1761

# Observationes de comparatione arcuum curvarum irrectificibilium

Leonhard Euler

# OBSERVATIONES DE COMPARATIONE ARCVVM CVRVARVM IRRECTIFICABILIVM.

Auctore

*L. EVLERO.*

Speculationes mathematicae, si ad earum vtilitatem respicimus, ad duas classes reduci debere videntur, ad priorem referendae sunt eae, quae cum ad vitam communem, tum ad alias artes, insigne aliquod commodum afferunt, quarum propterea pretium ex magnitudine huius commodi statui solet. Altera autem classis eas complectitur speculationes, quae etsi cum nullo insigni commodo sint coniunctae, tamen ita sunt comparatae, vt ad fines analyseos promouendos, viresque ingenii nostri acuendas occasionem praebeant. Cum enim plurimas inuestigationes, vnde maxima vtilitas expectari posset, ob solum analyseos defectum deserere cogamur, non minus pretium iis speculationibus statuendum videtur, quae haud contemnenda analyseos incrementa pollicentur. Ad hunc autem scopum imprimis accommodatae videntur eiusmodi obseruationes, quae, cum quasi casu sint factae, et a posteriori detectae, ratio ad easdem a priori, ac per viam directam, perueniendi minus, vel neutiquam, est perspecta. Sic enim cognita iam

iam veritate facilius in eas methodos inquirere licebit, quae ad eam directe sint perducturae, nouis autem methodis inuestigandis analyseos fines non mediocriter promoueri, nullum plane est dubium.

Huiusmodi autem obseruationes, quae nulla certa methodo sunt factae, quarumque ratio non parum abscondita videtur, nonnullas deprehendi in opere Ill. Comitis *Fagnani* nuper in lucem edito; quae idcirco omni attentione dignae sunt censendae, neque studium, quod in vlteriori earum inuestigatione consumitur, inutiliter collocatum erit iudicandum. Commemorantur autem in hoc libro quaedam eximiae proprietates, quibus curuae *Ellipsis*, *Hyperbola* et *Lemniscata* sunt praeditae, harumque curuarum arcus diuersi inter se comparantur: cum igitur ratio harum proprietatum maxime occulta videatur, haud alienum fore arbitror, si eas diligentius examinauero, et quae mihi insuper circa has curuas elicere contigit, cum publico communicauero.

Quod igitur primum ad has curuas attinet, notum est, earum rectificationem omnes analyseos vires transcendere, ita vt earum arcus non solum non algebraice exprimi, sed etiam nequidem ad quadraturam circuli, vel hyperbolae, reduci queant. Quare eo magis mirum videri debet, quod Ill. Comes *Fagnani* inuenit, Ellipsi et Hyperbola infinitis modis eiusmodi binos arcus exhiberi posse, quorum differentia geometrice assignari queat; in curua lemniscata autem infinitis modis eiusmodi dari arcus binos, qui inter se vel sint aequales, vel alter ad alterum rationem duplam teneat,

H 2 vnde

vnde deinceps modum colligit in hac curua etiam eiusmodi arcus aſſignandi, qui aliam inter ſe rationem teneant.

Pro Ellipſi quidem et Hyperbola nihil admodum mihi praeterea ſcrutari licuit; vnde contentus ero faciliorem conſtructionem eorum arcuum dediſſe, quorum differentia, geometrice exhiberi queat. Pro curua autem lemniſcata iisdem veſtigiis inſiſtens, multo plures, imo infinitas, elicui formulas, quarum beneficio non ſolum infinitis modis eiusmodi binos acus definire poſſum, qui inter ſe vel ſint aequales, vel rationem teneant duplam, ſed etiam qui ſint inter ſe in ratione quacunque numeri ad numerum.

## I. De Ellipſi.

1. Sit quadrans ellipticus ABC, cuius centrum in C, eiusque ſemiaxes ponantur $CA = 1$, et $CB = c$; ſumta ergo abſciſſa quacunque $CP = x$, erit applicata ei reſpondens $PM = y = c\sqrt{(1 - xx)}$; cuius differentiale cum ſit $dy = -\frac{c\,x\,dx}{\sqrt{(1-xx)}}$, erit abſciſſae $CP = x$ arcus ellipticus reſpondens $BM = \int \frac{dx\sqrt{(1-(1-cc)xx)}}{\sqrt{(1-xx)}}$. Ponatur breuitatis gratia $1 - cc = n$, vt ſit arcus $BM = \int dx \sqrt{\frac{1-nxx}{1-xx}}$ ſumtaque alia quauis abſciſſa $CQ = u$, erit ſimili modo arcus ei reſpondens $BN = \int du \sqrt{\frac{1-nuu}{1-uu}}$. His poſitis quaeritur, quomodo hae duae abſciſſae $x$ et $u$ inter ſe comparatae eſſe debeant, vt arcuum ſumma

$$BM + BN = \int dx \sqrt{\frac{1-nxx}{1-xx}} + \int du \sqrt{\frac{1-nuu}{1-uu}}$$

integrabilis euadat; ſeu geometrice exhiberi queat.

2. Quae-

2. Quaeſtio ergo huc redit, vt determinetur, cuiusmodi functio ipſius $x$ loco $u$ ſubſtitui debeat, vt formula differentialis $dx\sqrt{\frac{1-nxx}{1-xx}} + du\sqrt{\frac{1-nuu}{1-uu}}$ integrationem admittat. Facile autem perſpicitur, ſi haec quaeſtio in genere conſideretur, eius ſolutionem vtriusque formulae integratione inniti; ideoque aeque analyſeos fines transgredi, atque ipſam ellipſeos rectificationem. Cum igitur ſolutio generalis nullo modo expectari queat, in ſolutiones particulares erit inquirendum, quae vti nulla certa ratione reperiri poſſunt, ita etiam plurimum caſui et coniecturae erit tribuendum; ex quo earum verum fundamentum etiamſi ipſae ſint cognitae.

3. Primum quidem ſtatim occurrit caſus $u = -x$, quo formula noſtra differentialis in nihilum abit; ſed quia hinc duo Ellipſeos arcus aequales et ſimiles oriuntur, vti hic caſus nimis eſt obuius, ita etiam quaeſtioni propoſitae minime ſatisfacere eſt cenſendus. Cum igitur tentaminibus totum negotium abſolui debeat, fingatur $\sqrt{\frac{1-nxx}{1-xx}} = \alpha u$, et $\alpha$ ita concipiatur, vt viciſſim fiat $\sqrt{\frac{1-nuu}{1-uu}} = \alpha x$, ſic enim habebitur $BM + BN = \alpha\int u\,dx + \alpha\int x\,du = \alpha xu + \text{Conſt.}$ omnino vti poſtulatur. Pro valore autem ipſius $\alpha$ habebimus tam $1 - nxx - \alpha\alpha uu + \alpha\alpha uuxx = 0$, quam $1 - nuu - \alpha\alpha xx + \alpha\alpha xxuu = 0$; vnde patet, ſtatui debere $\alpha\alpha = n$ et $\alpha = \sqrt{n}$, ita vt $u = \sqrt{\frac{1-nxx}{n-nxx}}$ et $BM + BN = xu\sqrt{n} + \text{Conſt.}$

4. Etſi autem hoc modo quaeſtioni ſatisfactum videtur, tamen iſtae determinationes in Ellipſi locum habere nequeunt. Nam cum ſit $n < 1$ quia $n = 1 - cc$

erit $n - nxx < 1 - nxx$ ideoque $u > 1$, abſciſſa ergo CQ ſemi axem CA ſuperaret, eique propterea arcus imaginarius reſponderet; ita vt hinc nulla concluſio conformis deduci poſſet.

5. Tentemus ergo alias formulas, ſitque tam $\sqrt{\frac{1-nxx}{1-xx}} = \frac{\alpha}{u}$, quam $\sqrt{\frac{1-nuu}{1-uu}} = \frac{\alpha}{x}$, vnde ob $\alpha\alpha - \alpha\alpha xx - uu + nxxuu = 0$ et $\alpha\alpha - \alpha\alpha uu - xx + nxxuu = 0$ colligimus $\alpha = 1$, ita vt ſit $1 - uu - xx + nxxuu = 0$, ideoque $u = \sqrt{\frac{1-xx}{1-nxx}}$. Hinc autem prodit $BM + BN = \int\frac{dx}{u} + \int\frac{du}{x} = \int\frac{xdx + udu}{xu}$. Verum aequatio $uu + xx = 1 + nxxuu$ differentiata dat:

$xdx + udu = nxu(xdu + udx)$ ſeu $\frac{xdx + udu}{xu} = n(xdu + udx$ vnde concludimus $BM + BN = n\int(xdu + udx) = nxu + \text{Conſt.}$

6. Haec ſolutio nullo incommodo laborat, cum enim ſit $n < 1$, erit $1 - nxx > 1 - xx$, ideoque $u < 1$; vti natura rei poſtulat. Sumta ergo abſciſſa quacunque $CP = x$, capiatur altera $CQ = u = \sqrt{\frac{1-xx}{1-nxx}}$; eritque ſumma arcuum $BM + BN = nxu + \text{Conſt.}$ Ad quam conſtantem definiendam ſit $x = 0$, vt fiat $BM = 0$; eritque $u = 1$, et arcus BN abit in quadrantem BMNA; vnde fit $0 + BMNA = 0 + \text{Conſt.}$ ſicque haec conſtans erit $= BMNA$. Quo valore eius loco ſubſtituto habemus $BM + BN = nxu + BMNA$, ideoque

$$BM - AN = nxu = (1-u)xu = BN - AM$$

7. Dato ergo in quadrante elliptico ACB punčto quocunque M, aſſignare valemus alterum punčtum

ctum N, ita vt differentia arcuum $BM - AN$, vel quae huic est aequalis $BN - AM$ geometrice exprimi queat. Quod quo facilius praestari possit, ducamus ad Ellipsin in puncto $M$ normalem $MS$, erit subnormalis $PS = ccx$, et ob $PM = c\sqrt{(1 - xx)}$ ipsa normalis $MS = c\sqrt{(1 - xx + ccxx)} = c\sqrt{(1 - nxx)}$; ideoque pro altero puncto N abscissa erit $CQ = u = \frac{PM}{MS} \cdot CA$. Vel in normalem $MS$ productam ex $C$ demittatur perpendicularis $CR$, quae producatur in V, vt sit $CV = CA = 1$, et ob $\frac{CR}{CS} = \frac{PM}{MS}$ erit $CQ = \frac{CR}{CS} \cdot CV$. Quare ex puncto V in axem CA ducatur perpendicularis $VQ$, quae punctum Q, et producta ipsum punctum N designabit.

8. Cum sit $PS = ccx$, erit $CS = x - ccx = nx$, ideoque $CR = \frac{CQ \cdot CS}{CV} = \frac{u \cdot nx}{1} = nux$. Hoc ergo ipsum perpendiculum $CR$ differentiam arcuum $BM - AN$ seu $BN - AM$ exhibebit. Arcuum ergo hoc modo designatorum differentia erit $= nx\sqrt{\frac{1 - xx}{1 - nxx}}$, quae igitur euanescit tam casu $x = 0$, quam $x = 1$; quibus puncta M et N in ipsa puncta B et A incidunt. Maxima autem haec differentia euadit, si $nx^4 - 2xx + 1 = 0$, hoc est si $x = \frac{1}{\sqrt{(1 + c)}}$, quo casu fit $x = u$, et ambo puncta M et N in vnum punctum O coëunt: eritque hoc casu differentia arcuum $BO - AO = nxx = 1 - c$, ideoque ipsi semiaxium differentiae $CA - CB$ fiet aequalis: ita vt sit $CA + AO = CB + BO$.

9. Si punctum M in ipso hoc puncto O capiatur, vt sit $CP = x = \frac{1}{\sqrt{(1 + c)}}$ erit $PM = \frac{c\sqrt{c}}{\sqrt{(1 + c)}}$, et $PS = \frac{c c}{\sqrt{(1 + c)}}$

hinc-

hincque $MS = c\sqrt{c}$, vnde variis modis ſitus puncti O commode definiri poterit. Cum autem ſit $CM = CO = \frac{\sqrt{(1+c^3)}}{\sqrt{(1+c)}} = \sqrt{(1-c+cc)} = \sqrt{(1+cc-2c\cos. 60^\circ)}$, vnde facilis conſtructio deducitur: ſequentia ergo Theoremata ſubiungere viſum eſt, quorum demonſtratio ex allatis eſt manifeſta.

## Theorema 1.

Tab. I. Fig. 4. 10. In quadrante elliptico ACB, ſi ad punctum quoduis M ducatur tangens HMK, quae cum altero axe CB in H concurrat, eaque alteri ſemiaxi CA aequalis capiatur, vt ſit $HK = CA$; tum vero per K axi CB parallela agatur KN ellipſin ſecans in N; arcuum BM et AN differentia $BM - AN$ geometrice aſſignari poterit; demiſſo enim ex Centro C in tangentem perpendiculo CT, erit iſta arcuum differentia $BM - AN = MT$.

Fig. 3. et 4. Demonſtratio ex figura ſponte patet, cum tangens HMK ſit rectae illi CRV parallela et aequalis, tum vero perſpicuum eſt, eſſe $MT = CR$.

## Theorema 2.

Fig. 5. 11. Si ſuper quadrantis elliptici ACB altero ſemiaxe CA triangulum aequilaterum CAE conſtituatur, et in eius latere AE portio capiatur $AF = CB$, iunctaeque CF aequalis applicetur in ellipſi recta CO, punctum O hanc habebit proprietatem, vt ſit $CA +$ arcu $AO = CB +$ arcu $BO$.

Demonſtratio ex §. 9. euidens eſt. Cum enim ſit $CA = 1$, $AF = c$ et ang. $CAF = 60^\circ$ erit $CF = \sqrt{(1+cc-2c\cos. 60^\circ)}$ ideoque $= CO$.

11. De

## II. De Hyperbola.

12. Sit C centrum hyperbolae AMN eiusque semiaxis transuersus $CA = 1$, semiaxis coniugatus $= c$; erit sumta abscissa quacunque $CP = x$, applicata $PM = c\sqrt{(xx-1)}$, eiusque differentiale $= \frac{c\,x\,dx}{\sqrt{(xx-1)}}$; vnde fit arcus $AM = \frac{\int dx \sqrt{((1+cc)xx-1)}}{\sqrt{(xx-1)}}$. Ponatur breuitatis gratia $1+cc = n$; erit $AM = \int dx \sqrt{\frac{nxx-1}{xx-1}}$. Simili ergo modo si capiatur alia quaeuis abscissa $CQ = u$, erit arcus ei respondens $AN = \int du \sqrt{\frac{nuu-1}{uu-1}}$. Tab. I. Fig. 6.

13. His positis ista nobis proposita sit quaestio, vt dato puncto M alterum N ita definiatur, vt summa arcuum $AM + AN$, seu expressio $\int dx \sqrt{\frac{nxx-1}{xx-1}} + \int du \sqrt{\frac{nuu-1}{uu-1}}$ absolute integrationem admittat; quod quidem euenire casu $u = -x$ sponte patet; verum hinc nihil ad institutum nostrum concludere licet.

14. Ponamus ergo $\sqrt{\frac{nxx-1}{xx-1}} = u\sqrt{n}$, cum hinc vicissim fiat $\sqrt{\frac{nuu-1}{uu-1}} = x\sqrt{n}$, vtrinque enim prodit haec aequatio $nuuxx - n(uu+xx) + 1 = 0$. Facta autem hac hypothesi prodit summa arcuum $AM + AN = \int u\,dx\sqrt{n} + \int x\,du\sqrt{n} = ux\sqrt{n} + \text{Const.}$ Haec ergo integrabilitas vt locum habeat, oportet sit $u = \sqrt{\frac{nxx-1}{nxx-n}}$, vnde cum ob $n > 1$ prodeat quoque $u > 1$, ex dato puncto M semper alterum punctum N assignari poterit.

15. Ad constantem definiendam patet casum $x = 1$, quo punctum M in verticem A incidit, nihil iuuare, cum inde oriatur $u = \infty$, punctumque N in infinitum remo-

remoueatur. Quocirca vt haec constans debite determinetur, alium casum considerari oportet; potior autem non occurrit, quam is, vbi puncta M et N in vnum coalescunt, seu quo fit $u = x$, et $nx^4 - 2nxx + 1 = 0$. Hinc autem oritur $xx = 1 + \frac{c}{\sqrt{(1+cc)}}$ et $x = \sqrt{(1 + \frac{c}{\sqrt{(1+cc)}})}$

16. Sit igitur O hoc punctum, in quo ambo puncta M et N coalescunt, ductaque applicata OI erit abscissa $CI = \sqrt{(1 + \frac{c}{\sqrt{(1+cc)}})}$ et $2AO = c + \sqrt{(1+cc)}$ $+ \text{Const.}$ Hinc ergo obtinemus constantem quaesitam $= 2AO - c - \sqrt{(1+cc)}$ ob $\sqrt{n} = \sqrt{(1+cc)}$. Quo valore substituto erit pro quibusuis punctis M et N diuersis, ita sumtis, vt sit $u = \sqrt{\frac{nxx-1}{nxx-n}}$ summa arcuum $AM + AN = ux\sqrt{n} + 2AO - c - \sqrt{(1+cc)}$ seu $ON - OM = ux\sqrt{n} - c - \sqrt{(1+cc)}$. Sic igitur duos arcus nacti sumus ON et OM, quorum differentia $ON - OM$ geometrice assignari potest.

17. Quo autem facilius pateat, quomodo tam
Fig. 7. punctum O, quam ex puncto M punctum N definiri queat; erigatur in A perpendiculum $AD = c$, eritque recta CD hyperbolae asymtota; tum positis $CP = x$; $PM = y$, ducatur tangens MT, erit ob $y = c\sqrt{(xx-1)}$ et $dy = \frac{cxdx}{\sqrt{(xx-1)}}$ subtangens $PT = \frac{y\sqrt{(xx-1)}}{cx} = x - \frac{1}{x}$; et $CT = \frac{1}{x}$; et ipsa tangens $MT = \frac{y\sqrt{(nxx-1)}}{cx}$. Hinc prodit $\sqrt{\frac{xx-1}{nxx-1}} = \frac{PT}{MT}$, ideoque $u = \frac{MT}{PT\sqrt{(1+cc)}} = \frac{CA^2 \cdot MT}{CD \cdot PT} = CQ$.

18. Ducatur ex centro C tangenti TM parallela $CR = CD$, demissoque ex R in axem perpendiculo RS, erit $CS = \frac{CD \cdot PT}{MT}$, ideoque $CQ = \frac{CA^2}{CS}$. Quare CQ capienda erit tertia proportionalis ad CS et CA. Commodius

modius autem res ſequenti modo ſine tangentium adminiculo expedietur: nam cum ſit $QN = \frac{cc}{\sqrt{n(xx-1)}} = \frac{c^3}{y\sqrt{n}}$ erit $PM.QN = \frac{c^3}{\sqrt{(1+cc)}} = \frac{AD^3}{CD}$ vel demiſſo ex A in aſymtotam perpendiculo AE erit $PM.QN = AD.DE$ ob $DE = \frac{AD^2}{CD}$, vnde ſequens Theorema conficitur.

## Theorema 3.

19. Exiſtente AOZ hyperbola, C eius centro, A vertice, et CDZ eius aſymtota, ad quam ex A axi perpendiculariter ducta ſit recta AD, itemque AE ad aſymtotam perpendicularis; ſi applicata conſtituatur IO media proportionalis inter AD et DE, atque vtrinque applicatae PM et QN ita ſtatuantur, vt inter eas ſit IO media proportionalis; tum arcuum ON et OM differentia geometrice aſſignari poterit. Erit enim Fig. 8.

$$ON - OM = \frac{CP.CQ - CI.CT}{CE}.$$

Demonſtratio ex §. praec. eſt manifeſta. Cum enim punctis M et N in O coëuntibus ſit $IO.TO = AD\,DE$, erit IO media proportionalis inter AD et DE; hacque inuenta eſſe oportet $PM.QN = OI\,OT$. Tum vero ex §. 16. intelligitur eſſe $ON - OM = (CP.CQ - CI.CI)\sqrt{n}$, et ob $\sqrt{n} = CD$, erit homogeneitatem implendo $ON - OM = (CP.CQ - CI.CT)\frac{CD}{CA^2}$. At eſt $\frac{CA^2}{CD} = CE$, ſicque conſtat theorematis veritas.

## III. De Curua Lemniscata.

20. Haec curua ob plurimas, quibus praedita eſt, inſignes proprietates inter Geometras eſt celebrata, im-

Tab. II. Fig. 1. imprimis autem quod eius arcus arcubus curuae elasticae sunt aequales. Natura autem huius curuae ita est comparata, vt positis coordinatis orthogonalibus $CP = x$, $PM = y$, ista aequatione exprimatur $(xx + yy)^2 = xx - yy$. Vnde patet hanc curuam esse lineam quarti ordinis, quae in C, quod punctum eius centrum dicitur, cum axe CA angulum semirectum constituit, in A autem sumta $CA = 1$, axem normaliter traiicit. Figura autem CMNA quartam partem totius lemniscatae exhibet, cui tres reliquae partes circa centrum C aequales sunt concipiendae; id quod inde liquet, quod siue abscissa $x$, siue applicata $y$, siue vtraque, negatiuum valorem induat, aequatio eadem manet.

21. Quod igitur ad expressionem arcus cuiusque CM huius curuae attinet, is commodissime ex corda CM definitur. Si enim hanc cordam ponamus $CM = z$, ob $xx + yy = zz$ habebimus:

$$z^4 = xx - yy = 2xx - zz = zz - 2yy$$

vnde elicimus

$$x = z\sqrt{\frac{1 + zz}{2}} \text{ et } y = z\sqrt{\frac{1 - zz}{2}}$$

et differentiando

$$dx = \frac{dz(1 + 2zz)}{\sqrt{2(1 + zz)}} \text{ et } dy = \frac{dz(1 - 2zz)}{\sqrt{2(1 - zz)}}$$

Hinc ergo elementum arcus CM colligitur

$$\sqrt{(dx^2 + dy^2)} = dz\sqrt{\frac{(1 - zz)(1 + 2zz)^2 + (1 + zz)(1 - 2zz)^2}{2(1 + zz)(1 - zz)}}$$

siue $\sqrt{(dx^2 + dy^2)} = \frac{dz}{\sqrt{(1 - z^4)}}$

22. Si ergo corda quaecunque ex centro C educta ponatur $CM = z$, erit arcus ab ea subtensus

CM =

$CM = \int \frac{dz}{\sqrt{(1-z^4)}}$ Simili ergo modo si alia quaeuis corda CN dicatur $= u$, erit arcus ab ea subtensus $CN = \int \frac{du}{\sqrt{(1-u^4)}}$; cuius complementum ad totum quadrantem est arcus AN. Iam Ill. Comes *Fagnani* docuit, cuiusmodi functio ipsius $z$ capi debeat pro $u$, vt vel arcus AN aequalis fiat arcui CM, vel vt arcus CN sit duplus arcus CM, vel etiam vt arcus AN sit aequalis duplo arcui CM. Hos ergo casus primo exponam, deinceps autem, quae mihi circa alias huiusmodi arcuum proportiones eruere contigit, in medium sum allaturus.

## Theorema 4.

23. In curua lemniscata hactenus descripta, si applicetur corda quaecunque $CM = z$, aliaque insuper applicetur, quae sit $CN = u = \sqrt{\frac{1-zz}{1+zz}}$; erit arcus CM aequalis arcui AN, vel etiam arcus CN aequalis arcui AM.

## Demonstratio.

Cum sit corda $CM = z$, erit arcus $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$, et ob cordam $CN = u$ erit arcus $CN = \int \frac{du}{\sqrt{(1-u^4)}}$. At est $u = \sqrt{\frac{1-zz}{1+zz}}$; vnde fit $du = \frac{-2zdz}{(1+zz)\sqrt{(1-z^4)}}$. Praeterea vero est $u^4 = \frac{1-2zz+z^4}{1+2zz+z^4}$, ideoque $1-u^4 = \frac{4zz}{(1+zz)^2}$ et $\sqrt{(1-u^4)} = \frac{2z}{1+zz}$. Quibus valoribus substitutis habebitur arcus $CN = -\int \frac{dz}{\sqrt{(1-z^4)}} = -$ arc. CM + Const. ita vt sit arc. CN + arc. CM = Const. Ad hanc constantem definiendam perpendatur casus quo $z = 0$, ideoque et arcus $CM = 0$, hoc autem casu fit corda

 CN =

$CN = u = 1 = CA$; ideoque arcus $CN$ abit in quadrantem $CMNA$, ex quo habebitur pro hoc casu $CMNA + o =$ Const. Hoc ergo valore substituto prodibit in genere

arc. $CN +$ arc. $CM =$ arc. $CMNA$

hincque arc. $CM =$ arc. $AN$, et arcum $MN$ vtrinque addendo arc. $CMN =$ arc. $ANM$. Q. E. D.

## Coroll. 1.

24. Dato ergo quocunque arcu $CM$ in centro $C$ terminato, cuius corda est $CM = z$, ei ab altera parte seu vertice $A$ abscindetur arcus aequalis $AN$, sumendo cordam $CN = u = \sqrt{\frac{1-zz}{1+zz}}$, seu $CN = CA \sqrt{\frac{CA^2 - CM^2}{CA^2 + CM^2}}$, homogeneitatem supplendo per axem $CA = 1$.

## Coroll. 2.

25. Cum sit $u = \sqrt{\frac{1-zz}{1+zz}}$, erit vicissim $z = \sqrt{\frac{1-uu}{1+uu}}$, vnde cordas $CM$ et $CN$ inter se permutare licet, ita vt si ambae cordae $CM = z$ et $CN = u$ ita fuerint comparatae, vt sit $uuzz + uu + zz = 1$, etiam puncta $M$ et $N$ inter se permutari queant, indeque prodeat tam arcus $CM =$ arc. $AN$, quam arc. $CN =$ arc. $AM$.

## Coroll. 3.

26. Cum sit $CN = u = \sqrt{\frac{1-zz}{1+zz}}$, erit $\sqrt{\frac{1+uu}{2}} = \frac{1}{\sqrt{(1+zz)}}$ et $\sqrt{\frac{1-uu}{2}} = \frac{z}{\sqrt{(1+zz)}}$. Vnde cum ex natura curuae lemniscatae pro puncto $N$ coordinatae sint $CQ = u\sqrt{\frac{1+uu}{2}}$ et $QN = u\sqrt{\frac{1-uu}{2}}$, erit $CQ = \frac{u}{\sqrt{(1+zz)}}$

et

et $QN = \frac{uz}{\sqrt{(1+zz)}}$, ideoque $\frac{QN}{CQ} = z$. Quare si in A ad axem CA erigatur normalis AT, donec cordae CN productae occurrat in T, erit $AT = z = CM$.

## Coroll. 4.

27. Ex dato ergo puncto M alterum punctum N ita facillime definitur: capiatur tangens AT aequalis cordae CM, ductaque recta CT curuam in puncto quaesito N secabit. Ob eandem autem rationem patet, si corda CM producatur, donec tangenti in A occurrat in S, erit pariter $AS = CN$.

## Coroll. 5.

28. Manifestum etiam est puncta M et N in vnum punctum O coire posse, in quo propterea totus quadrans COA in duas partes aequales diuiditur. Inuenietur ergo hoc punctum O, si ponatur $u = z$, vnde fit $z^4 + 2zz = 1$, hincque $zz + 1 = \sqrt{2}$; prodit ergo corda $CO = \sqrt{(\sqrt{2} - 1)}$, cui simul tangens AI erit aequalis, vnde simul positio huius puncti O facile assignatur.

## Coroll. 6.

29. Notato ergo hoc puncto O, quo totus quadrans COA in duas partes aequales CMO et ANO diuiditur, erit quoque puncti M et N per regulam expositam definitis arc. MO = arc. ON: ita vt item hoc punctum O omnes arcus MN in duas partes aequales dispescat.

Theore-

## Theorema. 5.

Tab. II. Fig. 2. 30. In curua lemniscata cuius axis $CA = 1$, si applicata sit corda quaecunque $CM = z$, aliaque insuper chorda applicetur $CM^2 = u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$, erit arcus a corda hac $u$ subtensus $CM^2$ duplo maior quam arcus ab illa corda subtensus $CM$.

## Demonstratio.

Cum sit corda $CM = z$, erit arcus $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$ similiterque ob cordam $CM^2 = u$ erit arcus $CM^2 = \int \frac{du}{\sqrt{(1-u^4)}}$. Quia autem est $u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$, erit $uu = \frac{4zz-4z^6}{1+2z^4+z^8}$ ideoque $\sqrt{(1-uu)} = \frac{1-2zz-z^4}{1+z^4}$ et $\sqrt{(1+uu)} = \frac{1+2zz-z^4}{1+z^4}$ vnde fit $\sqrt{(1-u^4)} = \frac{1-6z^4+z^8}{(1+z^4)^2}$. Tum vero differentiando colligitur $du = \frac{2dz(1-z^8)-4z^4dz(1+z^4)-8z^4dz(1-z^4)}{(1+z^4)^2\sqrt{(1-z^4)}}$ seu $du = \frac{2dz-12z^4dz+2z^8dz}{(1+z^4)^2\sqrt{(1-z^4)}} = \frac{2dz(1-6z^4+z^8)}{(1+z^4)^2\sqrt{(1-z^4)}}$.

Hinc ergo nanciscimur $\frac{du}{\sqrt{(1-u^4)}} = \frac{2dz}{\sqrt{(1-z^4)}}$ et integrando arc. $CM^2 = 2$ arc. $CM +$ Const. Cum autem posito $z = 0$ fiat etiam $u = 0$, ideoque ambo arcus $CM$ et $CM^2$ euanescant, constans quoque in nihilum abit. Sicque sumta corda $CM^2 = u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$ erit arcus $CM^2 = 2$ *arc.* $CM$. Q. E. D.

## Coroll. 1.

31. Si capiatur corda $CN = \sqrt{\frac{1-zz}{1+zz}}$, erit arcus $AN =$ arc. $CM$, hincque etiam arcus $CM^2$ erit $= 2$ arc. $AN$. Simili modo si capiatur corda $CN^2 = \sqrt{\frac{1-uu}{1+uu}}$, erit arcus $AN^2 =$ arc. $CM^2$, sicque etiam a ver-

a vertice A erit arc. $AN^2 = 2$ arc. AN. Hoc ergo modo obtinentur quatuor arcus inter se aequales scilicet arc. CM, arc. $MM^2$, arc. AN, et arc. $NN^2$.

## Coroll. 2.

32. Cum autem sit $u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$; $\sqrt{(1-uu)} = \frac{1-2zz-z^4}{1+z^4}$ et $\sqrt{(1+uu)} = \frac{1+2zz-z^4}{1+z^4}$, hae quatuor cordae ita habebuntur expressae vt sit:
$CM = z$; $CN = \sqrt{\frac{1-zz}{1+zz}}$; $CM^2 = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$, $CN^2 = \frac{1-2zz-z^4}{1+2zz-z^4}$

## Coroll. 3.

33. Conueniant ambo puncta $M^2$ et $N^2$ in curvae puncto medio O, pro quo supra vidimus esse cordam $CO = \sqrt{(\sqrt{2}-1)}$ atque hoc casu tota curua COA in quatuor partes aequales dispescetur in punctis M. O et N. Hoc igitur euenit si sit $CM^2 = CN^2 = \sqrt{(\sqrt{2}-1)}$: ita vt posito breuitatis gratia $\sqrt{(\sqrt{2}-1)} = \alpha$, habeamus. $1-2zz-z^4 = \alpha + 2\alpha zz - \alpha z^4$ seu $z^4 = \frac{-2(1+\alpha)zz+1-\alpha}{1-\alpha}$ et $zz = \frac{-(1+\alpha)+\sqrt{2(1+\alpha\alpha)}}{1-\alpha}$ vel $zz = \frac{-1-\sqrt{(\sqrt{2}-1)}+\sqrt{2}\sqrt{2}}{1-\sqrt{(\sqrt{2}-1)}}$. Vnde colligimus $CM = z = \sqrt{\frac{-1-\alpha+\sqrt{2(1+\alpha\alpha)}}{1-\alpha}}$ et $CN = \sqrt{\frac{-1+\alpha+\sqrt{2(1+\alpha\alpha)}}{1+\alpha}}$. Tab. II. Fig. 3.

## Coroll. 4.

34. Coalescant ambo puncta $M^2$ et N, et puncta M et $N^2$ pariter coibunt, sicque tota curua CMNA in punctis M et N trifariam secabitur. Pro hoc ergo casu habebitur vel $\frac{2z\sqrt{(1-z^4)}}{1+z^4} = \sqrt{\frac{1-zz}{1+zz}}$ vel $z = \frac{1-2zz-z^4}{1+2zz-z^4}$ quarum posterior dat $1-z-2zz-2z^3-z^4+z^5 = 0$, Fig. 4.

 haec-

haecque per $1+z$ diuisa: $1-2z-2z^3+z^4=0$; cuius concipiantur factores $(1-\mu z+zz)(1-\nu z+zz)=0$, eritque $\mu+\nu=2$ et $\mu\nu=-2$, vnde fit $\mu-\nu=2\sqrt{3}$, hincque $\mu=1+\sqrt{3}$ et $\nu=1-\sqrt{3}$. Erit ergo $z=\frac{1+\sqrt{3}\pm\sqrt{2\sqrt{3}}}{2}=CM$; et ob $zz=\frac{4+4\sqrt{3}\pm 2(1+\sqrt{3})\sqrt{2\sqrt{3}}}{4}$ orietur $CN=\sqrt{\frac{1-zz}{1+zz}}=\sqrt{\frac{-2\sqrt{3}\mp(1+\sqrt{3})\sqrt{2\sqrt{3}}}{4+2\sqrt{3}\pm(1+\sqrt{3})\sqrt{2\sqrt{3}}}}=\sqrt{\frac{\mp\sqrt{2\sqrt{3}}}{1+\sqrt{3}}}$; Est itaque $CM=\frac{1+\sqrt{3}-\sqrt{2\sqrt{3}}}{2}$ et $CN=\sqrt{\frac{\sqrt{2\sqrt{3}}}{1+\sqrt{3}}}$.

## Coroll. 5.

Tab. II. Fig. 2. 35. Dato etiam quocunque arcu $CM^2$, inueniri potest eius semissis $CM$: si enim arcus illius ponatur corda $CM^2=u$, et arcus quaesiti corda $CM=z$, erit $u=\frac{2z\sqrt{(1-z^4)}}{1+z^4}$ et $1-\frac{4zz}{uu}+2z^4+\frac{4z^6}{uu}+z^8=0$, cuius factores concipiantur $(1-\mu zz-z^4)(1-\nu zz-z^4)=0$: vnde obtinetur $\mu+\nu=\frac{4}{uu}$ et $\mu\nu=4$; erit ergo $\mu-\nu=4\sqrt{(\frac{1}{u^4}-1)}=\frac{4}{uu}\sqrt{(1-u^4)}$ hincque $\mu=\frac{2+2\sqrt{(1-u^4)}}{uu}$ et $\nu=\frac{2-2\sqrt{(1-u^4)}}{uu}$: ergo $zz=\frac{-1-\sqrt{(1-u^4)}+\sqrt{2(1+\sqrt{(1-u^4)})}}{uu}$: vnde pro $z$ duplex valor realis elicitur:

alter $z=\frac{\sqrt{(-1-\sqrt{(1-u^4)}+\sqrt{2(1+\sqrt{(1-u^4)})})}}{u}=\frac{\sqrt{(1-\sqrt{(1-uu)})}(\sqrt{(1+uu)}-1)}{u}$

alter $z=\frac{\sqrt{(-1+\sqrt{(1-u^4)}+\sqrt{2(1-\sqrt{(1-u^4)})})}}{u}=\frac{\sqrt{(1+\sqrt{(1-uu)})}(\sqrt{(1+uu)}-1)}{u}$

## Coroll. 6.

Fig. 5. 36. Duplex hic valor reuera locum obtinet, cum enim eadem corda $CM^2$ et $Cm^2$ duos arcus diversos $CM^2$ et $CM^2m^2$ subtendat, alter valor ipsius $z$ praebebit cordam arcus $CM$, qui est semissis arcus $CM^2$, alter autem valor ipsius $z$ dat cordam arcus $CM$, qui est

est semissis arcus $CM^2m^2$: ac prior quidem valor pro illo casu, posterior vero pro hoc locum habet.

## Coroll. 7.

37. Hoc modo etiam lemniscata $CA$ in quinque partes aequales diuidi potest. Sit enim corda partis simplicis $C1 = z$; corda partis duplicatae $C2 = \frac{2z\sqrt{(1-z^4)}}{1+z^4} = u$, erit corda partis quadruplicatae $C4 = \frac{2u\sqrt{(1-u^4)}}{1+u^4} = \sqrt{\frac{1-zz}{1+zz}}$, quia est $A4 = C1$; vnde corda $z$ definitur, qua inuenta cum sit $C2 = A3$, erit corda $C3 = \sqrt{\frac{1-uu}{1+uu}}$. Fig. 6.

## Coroll. 8.

38. Cum hinc posita corda cuiuspiam $= z$, reperiri possint cordae arcuum dupli, quadrupli, octupli, sedecupli, etc. manifestum est hoc modo etiam lemniscatam in tot partes diuidi posse, quarum numerus sit $2^m + 2^n$. In hac autem formula continentur sequentes numeri

1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24, 32, 33 etc.

Verum hinc non semper omnia diuisionum puncta assignare licet.

## Scholion.

39. Haec igitur sunt, quae Ill. Comes *Fagnani* de curua lemniscata obseruauit, vel quae ex eius inuentis deriuare licet. Etsi enim tantum proposito arcu quocunque eius duplum assignare docuit, tamen hunc arcum iterum continuo duplicando, etiam cordae arcuum

 quadrupli,

quadrupli, octupli, sedecupli etc. inde colligentur: Namque si corda arcus simpli statuatur $=z$; arcus dupli $=u$, quadrupli $=p$; octupli $=q$, sedecupli $=r$ etc. erit:

$$u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$$

$$p = \frac{2u\sqrt{(1-u^4)}}{1+u^4} = \frac{4z(1+z^4)(1-6z^4+z^8)\sqrt{(1-z^4)}}{(1+z^4)^4+16z^4(1-z^4)^2}$$

$$q = \frac{2p\sqrt{(1-p^4)}}{1+p^4};\ r = \frac{2q\sqrt{(1-q^4)}}{1+q^4} \text{ etc.}$$

Aliorum autem arcuum multiplorum cordas ex his assignare non licet. Quemadmodum ergo arcuum quorumuis multiplorum cordae exprimantur, hic inuestigabo, vt hoc argumentum, quantum limites analyseos id quidem permittunt, penitus perficiatur. Primum quidem tentando elicui, si arcus simpli corda sit $=z$, tum arcus tripli cordam fore $= \frac{z(3-6z^4-z^8)}{1+6z^4-3z^8}$ verum postea rem sequenti modo generaliter expediri posse intellexi.

## Theorema 6.

Fig. 7. 40. Si corda arcus simplicis CM sit $=z$, et corda arcus $n$ cupli $CM^n = u$, erit corda arcus $(n+1)$ cupli

$$CM^{n+1} = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - uz\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}}$$

## Demonstratio.

Erit ergo ipse arcus simplex $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$; et arcus $n$ cuplus $CM^n = \int \frac{du}{\sqrt{(1-u^4)}} = n\int \frac{dz}{\sqrt{(1-z^4)}}$: ideoque habemus $du = \frac{n\,dz\sqrt{(1-u^4)}}{\sqrt{(1-z^4)}}$. Ponamus breuitatis gratia $z\sqrt{\frac{1-uu}{1+uu}} = P$, et $u\sqrt{\frac{1-zz}{1+zz}} = Q$, vt sit corda pro arcu $(n+1)$ cuplo

cuplo exhibita $CM^{n+1} = \frac{P+Q}{1-PQ}$, quae dicatur $= s$ atque demonſtrari oportet, eſſe arcum huic cordae reſpondentem $\int \frac{ds}{\sqrt{(1-s^4)}} = (n+1)\int \frac{dz}{\sqrt{(1-z^4)}}$ ſeu $\frac{ds}{\sqrt{(1-s^4)}} = \frac{(n+1)dz}{\sqrt{(1-z^4)}}$. Cum autem ſit $s = \frac{P+Q}{1-PQ}$ erit $ds = \frac{dP(1+QQ)+dQ(1+PP)}{(1-PQ)^2}$: tum vero reperitur

$1-s^4 = \frac{(1-PQ)^4-(P+Q)^4}{(1-PQ)^4} = \frac{(1+PP+QQ+PPQQ)(1-PP-QQ-4PQ+PPQQ)}{(1-PQ)^4}$

ergo $\sqrt{(-s^4)} = \frac{\sqrt{(1+PP)(1+QQ)(1-PP-QQ-4PQ+PPQQ)}}{(1-PQ)^2}$

ex quo elicitur:

$$\frac{ds}{\sqrt{(1-s^4)}} = \frac{dP\sqrt{\frac{1+QQ}{1+PP}} + dQ\sqrt{\frac{1+PP}{1+QQ}}}{\sqrt{(1-PP-QQ-4PQ+PPQQ)}}$$

cuius expreſſionis ergo valorem inueſtigemus:

Ac primo quidem eſt $1+PP = \frac{1+uu+zz+uuzz}{1+uu}$ et $1+QQ = \frac{1+uu+zz-uuzz}{1+zz}$, ita vt ſit $\frac{1+PP}{1+QQ} = \frac{1+zz}{1+uu}$,

ideoque $\frac{ds}{\sqrt{(1-s^4)}} = \frac{dP\sqrt{\frac{1+uu}{1+zz}} + dQ\sqrt{\frac{1+zz}{1+uu}}}{\sqrt{(1-PP-QQ+PPQQ-4PQ)}}$.

Deinde vero ob

$1-PP = \frac{1+uu-zz+uuzz}{1+uu}$ et $1-QQ = \frac{1+zz-uu+uuzz}{1+zz}$

erit

$(1-PP)(1-QQ) = 1-P^2-Q^2+P^2Q^2 = \frac{1-z^4-u^4+4uuzz+u^4z^4}{(1+zz)(1+uu)}$

et $4PQ = \frac{4uz\sqrt{(1-z^4)(1-u^4)}}{(1+zz)(1+uu)}$; hincque concluditur denominator $\sqrt{(1-PP-QQ+PPQQ-4PQ)}$

$= \frac{\sqrt{(1-z^4-u^4+4uuzz+u^4z^4-4uz\sqrt{(1-z^4)(1-u^4)})}}{\sqrt{(1+zz)(1+uu)}}$

$= \frac{\sqrt{(1-z^4)(1-u^4)}-2uz}{\sqrt{(1+zz)(1+uu)}}$, ex quo obtinebitur

$$\frac{ds}{\sqrt{(1-s^4)}} = \frac{dP(1+uu)+dQ(1+zz)}{\sqrt{(1-z^4)(1-u^4)}-2uz}.$$

Iam vero differentiando elicimus

$$dP = dz\sqrt{\frac{1-uu}{1+uu}} - \frac{2zudu}{(1+uu)\sqrt{(1-u^4)}}$$
$$dQ = du\sqrt{\frac{1-zz}{1+zz}} - \frac{2zudz}{(1+zz)\sqrt{(1-z^4)}},$$

 quare

quare ob $du = \frac{n\,dz\,\surd(1-u^4)}{\surd(1-z^4)}$, erit

$$dP = dz\surd\frac{1-uu}{1+uu} - \frac{2nuz\,dz}{(1+uu)\surd(1-z^4)}$$

$$dQ = \frac{n\,dz\,\surd(1-u^4)}{1+zz} - \frac{2uz\,dz}{(1+zz)\surd(1-z^4)}$$

vnde conficitur numerator $dP(1+uu)+dQ(1+zz)$ $= dz\surd(1-u^4) - \frac{2uuz\,dz}{\surd(1-z^4)} + n\,dz\surd(1-u^4) - \frac{2uz\,dz}{\surd(1-z^4)}$ siue $dP(1+uu)+dQ(1+zz) = (n+1)\,dz\surd(1-u^4) - \frac{2(n+1)uz\,dz}{\surd(1-z^4)}$ $= \frac{(n+1)dz}{\surd(1-z^4)}(\surd(1-z^4)(1-u^4) - 2uz)$ vnde perspicuum est esse $\frac{ds}{\surd(1-s^4)} = \frac{(n+1)dz}{\surd(1-z^4)}$ et arc. $CM^{n+1} = (n+1)$ arc. $CM$ Q. E. D.

## Coroll. 1.

41. *Si* a vertice A abscindantur arcus $Am$, $Am^n$, $Am^{n+1}$ arcubus $CM$, $CM^n$, $CM^{n+1}$ respectiue aequales, erit $Cm$ corda complementi arcus $CM$, $Cm^n$ corda complementi arcus $CM^n$; $Cm^{n+1}$ corda complementi arcus $CM^{n+1}$. Erunt autem ob cordas $CM = z$; $CM^n = u$; $CM^{n+1} = s$, complementorum cordae $Cm = \surd\frac{1-zz}{1+zz}$; $Cm^n = \surd\frac{1-uu}{1+uu}$; $Cm^{n+1} = \surd\frac{1-ss}{1+ss}$. Cum autem sit $s = \frac{z\surd\frac{1-uu}{1+uu} + u\surd\frac{1-zz}{1+zz}}{1 - zu\surd\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}} = \frac{P+Q}{1-PQ}$ erit

$$\surd\frac{1-ss}{1+ss} = \surd\frac{1-PP-QQ-PQ+PPQQ}{(1+PP)(1+QQ)} = \frac{\surd(1-z^4)(1-u^4)-2uz}{1+uu+zz-uuzz}$$

quae ad hanc formam reducitur

$$\surd\frac{1-ss}{1+ss} = \frac{\surd\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)} - uz}{1+uz\surd\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)}}.$$

## Coroll. 2.

42. Si igitur ponatur:
corda arcus fimplicis $= z$; corda complementi $= Z$.
corda arcus $n$ cupli $= u$; corda complementi $= V$
vt fit $Z = \sqrt{\frac{1-zz}{1+zz}}$ et $V = \sqrt{\frac{1-uu}{1+uu}}$; erit
corda arcus $(n+1)$ cupli $= \frac{zV+uZ}{1-zuZV}$
corda complementi $= \frac{ZV-zu}{1-zuZV}$.

## Coroll. 3.

43. Inuentio ergo cordarum arcuum quorumuis multiplorum vna cum cordis complementi ita fe habebit:

| Corda arcus | | corda complementi |
|---|---|---|
| fimpli | $= a$ - - - | fimpli $= A$ |
| dupli | $= b = \frac{2aA}{1-aaAA}$ - - | dupli $= \frac{AA-aa}{1+aaAA} = B$ |
| tripli | $= c = \frac{aB+bA}{1-abAB}$ - | tripli $= \frac{AB-ab}{1+abAB} = C$ |
| quadrupli | $= d = \frac{aC+cA}{1-acAC}$; | quadrupli $= \frac{AC-ac}{1+acAC} = D$ |
| quintupli | $= e = \frac{aD+dA}{1-adAD}$; | quintupli $= \frac{AD-ad}{1+adAD} = E$ |

## Coroll. 4.

44. Simili modo fi corda arcus $m$ cupli fit $= r$, corda complementi $= R$; et corda arcus $n$ cupli $= s$ eiusque corda complementi $= S$, vt fit $R = \sqrt{\frac{1-rr}{1+rr}}$ et $S = \sqrt{\frac{1-ss}{1+ss}}$, erit corda arcus $(m+n)$ cupli $= \frac{rS+sR}{1-rsRS}$ et corda complementi $= \frac{RS-rs}{1+rsRS}$. Quin etiam fumendo pro $n$ numerum negatiuum, quia tum corda $s$ abit in

in fui negatiuum, corda differentiae illorum arcuum exhiberi poterit, erit fcilicet corda arcus $(m-n)$ cupli $= \frac{rS - sR}{1 + rsRS}$ et corda complementi eius $= \frac{RS + rs}{1 - rsRS}$.

## Coroll. 5.

45. Sumtis ergo denominationibus, quae in coroll. 3 funt adhibitae, erit quoque

$$d = \frac{2bB}{1 - bbBB} \quad \text{et} \quad D = \frac{BB - bb}{1 + bbBB}$$

$$e = \frac{bC + cB}{1 - bcBC} \quad \text{et} \quad E = \frac{BC - bc}{1 + bcBC}$$

## Coroll. 6.

46. Ex his colligitur fi corda arcus fimplicis ftatuatur $= z$; valores cordarum in coroll 3 adhibitarum fore

$$a = z; \qquad A = \sqrt{\frac{1 - zz}{1 + zz}}$$

$$b = \frac{2z\sqrt{(1 - z^4)}}{1 + z^4}; \qquad B = \frac{1 - 2zz - z^4}{1 + 2zz - z^4}$$

$$c = \frac{z(3 - 6z^4 - z^8)}{1 + 6z^4 - 3z^8}; \qquad C = \frac{(1 + z^4)^2 - 4zz(1 - zz)^2}{(1 + z^4)^2 + 4zz(1 - zz)^2} \sqrt{\frac{1 - zz}{1 + zz}}$$

$$d = \frac{4z(1 + z^4)(1 - 6z^4 + z^8)\sqrt{(1 - z^4)}}{(1 + z^4)^4 + 16z^4(1 - z^4)^2}; \quad D = \frac{(1 - 6z^4 + z^8)^2 - 8zz(1 - z^4)(1 + z^4)^2}{(1 - 6z^4 + z^8)^2 + 8zz(1 - z^4)(1 + z^4)^2}$$

## Scholion I.

47. Ratio compofitionis formularum $\frac{rS + sR}{1 - rsRS}$ et $\frac{RS - rs}{1 + rsRS}$ imprimis ideo notari meretur, quod fimilis eft regulae, qua tangens fummae vel differentiae duorum angulorum definiri folet. Si enim fit $rS = \text{tang.}\,\alpha$, et $sR = \text{tang.}\,\beta$ erit $\frac{rS + sR}{1 - rsRS} = \text{tang.}\,(\alpha + \beta)$, et pro diffe-

differentia in coroll. 4 exhibita $\frac{rS - sR}{1 + rsRS} = \text{tang.}\,(\alpha - \beta)$. Similique modo ſi ponatur $RS = \text{tang.}\,\gamma$ et $rs = \text{tang.}\,\delta$ erit $\frac{RS - rs}{1 + rsRS} = \text{tang.}\,(\gamma - \delta)$ et $\frac{RS + rs}{1 - rsRS} = \text{tang.}\,(\gamma + \delta)$. Commodius autem iſta compoſitionis ratio repraeſentabitur, ſi ponatur

Corda arcus $m$ cupli $r = M\,\text{ſin}\,\mu$, corda complementi $R = M\,\text{coſ}\,\mu$

Corda arcus $n$ cupli $s = N\,\text{ſin}\,\nu$; corda compl. $S = N\,\text{coſ}\,\nu$ tum enim erit

Corda arcus $(m + n)$ cupli $= \frac{MN\,\text{ſin}\,(\mu + \nu)}{1 - M^2N^2\,\text{ſin}\,\mu\,\text{ſin}\,\nu\,\text{coſ}\,\mu\,\text{coſ}\,\nu}$

Corda eius complementi $= \frac{MN\,\text{coſ.}\,(\mu + \nu)}{1 + M^2N^2\,\text{ſin}\,\mu\,\text{ſin}\,\nu\,\text{coſ}\,\mu\,\text{coſ}\,\nu}$

Corda arcus $(m - n)$ cupli $= \frac{MN\,\text{ſin}\,(\mu - \nu)}{1 + M^2N^2\,\text{ſin}\,\mu\,\text{ſin}\,\nu\,\text{coſ}\,\mu\,\text{coſ}\,\nu}$

Corda eius complementi $= \frac{MN\,\text{coſ.}\,(\mu - \nu)}{1 - M^2N^2\,\text{ſin}\,\mu\,\text{ſin}\,\nu\,\text{coſ}\,\mu\,\text{coſ}\,\nu}$

Cum autem ſit $1 - rr - RR = rrRR$, erit $1 - MM = M^4\,\text{ſin}\,\mu^2\,\text{coſ}\,\mu^2$, ideoque $M^2\,\text{ſin}\,\mu\,\text{coſ}\,\mu = \surd(1 - MM)$ et $N^2\,\text{ſin}\,\nu\,\text{coſ}\,\nu = \surd(1 - NN)$, vnde iſtarum formularum denominatores abibunt in

$$1 - \surd(1 - MM)(1 - NN) \text{ et } 1 + \surd(1 - M^2)(1 - NN)$$

Praeterea vero ex illa aequatione $1 - MM = M^4\,\text{ſin}\,\mu^2\,\text{coſ.}\,\mu^2$ fit $\frac{1}{MM} = \frac{1}{2} + \frac{1}{2}\surd(1 + \text{ſin}\,2\mu\,.\,\text{ſin}\,2\mu)$ ob $\text{ſin}\,2\mu = 2\,\text{ſin}\,\mu\,\text{coſ}\,\mu$. Verum hinc illae formulae non concinniores euadunt.

## Scholion 2.

48. Ex his obſeruationibus calculus integralis non contemnenda augmenta conſequitur, ſiquidem hinc plu-

 rima-

rimarum aequationum differentialium integrales particulares exhibere valemus, quarum integratio in genere vix ſperari poteſt. Sic propoſita aequatione differentiali

$$\frac{du}{\sqrt{(1-u^4)}} = \frac{dz}{\sqrt{(1-z^4)}}$$

praeterquam quod caſus integralis $u = z$ per ſe eſt obvius, nouimus ei quoque ſatisfacere $u = -\sqrt{\frac{1-zz}{1+zz}}$. In genere igitur cum integratio conſtantem arbitrariam puta C inuoluat, erit $u$ aequalis functioni cuipiam, quantitatum $z$ et C; quae tamen nihilominus ita erit comparata, vt pro certo quodam ipſius C valore fiat $u = z$, itemque pro alio quodam ipſius C valore, $u = -\sqrt{\frac{1-zz}{1+zz}}$. Duo ergo dantur valores, quae conſtanti huic C tributi functionem illam in expreſſionem algebraicam adeo ſimplicem conuertunt.

Simili modo propoſita hac aequatione

$$\frac{du}{\sqrt{(1-u^4)}} = \frac{2dz}{\sqrt{(1-z^4)}}$$

duos habemus valores, quos ei ſatisfacere nouimus:

$$u = \frac{2z\sqrt{(1-z^4)}}{1+z^4} \text{ et } u = \frac{-1+2zz+z^4}{1+2zz-z^4}$$

pariterque geminos valores exhibere docuimus, qui in genere huic aequationi ſatisfaciant

$$\frac{m\,du}{\sqrt{(1-u^4)}} = \frac{n\,dz}{\sqrt{(1-z^4)}}$$

vnde

vnde via ad harum formularum integralia generalia inuenienda non parum praeparata videtur.

Deinde quae fupra de ellipfi et hyperbola funt allata, fequentes aequationum differentialium integrationes fpeciales fuppeditant.

Propofita enim ex §. 3 hac aequatione

$$dx\sqrt{\frac{1-nxx}{1-xx}}+du\sqrt{\frac{1-nuu}{1-uu}}=(xdu+udx)\sqrt{n}$$

nouimus ei fatisfacere hanc aequationem integralem

$$1-nxx-nuu+nuuxx=0$$

Ifti autem aequationi ex § 5 petitae

$$dx\sqrt{\frac{1-nxx}{1-xx}}+du\sqrt{\frac{1-nuu}{1-uu}}=n(xdu+udx)$$

fatisfacere inuenta eft haec aequatio

$$1-xx-uu+nuuxx=0$$

Deinde fequenti aequationi ex hyperbola §. 14 petitae

$$dx\sqrt{\frac{nxx-1}{xx-1}}+du\sqrt{\frac{nuu-1}{uu-1}}=(xdu+udx)\sqrt{n}$$

fatisfacit quoque $1-nxx-nuu+nuuxx=0$, quae quidem cum priore ex ellipfi petita congruit cum fit

$$\sqrt{\frac{nxx-1}{xx-1}}=\sqrt{\frac{1-nxx}{1-xx}}.$$

Hinc autem facile concludere licet, huic aequationi

$$dx\sqrt{\frac{f-gxx}{h-kxx}}+du\sqrt{\frac{f-guu}{h-kuu}}=(xdu+udx)\sqrt{\frac{g}{h}}$$

fatisfacere hanc integralem fpecialem

$$fh - gh(xx + uu) + gkxxuu = o$$

Isti autem aequationi alteri

$$dx\sqrt{\frac{f - gxx}{h - kxx}} + du\sqrt{\frac{f - guu}{h - kuu}} = (xdu + udx)\frac{g}{\sqrt{fk}}$$

satisfacere hanc integralem specialem

$$fh - fk(xx + uu) + gkxxuu = o$$

Haec igitur ideo proponenda censui, quod ansam mihi praebere videntur subsidia Analyseos vlterius excolendi.

DE

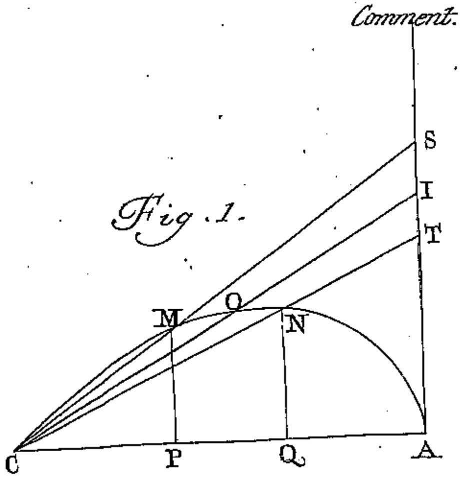



Fig. 1.

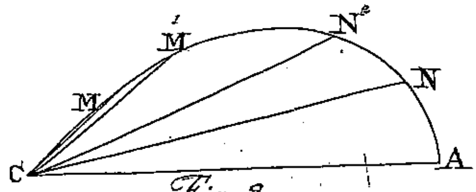



Fig. 2.

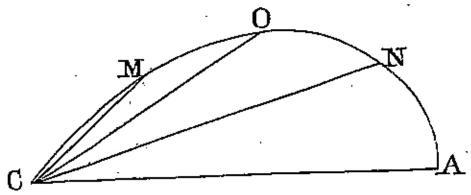



Fig. 3.

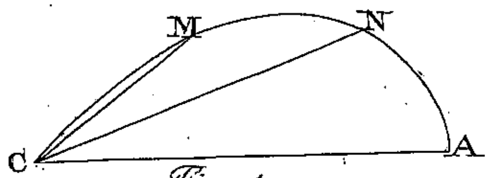



Fig. 4.

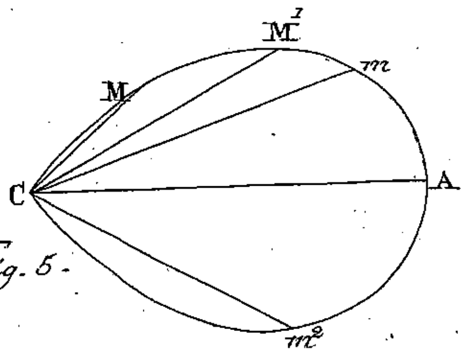



Fig. 5.

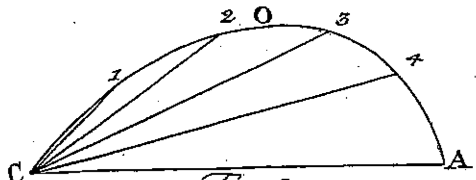



Fig. 6.

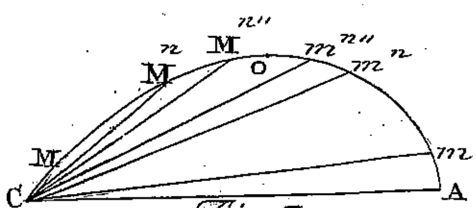



Fig. 7.

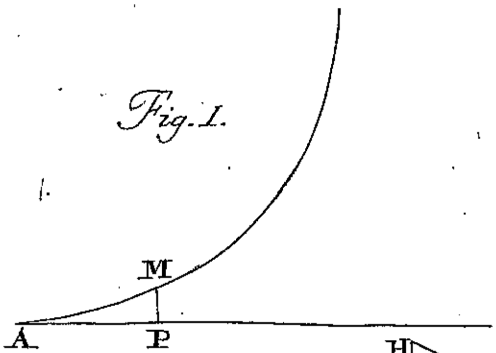


Fig. 1.
M
A
P

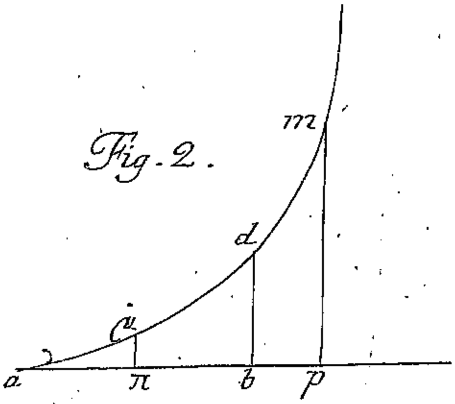
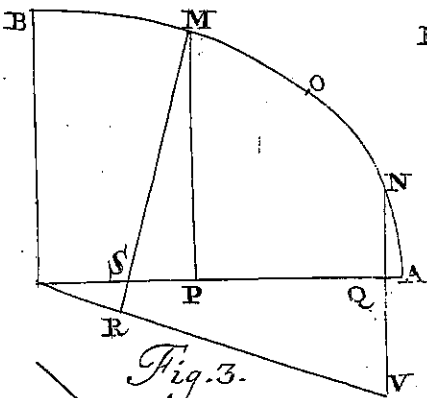
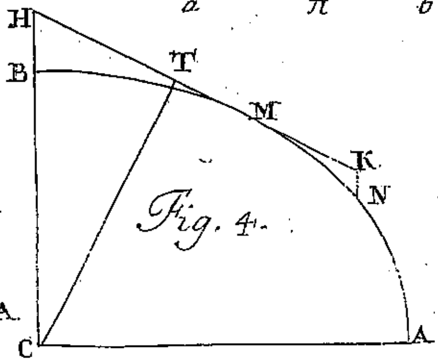
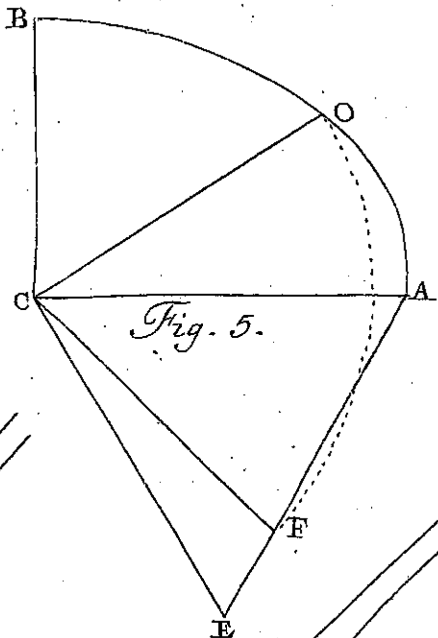


Fig. 2.
m
d
a
π
b
p


B
M
O
N
S
P
Q
A
R
V
Fig. 3.


H
T
B
M
K
N
Fig. 4.
C
A


B
O
C
A
Fig. 5.
F
E

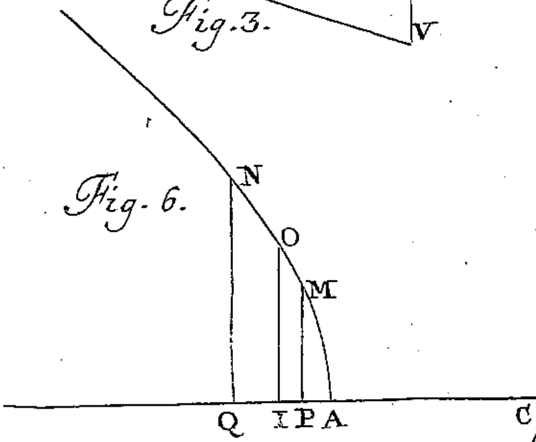


Fig. 6.
N
O
M
Q
I
P
A
C

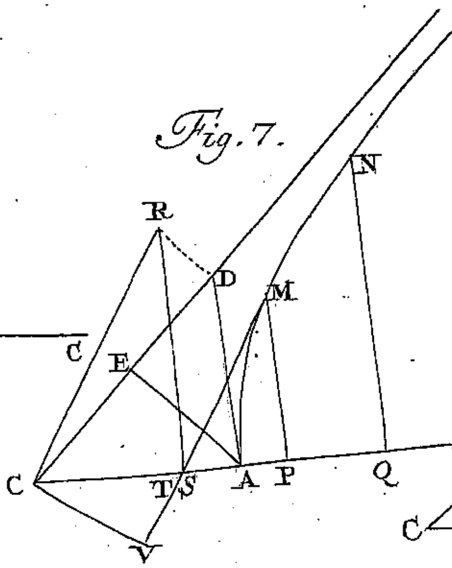
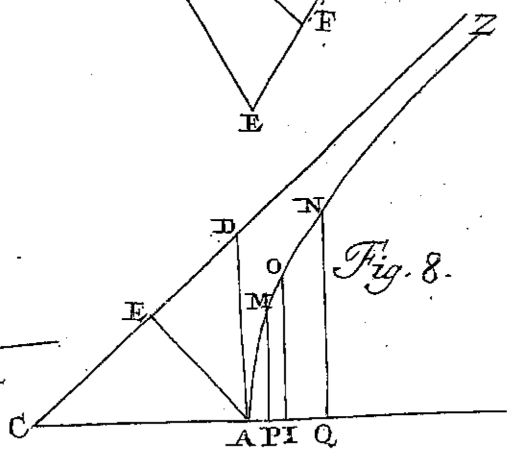


Fig. 7.
N
R
D
M
E
C
T
S
A
P
Q
V


Z
D
N
Fig. 8.
O
M
E
C
A
P
I
Q