}dx(1-x)^{c-n-1}}{\int x^{n-1}dx(1-x)^{e-n-1}}$$

$$P=\frac{\int x^{n+b-1}dx(1-x)^{c-b-n-1}}{\int x^{n+a-1}dx(1-x)^{e-b-n-1}}\cdot\frac{\int x^{n-1}dx(1-x)^{b-1}}{\int x^{n-1}dx(1-x)^{a-1}}$$

$$P=\frac{\int x^{e-1}dx(1-x)^{n+b-e-1}}{\int x^{c-1}dx(1-x)^{n+b-c-1}}\cdot\frac{\int x^{n-1}dx(1-x)^{b-1}}{\int x^{n-1}dx(1-x)^{a-1}}$$

$$P=\frac{\int x^{n-1}dx(1-x)^{a-1}}{\int x^{c-1}dx(1-x)^{a-1}}\cdot\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{n-1}}=\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{c-1}dx(1-x)^{a-1}}$$

quae poſtrema iam in praecedentibus continetur. Hic autem monendum eſt, ſuperfluum eſſe hic rationem exponentium definire, vti ſupra factum eſt. Cum enim valor P certo ſit finitus, ſi $a+c=b+e$; ſi quaepiam formularum integralium habeat exponentes negatiuos infra -1; tum eam ad exponentes maiores reducere licet, ac tum verus valor ipſius P obtinebitur. Formulae autem ſimpliciores continentur in hoc Theoremate:

Theorema 4.

40. Si fuerit $a+c=b+e=s$; tum erit

$$\frac{\int x^{a-1}dx(1-x)^{c-1}}{\int x^{b-1}dx(1-x)^{e-1}}=\frac{\int x^{a-1}dx(1-x)^{s-a-b-1}}{\int x^{b-1}dx(1-x)^{s-a-b-1}}$$

ſi quidem poſt integrationem ſtatuatur $x=1$.

Demon-

Demonſtratio.

Eſt enim ex praecedentibus formulis

$$\frac{\int x^{a-1}dx(1-x)^{c-1}}{\int x^{b-1}dx(1-x)^{e-1}} = \frac{\int x^{a-1}dx(1-x)^{c-b-1}}{\int x^{b-1}dx(1-x)^{e-a-1}}.$$

At ob $a+c=b+e=s$, eſt $c=s-a$; et $e=s-b$; vnde erit $c-b=e-a=s-a+b$, vnde forma propoſita conficitur. Q. E. D.

Coroll. 1.

41. Hic licet tam numeros a et c, quam b et e inter ſe permutare, vnde quatuor obtinentur formulae primae aequales; ſcilicet ſingulae harum formularum:

$$\frac{\int x^{a-1}dx(1-x)^{s-a-b-1}}{\int x^{b-1}dx(1-x)^{s-a-b-1}};\quad \frac{\int x^{a-1}dx(1-x)^{s-a-e-1}}{\int x^{e-1}dx(1-x)^{s-a-e-1}}$$

$$\frac{\int x^{c-1}dx(1-x)^{s-b-c-1}}{\int x^{b-1}dx(1-x)^{s-b-c-1}};\quad \frac{\int x^{c-1}dx(1-x)^{s-c-e-1}}{\int x^{e-1}dx(1-x)^{s-c-e-1}}$$

aequales ſunt huic formae $\frac{\int x^{a-1}dx(1-x)^{c-1}}{\int x^{b-1}dx(1-x)^{e-1}}$.

Coroll. 2.

42. Valor autem vniuscuiusque harum formularum aequalis eſt huic producto ex factoribus infinitis conſtanti:

$$\frac{be}{ac}\cdot\frac{(b+1)(e+1)}{(a+1)(c+1)}\cdot\frac{(b+2)(e+2)}{(a+2)(c+2)}\cdot \text{etc.}$$

Coroll. 3.

43. Si sit $e = 1$; ideoque $b = s - 1$; $d = s - c$; erit, posito

$$P = \frac{1(s-1)}{c(s-c)} \cdot \frac{2\,s}{(c+1)(s-c+1)} \cdot \frac{3(s+1)}{(c+2)(s-c+2)} \cdot \text{etc.}$$

ob $\int x^{b-1} dx (1-x)^{e-1} = \int x^{s-2} dx = \frac{1}{s-1}$,

$$P = (s-1) \int x^{s-c-1} dx (1-x)^{c-1}$$

$$P = (c-1) \int x^{s-c-1} dx (1-x)^{c-2} = (s-1)$$

$$P = (s-c-1) \int x^{c-1} dx (1-x)^{s-c-2}$$

$$P = \frac{\int x^{s-c-1} dx (1-x)^{c-s}}{\int x^{s-2} dx (1-x)^{c-s}} = \frac{\int x^{c-1} dx (1-x)^{-c}}{\int x^{s-2} dx (1-x)^{-c}} = (s-1) \int x^{s-c-1} dx (1-x)^{c-2}.$$

Scholion.

44. Quoniam vero huiusmodi formularum integralium comparationes iam plures exposui, hic imprimis nonnullos valores prae reliquis notabiles persequar, et quemadmodum ii per formulas integrales exprimi queant, ostendam. Notatu autem potissimum digna sunt illa producta infinita, quibus sinus et cosinus cuiusque anguli exprimitur. Denotante enim ϱ angulum rectum, et Φ angulum quemcunque, constat esse

$$\sin.\Phi = \Phi \left(1 - \frac{\Phi\Phi}{4\varrho\varrho}\right)\left(1 - \frac{\Phi\Phi}{16\varrho\varrho}\right)\left(1 - \frac{\Phi\Phi}{36\varrho\varrho}\right)\left(1 - \frac{\Phi\Phi}{64\varrho\varrho}\right) \text{etc.}$$

et $\cos.\Phi = \left(1 - \frac{\Phi\Phi}{\varrho\varrho}\right)\left(1 - \frac{\Phi\Phi}{9\varrho\varrho}\right)\left(1 - \frac{\Phi\Phi}{25\varrho\varrho}\right)\left(1 - \frac{\Phi\Phi}{49\varrho\varrho}\right)$ etc.

Quodsi iam ponatur $\Phi = \frac{m}{n}\varrho$, erit

$$\left(1 - \frac{mm}{4nn}\right)\left(1 - \frac{mm}{16nn}\right)\left(1 - \frac{mm}{36nn}\right) \text{etc.} = \frac{n}{m\varrho} \sin.\frac{m}{n}\varrho$$

$$\left(1 - \frac{mm}{nn}\right)\left(1 - \frac{mm}{9nn}\right)\left(1 - \frac{mm}{25nn}\right) \text{etc.} = \cos.\frac{m}{n}\varrho$$

Vel

Vel si angulus duobus rectis aequalis π introducatur, et ob $\varrho = \frac{1}{2}\pi$ pro m scribatur $2m$, erit factores euoluendo

$$\frac{(n-m)(n+m)}{n\,.\,n} \cdot \frac{(2n-m)(2n+m)}{2n\,.\,2n} \cdot \frac{(3n-m)(3n+m)}{3n\,.\,3n} \text{ etc.} = \frac{n}{m\pi} \text{ sin. } \frac{m}{n}\pi$$

$$\frac{(n-2m)(n+2m)}{n\,.\,n} \cdot \frac{(3n-2m)(3n+2m)}{3n\,.\,3n} \cdot \frac{(5n-2m)(5n+2m)}{5n\,.\,5n} \text{ etc.} = \text{cos. } \frac{m}{n}\pi.$$

Reducendo autem differentias ad vnitatem; erit

$$\frac{(1-\frac{m}{n})(1+\frac{m}{n})}{1\,.\,1} \cdot \frac{(2-\frac{m}{n})(2+\frac{m}{n})}{2\,.\,2} \cdot \frac{(3-\frac{m}{n})(3+\frac{m}{n})}{3\,.\,3} \text{ etc.} = \frac{n}{m\pi} \text{ sin. } \frac{m}{n}\pi$$

$$\frac{(\frac{1}{2}-\frac{m}{n})(\frac{1}{2}+\frac{m}{n})}{\frac{1}{2}\,.\,\frac{1}{2}} \cdot \frac{(\frac{3}{2}-\frac{m}{n})(\frac{3}{2}+\frac{m}{n})}{\frac{3}{2}\,.\,\frac{3}{2}} \text{ etc.} = \text{cos. } \frac{m}{n}\pi.$$

Problema 6.

45. Inuenire formulam integralem, cuius valor casu $x = 1$ praebeat sin. $\frac{m}{n}\pi$.

Solutio.

Cum sit

$$\frac{n}{m\pi} \text{ sin } \frac{m}{n}\pi = \frac{(1-\frac{m}{n})(1+\frac{m}{n})}{1\,.\,1} \cdot \frac{(2-\frac{m}{n})(2+\frac{m}{n})}{2\,.\,2} \text{ etc.}$$

comparetur hoc productum infinitum cum forma generali

$$P = \frac{be}{ac} \cdot \frac{(b+1)(e+1)}{(a+1)(c+1)} \cdot \frac{(b+2)(e+2)}{(a+2)(c+2)} \text{ . etc.}$$

cuius valor ante §. 41. pluribus modis in formulis integralibus est exhibitus. Statui ergo oportet $a = 1$; $c = 1$; $b = 1 - \frac{m}{n}$; et $e = 1 + \frac{m}{n}$: eritque $s = a + c = b + e = 2$; tum vero

$$s - a - b - 1 = -1 + \frac{m}{n}; \quad s - a - e - 1 = -1 - \frac{m}{n},$$

$$s - b - c - 1 = -1 + \frac{m}{n}; \quad s - c - e - 1 = -1 - \frac{m}{n}.$$

Hinc

Hinc ergo pro P prodit sequens expressio:

$$P = 1 \frac{\int dx(1-x)^0}{\int x^{\frac{-m}{n}} dx (1-x)^{\frac{m}{n}}} = \frac{1}{\int x^{\frac{-m}{n}} dx (1-x)^{\frac{m}{n}}}.$$

ad quam reliquae omnes facile reducuntur. Haec igitur forma dat

$$\int x^{-\frac{m}{n}} dx (1-x)^{\frac{m}{n}} = \int \frac{x^{\frac{m}{n}} dx}{(1-x)^{\frac{m}{n}}} = \frac{m\pi}{n \sin.\frac{m}{n}\pi},$$

et posito $x = y^n$, habebitur

$$\int \frac{y^{m+n-1} dy}{(1-y^n)^{\frac{m}{n}}} = \frac{m\pi}{nn \sin.\frac{m}{n}\pi} \text{ seu } \int \frac{y^{m-1} dy}{(1-y^n)^{\frac{m}{n}}} = \frac{\pi}{n \sin.\frac{m}{n}\pi}.$$

Inuenimus ergo $\sin.\frac{m}{n}\pi = \frac{\pi}{n} : \int \frac{y^{m-1} dy}{(1-y^n)^{\frac{m}{n}}}$. Q. E. I.

Coroll. 1.

46. Per Theorema ergo primum haec forma $\int y^{m-1} dy (1-y^n)^{-\frac{m}{n}}$ ob $k = -\frac{m}{n}$ conuertitur in hanc $\int \frac{y^{m-1} dy}{1+y^n}$, ideoque habebitur

$$\int \frac{y^{m-1} dy}{1+y^n} = \frac{\pi}{n \sin.\frac{m}{n}\pi} \text{ casu } y = \infty,$$

quae forma ob simplicitatem imprimis est notatu digna.

Coroll. 2.

47. Habemus ergo has duas aequalitates admodum notabiles:

$$\frac{m\pi}{n \sin.\frac{m}{n}\pi} = \int \frac{my^{m-1} dy}{(1-y^n)^{\frac{m}{n}}} \text{ posito } y = 1$$

$$\text{et } \frac{m\pi}{n \sin.\frac{m}{n}\pi} = \int \frac{my^{m-1} dy}{1+y^n} \text{ posito } y = \infty$$

quibus

quibus igitur casibus vtriusque formulae integrale satis commode exhiberi potest.

Coroll. 3.

48. Cum ergo posito $x=1$, et $y=\infty$, sit

$$\frac{\pi}{n\,\text{sin}\,\frac{m}{n}\pi}=\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}}=\int\frac{y^{m-1}dy}{1+y^n}$$

si pro m scribatur $2in+m$, ob sin. $\frac{2in+m}{n}\pi=\text{sin.}\frac{m}{n}\pi$, erit quoque

$$\int\frac{x^{2in+m-1}dx}{(1-x^n)^{\frac{2in+m}{2}}}=\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}}=\int\frac{\pi}{n\,\text{sin.}\frac{m}{n}\pi}\ \text{et}$$

$$\int\frac{y^{2in+m-1}dy}{1+y^n}=\int\frac{y^{m-1}dy}{1+y^n}=\frac{\pi}{n\,\text{sin.}\frac{m}{n}\pi}$$

denotante i numerum integrum quemcunque.

Coroll. 4.

49. Quia porro denotante i numerum integrum quemcunque, si pro m scribatur $2in-m$, est sin. $\frac{2in-m}{n}\pi$ $=-$sin. $\frac{m}{n}\pi$, erit

$$\int\frac{x^{2in-m-1}dx}{(1-x^n)^{\frac{2in-m}{n}}}=-\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}}=-\frac{\pi}{n\,\text{sin.}\frac{m}{n}\pi}\ \text{et}$$

$$\int\frac{y^{2in-m-1}dy}{1+y^n}=-\int\frac{y^{m-1}dy}{1+y^n}=-\frac{\pi}{n\,\text{sin.}\frac{m}{n}\pi}$$

 Deinde

Deinde si pro m scribatur $(2i-1)n-m$, ob sin. $\frac{(2i-1)n-m}{n}\pi = \sin\frac{m}{n}\pi$, erit

$$\int\frac{x^{(2i-1)n-m-1}dx}{(1-x^n)^{\frac{(2i-1)n-m}{n}}} = \int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = \frac{\pi}{n\sin.\frac{m}{n}\pi}$$

$$\int\frac{y^{(2i-1)n-m-1}dy}{1+y^n} = \int\frac{y^{m-1}dy}{1+y^n} = \frac{\pi}{n\sin.\frac{m}{n}\pi}.$$

Denique erit eodem modo

$$\int\frac{x^{(2i-1)n+m-1}dx}{(1-x^n)^{\frac{(2i-1)n+m}{n}}} = -\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = -\frac{\pi}{n\sin.\frac{m}{n}\pi}$$

$$\int\frac{y^{(2i-1)n+m-1}dy}{1+y^n} = -\int\frac{y^{m-1}dy}{1+y^n} = -\frac{\pi}{n\sin.\frac{m}{n}\pi}.$$

Coroll. 5.

50. Cum formula integralis $\int\frac{y^{m-1}dy}{1+y^n}$ saepius occurrat, operae pretium erit, eius valores pro praecipuis casibus exponere, posito $y=\infty$. Erit ergo

$$\int\frac{dy}{1+yy} = \frac{\pi}{2\sin.\frac{\pi}{2}} = \frac{\pi}{2} \quad \text{ob} \quad \sin.\frac{\pi}{2} = 1$$

$$\int\frac{dy}{1+y^3} = \frac{\pi}{3\sin.\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} \quad \text{ob} \quad \sin.\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\int\frac{y\,dy}{1+y^3} = \frac{\pi}{3\sin.\frac{2\pi}{3}} = \frac{2\pi}{3\sqrt{3}} \quad \text{ob} \quad \sin.\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

—∫

$$\int \frac{dy}{1+y^4} = \frac{\pi}{4 \sin. \frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}}$$

$$\int \frac{y^2\, dy}{1+y^4} = \frac{\pi}{4 \sin. \frac{3\pi}{4}} = \frac{\pi}{2\sqrt{2}}$$

$$\int \frac{dy}{1+y^5} = \int \frac{y^3\, dy}{1+y^5} = \frac{\pi}{5 \sin. \frac{\pi}{5}}$$

$$\int \frac{y\, dy}{1+y^5} = \int \frac{y^2\, dy}{1+y^5} = \frac{\pi}{5 \sin. \frac{2\pi}{5}}$$

$$\int \frac{dy}{1+y^6} = \int \frac{y^4\, dy}{1+y^6} = \frac{\pi}{6 \sin. \frac{\pi}{6}} = \frac{\pi}{3}$$

et ita porro.

Problema 2.

51. Inuenire formulam integralem, cuius valor casu $x=1$ praebeat $\cos. \frac{m}{n}\pi$.

Solutio.

Cum sit

$$\cos. \frac{m}{n}\pi = \frac{(\frac{1}{2}-\frac{m}{n})(\frac{1}{2}+\frac{m}{n})}{\frac{1}{2}\cdot\frac{1}{2}} \cdot \frac{(\frac{3}{2}-\frac{m}{n})(\frac{3}{2}+\frac{m}{n})}{\frac{3}{2}\cdot\frac{3}{2}} \text{ etc.}$$

comparetur cum hoc producto infinito forma generalis

$$P = \frac{b\,e}{a\,c} \cdot \frac{(b+1)(e+1)}{(a+1)(c+1)} \text{ etc.}$$

 hinc

hincque ſtatuatur $a=\frac{1}{2}$; $c=\frac{1}{2}$; $b=\frac{1}{2}-\frac{m}{n}$; $e=\frac{1}{2}+\frac{m}{n}$ ita, vt ſit $s=a+c=b+e=1$, atque

$$s-a-b-1=-1+\frac{m}{n}; \quad s-a-e-1=-1-\frac{m}{n};$$
$$s-b-c-1=-1+\frac{m}{n}; \quad s-c-e-1=-1-\frac{m}{n};$$

Eritque idcirco:

$$P=\frac{\int x^{-\frac{1}{2}}dx(1-x)^{-\frac{1}{2}}}{\int x^{-\frac{1}{2}-\frac{m}{n}}dx(1-x)^{-\frac{1}{2}+\frac{m}{n}}}=\frac{\int dx:\sqrt{(x-xx)}}{\int\frac{x^{\frac{m}{n}-\frac{1}{2}}dx}{(1-x)^{\frac{1}{2}+\frac{m}{n}}}}$$

At eſt $\int\frac{dx}{\sqrt{(x-xx)}}=\pi$, poſito $x=1$.

Vnde fit $P=\operatorname{cof.}\frac{m}{n}\pi=\dfrac{\pi}{\int\frac{x^{\frac{m}{n}-\frac{1}{2}}dx}{(1-x)^{\frac{1}{2}+\frac{m}{n}}}}$

Per reliquas vero formulas ipſius P habebitur:

$$P=\operatorname{cof.}\frac{m}{n}\pi=\frac{\int x^{-\frac{1}{2}}dx(1-x)^{-1+\frac{m}{n}}}{\int x^{-\frac{1}{2}-\frac{m}{n}}dx(1-x)^{-1+\frac{m}{n}}}=\frac{\int x^{-\frac{1}{2}}dx(1-x)^{-1-\frac{m}{n}}}{\int x^{-\frac{1}{2}+\frac{m}{n}}dx(1-x)^{-1-\frac{m}{n}}}$$

Q. E. I.

Coroll. 1.

52. Ponatur $x=y^2$, et prior forma abit in hanc:

$$\operatorname{cof.}\frac{m}{n}\pi=\frac{\pi}{2\int\frac{y^{\frac{2m}{n}}dy}{(1-yy)^{\frac{1}{2}+\frac{m}{n}}}}$$

ita

ita vt sit $\int \frac{x^{\frac{2m}{n}} dx}{(1-xx)^{\frac{1}{2}+\frac{m}{n}}} = \frac{\pi}{2 \operatorname{cos.} \frac{m}{n} \pi}$.

Coroll. 2.

53. Est vero etiam per Theorema primum

$$\int x^{\frac{m}{n}-\frac{1}{2}} dx (1-x)^{-\frac{1}{2}-\frac{m}{n}} = \int \frac{y^{\frac{m}{n}-\frac{1}{2}} dy}{1+y}, \text{ posito } y = \infty,$$

Cum igitur sit $\int \frac{y^{\frac{m}{n}-\frac{1}{2}} dy}{1+y} = \frac{\pi}{\operatorname{cos.} \frac{m}{n} \pi}$ ponatur y^n pro y et erit

$$\int \frac{y^{m+\frac{1}{2}n-1} dy}{1+y^n} = \frac{\pi}{n \operatorname{cos.} \frac{m}{n} \pi} = \int \frac{y^{\frac{1}{2}n-m-1} dy}{1+y^n}.$$

Coroll. 3.

54. Si et reliquae formulae per Theorema primum conuertantur, prodibit:

$$\int x^{-\frac{1}{2}} dx (1-x)^{-1+\frac{m}{n}} = \int \frac{y^{-\frac{1}{2}} dy}{(1+y)^{\frac{1}{2}+\frac{m}{n}}} = \int \frac{y^{\frac{m}{n}-1} dy}{(1+y)^{\frac{1}{2}+\frac{m}{n}}}$$

$$\int x^{-\frac{1}{2}-\frac{m}{n}} dx (1-x)^{-1+\frac{m}{n}} = \int \frac{y^{-\frac{1}{2}-\frac{m}{n}} dy}{\sqrt{(1+y)}} = \int \frac{y^{\frac{m}{n}-1} dy}{\sqrt{(1+y)}}$$

posito $y = \infty$. Posito ergo y^n pro y, erit

$$\operatorname{cos}.\frac{m\pi}{n} = \frac{\int\frac{y^{m-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int\frac{y^{m-1}dy}{\sqrt{(1+y^n)}}} = \frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int\frac{y^{\frac{1}{2}n-m-1}dy}{\sqrt{(1+y^n)}}}.$$

Coroll. 4.

54. Si pro m scribatur $\frac{1}{2}n-m$, ob $\operatorname{cos}(\frac{1}{2}n-m)\frac{\pi}{n} = \operatorname{sin}.\frac{m}{n}\pi$, obtinebitur primum $\frac{\pi}{n \operatorname{sin}\frac{m}{n}\pi} = \int\frac{y^{m-1}dy}{1+y^n}$ vt ante; reliquae vero formulae dabunt:

$$\operatorname{sin}.\frac{m}{n}\pi = \frac{\int\frac{y^{\frac{1}{2}n-m-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{\frac{1}{2}n-m-1}dy}{\sqrt{(1+y^n)}}} = \frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{m-1}dy}{\sqrt{(1+y^n)}}}$$

et quia pro cosinu licet m negatiue sumere erit quoque

$$\operatorname{sin}.\frac{m}{n}\pi = \frac{\int\frac{y^{m-\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{m-\frac{1}{2}n-1}dy}{\sqrt{(1+y^n)}}} = \frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{n-m-1}dy}{\sqrt{(1+y^n)}}}$$

Coroll.

Coroll. 5.

55. At vero etiam ex praeced. problemate aliam formulam pro cosinu licet elicere: Cum enim, posito $2m$ pro m, sit

$$\frac{\pi}{n\,\text{sin.}\frac{2m}{n}\pi} = \frac{\pi}{2n\,\text{sin.}\frac{m}{n}\pi\ \text{cos.}\frac{m}{n}\pi} = \int\frac{y^{2m-1}dy}{1+y^n}.$$

et $\int\frac{y^{m-1}dy}{1+y^n} = \frac{\pi}{n\,\text{sin.}\frac{m}{n}\pi}$ si haec forma per illam dividatur, habebimus:

$$2\,\text{cos.}\frac{m}{n}\pi = \frac{\int\frac{y^{m-1}dy}{1+y^n}}{\int\frac{y^{2m-1}dy}{1+y^n}} \quad \text{seu cos.}\frac{m}{n}\pi = \frac{\frac{1}{2}\int\frac{y^{m-1}dy}{1+y^n}}{\int\frac{y^{2m-1}dy}{1+y^n}}$$

Coroll. 6.

56 En ergo plures formas integrales, quae casu $y=\infty$ praebent sin, $\frac{m}{n}\pi$

I. $\dfrac{\pi}{n\int\frac{y^{m-1}dy}{1+y^n}}$;

II. $\dfrac{\int\frac{y^{\frac{1}{2}n-m-1}dy}{1+y^n}}{2\int\frac{y^{n-2m-1}dy}{1+y^n}}$;

III. $\dfrac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{n-m-1}dy}{\sqrt{(1+y^n)}}}$;

IV. $\dfrac{\int\frac{y^{m-\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{m-\frac{1}{2}n-1}dy}{\sqrt{(1+y^n)}}}$

V.

$$\text{V. } \frac{\int \frac{y^{\frac{1}{2}n-1}\,dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int \frac{y^{m-1}\,dy}{\sqrt{(1+y^n)}}}. \qquad \text{VI. } \frac{\int \frac{y^{\frac{1}{2}n-m-1}\,dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int \frac{y^{\frac{1}{2}n-m-1}\,dy}{\sqrt{(1+y^n)}}}.$$

vbi notandum eſt, in formis III et IV, item in V et VI, ſeorſim numeratores et denominatores inter ſe eſſe aequales.

Coroll. 7.

57. Simili modo totidem habebimus formulas pro cof. $\frac{m}{n}\pi$ quae ſunt:

$$\text{I. } \frac{\pi}{n\int \frac{y^{\frac{1}{2}n-m-1}\,dy}{1+y^n}}; \qquad \text{II. } \frac{\int \frac{y^{m-1}\,dy}{1+y^n}}{2\int \frac{y^{2m-1}\,dy}{1+y^n}}$$

$$\text{III. } \frac{\int \frac{y^{m-1}\,dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int \frac{y^{m-1}\,dy}{\sqrt{(1+y^n)}}} \qquad \text{IV. } \frac{\int \frac{y^{\frac{1}{2}n-1}\,dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int \frac{y^{\frac{1}{2}n-m-1}\,dy}{\sqrt{(1+y^n)}}}$$

$$\text{V. } \frac{\int \frac{y^{-m-1}\,dy}{(1+y^n)^{\frac{1}{2}-\frac{m}{n}}}}{\int \frac{y^{-m-1}\,dy}{\sqrt{(1+y^n)}}}; \qquad \text{VI. } \frac{\int \frac{y^{\frac{1}{2}n-1}\,dy}{(1+y^n)^{\frac{1}{2}-\frac{m}{n}}}}{\int \frac{y^{\frac{1}{2}n+m-1}\,dy}{\sqrt{(1+y^n)}}}.$$

Scho-

Scholion.

58. Hinc vero etiam plures formulas pro tangente anguli $\frac{m}{n}\pi$ deducere licet, quarum quae sunt simpliciores, hic exhibebo.

$$\text{tang.} - \frac{m}{n}\pi = \frac{\int \frac{y^{\frac{1}{2}n-m-1}dy}{1+y^n}}{\int \frac{y^{m-1}dy}{1+y^n}};$$

$$\text{tang.}\,\frac{m}{n}\pi = \frac{\int \frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int \frac{y^{m-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}} = \frac{\int \frac{y^{\frac{1}{2}n-m-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int \frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}$$

Deinde vero ex combinatione harum formularum insignes proprietates innotescent, veluti si $n=4$ et $m=1$, erit

$$\frac{1}{\sqrt{2}} = \frac{\pi}{4\int\frac{dy}{1+y^4}} = \frac{\int\frac{dy}{1+y^4}}{2\int\frac{y\,dy}{1+y^4}} = \frac{\int\frac{y\,dy}{\sqrt{(1+y^4)}}}{\int\frac{yy\,dy}{\sqrt{(1+y^4)}}}$$

$$= \frac{\int \frac{y\,dy}{\sqrt[4]{(1+y^4)^3}}}{\int \frac{dy}{\sqrt{(1+y^4)}}} = \frac{\int \frac{dy}{\sqrt[4]{(1+y^4)^3}}}{\int \frac{dy}{\sqrt{(1+y^4)}}}$$

vnde colligitur fore $\int \frac{y\,dy}{\sqrt[4]{(1+y^4)^3}} = \int \frac{dy}{\sqrt[4]{(1+y^4)^3}}$ casu $y = \infty$, seu esse $\int \frac{(1-y)\,dy}{\sqrt[4]{(1+y^4)^3}} = 0$; talibus autem proprietatibus eruendis hic non immoror.

SOLV-