



1760

Demonstratio theorematis circa ordinem in summis divisorum observatum

Leonhard Euler

Follow this and additional works at: <https://scholarlycommons.pacific.edu/euler-works>

 Part of the [Mathematics Commons](#)

Record Created:

2018-09-25

Recommended Citation

Euler, Leonhard, "Demonstratio theorematis circa ordinem in summis divisorum observatum" (1760). *Euler Archive - All Works*. 244.
<https://scholarlycommons.pacific.edu/euler-works/244>

This Article is brought to you for free and open access by the Euler Archive at Scholarly Commons. It has been accepted for inclusion in Euler Archive - All Works by an authorized administrator of Scholarly Commons. For more information, please contact mgibney@pacific.edu.

DEMONSTRATIO

THEOREMATIS CIRCA ORDINEM IN SUMMIS DIVISORVM OBSERVATVM.

AVCT. L. EVLERO.

Iam ab aliquo tempore incidi in theorema, quo natura numerorum non mediocriter illustrari est visa, cum in eo ordo contineatur, quem summae diuisorum, ex numeris serie naturali procedentibus ortae, inter se tenent. Ostendi enim, si singulorum numerorum naturalium 1, 2, 3, 4, 5, 6, 7, 8, etc. omnes diuisores in vnam summam colligantur, haeque diuisorum summae in seriem disponantur, quae erit

1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18 etc. hanc seriem esse recurrentem; eiusque singulos terminos ex praecedentibus secundum quandam scalam relationis determinari. Atque hic ordo non solum ideo maxime notatu dignus est visus, quod vix quisquam suspicatus fuerit, hanc seriem certae cuiusdam legi esse adstrictam, sed etiam, quod istius ordinis nullam demonstrationem firmam mihi quidem tum temporis reperire licuerit, etiamsi pluribus modis rem tentauerim. Perductus quidem fui ad huius ordinis observationem, dum sequentem formulam in infinitum productam sum contemplantus:

$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$
ex cuius evolutione per inductionem conclusi, fore

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{etc.}$$

76 DEMONSTRATIO THEOREMATIS

vbi exponentium ipsius x ordo eorum differentis sumendis fit manifestus; erit enim series differentiarum

1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, 13, 7, 15, 8, etc.

Excerptis enim terminis alternis patet, hanc seriem esse permixtam ex serie numerorum imparium, et ex serie numerorum omnium integrorum. Verum quod sit secundum hanc legem: $s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} +$ etc. siquidem fuerit $s = (1-x)(1-xx)(1-x^5)(1-x^4)(1-x^5)$ etc. per inductionem tantum collegi, neque aequalitatem harum duarum formularum solida demonstratione euincere potui. Quam ob causam etiam ordinem illum, quem in summis diuisorum hinc elicui, firmiter demonstrare non valui; sed eius demonstrationem iam tum inniti declaravi demonstrationi aequalitatis inter binas illas formulas infinitas modo exhibitas. Cum igitur nunc istam demonstrationem sum adeptus, ordinem quoque illum in summis diuisorum detectum non amplius illis veritatibus, quae agnoscuntur, neque tamen demonstrari possunt, accenseri conueniet, quemadmodum tum temporis sum arbitratus, sed iam merito ipsi locus inter veritates rigide demonstratas assignari poterit. Cuius rei ne vllum dubium relinquatur, singulas propositiones, quibus demonstratio huius veritatis innitur, hic ordine apponam atque demonstrabo:

PROPOSITIO I.

Si sit $s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)(1+\zeta)(1+\eta)$ etc. productum hoc, ex infinitis factoribus constans, in seriem sequentem conuertitur:

$$s = (1+\alpha) + \beta(1+\alpha) + \gamma(1+\alpha)(1+\beta) + \delta(1+\alpha)(1+\beta)(1+\gamma) + \varepsilon(1+\alpha)(1+\beta)(1+\gamma)(1+\delta) + \zeta(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon) + \text{etc.}$$

DEMON-

DEMONSTRATIO.

Cum enim seriei primus terminus sit $(1+\alpha)$ et secundus $= \beta(1+\alpha)$, erit summa primi et secundi $= (1+\alpha)(1+\beta)$: si iam addatur tertius terminus $\gamma(1+\alpha)(1+\beta)$, prodibit $(1+\alpha)(1+\beta)(1+\gamma)$: addatur insuper terminus quartus, qui est $\delta(1+\alpha)(1+\beta)(1+\gamma)$, erit summa $= (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)$. Atque sic in infinitum procedendo, summa totius seriei, seu omnium eius terminorum, perducetur ad hoc productum: $(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)(1+\zeta)$ etc. Unde manifestum est, si fuerit

$$s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)(1+\zeta) \text{ etc.}$$

fore vicissim:

$$s = (1+\alpha) + \beta(1+\alpha) + \gamma(1+\alpha)(1+\beta) + \delta(1+\alpha)(1+\beta)(1+\gamma) + \text{etc.}$$

PROPOSITIO II.

Si fuerit $s = (1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. productum hoc, ex infinitis factoribus constans, re-ducetur ad hanc seriem:

$$s = 1-x-xx(1-x)-x^3(1-x)(1-x^2)-x^4(1-x)(1-x^2)(1-x^3) \text{ etc.}$$

DEMONSTRATIO.

Si haec forma $s = (1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ etc. cum forma praecedente $s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)$ etc. comparetur, manifestum est fore:

$$\alpha = -x; \beta = -x^2; \gamma = -x^3; \delta = -x^4; \epsilon = -x^5; \text{ etc.}$$

His igitur valoribus in serie ibi data, quae producto s aequalis est inuenta, rite substitutis, patebit propositionis veritas, scilicet esse:

$$s = 1-x-xx(1-x)-x^3(1-x)(1-x^2)-x^4(1-x)(1-x^2)(1-x^3) \text{ etc.}$$

78 DEMONSTRATIO THEOREMATIS

PROPOSITIO III.

Si fuerit $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$ etc. erit hoc productum infinitum per multiplicationem euoluendo, terminosque secundum potestates ipsius x disponendo:

$$s = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - \text{etc.}$$

cuius seriei ratio est ea ipsa, quae supra est exposita.

DEMONSTRATIO.

Cum sit $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$ etc. erit $s = 1 - x - xx(1-x) - x^5(1-x)(1-x^2) - x^7(1-x)(1-x^2)(1-x^3) + \text{etc.}$

Ponatur $f = 1 - x - Axx$, erit:

$$A = 1 - x + x(1-x)(1-xx) + x^5(1-x)(1-x^2)(1-x^3) + \text{etc.}$$

Euoluantur singuli termini tantum secundum factorem $1-x$, ac sequenti modo disponantur:

$$A = \frac{-x}{1+x} + \frac{-x^2(1-xx)}{1+x(1-xx)} + \frac{-x^5(1-x^2)}{1+x^5(1-x^2)} + \frac{-x^7(1-x^2)(1-x^3)}{1+x^7(1-x^2)(1-x^3)} + \text{etc.}$$

eritque terminis subscriptis colligendis:

$$A = 1 - x^5 - x^5(1-x^2) - x^7(1-x^2)(1-x^3) - x^9(1-x^2)(1-x^3)(1-x^4) - \text{etc.}$$

Ponatur $A = 1 - x^5 - Bx^5$, erit

$$B = 1 - x^2 + x^2(1-x^2)(1-x^3) + x^4(1-x^2)(1-x^3)(1-x^4) + \text{etc.}$$

in quibus terminis subscriptis $1-x^2$ tantum euoluatur, ac fiet

$$B = \frac{-x^2}{1+x^2} + \frac{-x^4(1-x^3)}{1+x^2(1-x^3)} + \frac{-x^6(1-x^3)(1-x^4)}{1+x^6(1-x^3)(1-x^4)} + \text{etc.}$$

denuoque terminis subscriptis colligendis habebitur:

$$B = 1 - x^5 - x^6(1-x^3) - x^{11}(1-x^3)(1-x^4) - x^{14}(1-x^3)(1-x^4)(1-x^5) - \text{etc.}$$

Ponatur $B = 1 - x^5 - Cx^6$, erit

$$C = 1 - x^3 + x^3(1-x^3)(1-x^4) + x^6(1-x^3)(1-x^4)(1-x^5) + \text{etc.}$$

vbi in singulis terminis factor $1-x^3$ euoluatur, vt fiat scribendo vt supra:

$$C =$$

$$C = \frac{-x^3}{1+x^3(1+x^3)+x^6(1-x^4)(1-x^5)+x^9(1-x^4)(1-x^5)(1-x^6)} - \text{etc.}$$

vnde colligetur:

$$C = \frac{-x^3}{1+x^3(1+x^3)+x^6(1-x^4)(1-x^5)+x^9(1-x^4)(1-x^5)(1-x^6)} - \text{etc.}$$

Ponatur $C = 1 - x^7 - D x^{11}$, erit

$$D = \frac{-x^4}{1+x^4(1-x^5)+x^8(1-x^5)(1-x^6)+x^{12}(1-x^4)(1-x^5)(1-x^6)} + \text{etc.}$$

quae abit in hanc formam:

$$D = \frac{-x^4}{1+x^4(1-x^5)+x^8(1-x^5)(1-x^6)+x^{12}(1-x^4)(1-x^5)(1-x^6)} + \text{etc.}$$

ficque erit

$$D = \frac{-x^4}{1+x^4(1-x^5)+x^8(1-x^5)(1-x^6)+x^{12}(1-x^4)(1-x^5)(1-x^6)} - \text{etc.}$$

Quodsi porro ponatur $D = 1 - x^9 - E x^{14}$, reperietur simili modo:

$$E = 1 - x^{11} - F x^{17}; \text{ hincque ultra:}$$

$$F = 1 - x^{17} - G x^{23}; G = 1 - x^{15} - H x^{23}; H = 1 - x^{17} - I x^{26};$$

etc.

Restituamus iam successive hos valores, eritque:

$$s = 1 - x - A x x$$

$$A x^2 = x^2 (1 - x^3) - B x^7$$

$$B x^7 = x^7 (1 - x^5) - C x^{15}$$

$$C x^{15} = x^{15} (1 - x^7) - D x^{26}$$

$$D x^{26} = x^{26} (1 - x^9) - E x^{40}$$

etc.

Quamobrem habebimus:

$$s = 1 - x - x^2(1-x^3) + x^7(1-x^5) - x^{15}(1-x^7) + x^{26}(1-x^9) - x^{40}(1-x^{11}) + \text{etc.}$$

sive id ipsum, quod demonstrari oportet:

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.}$$

vnde simul lex exponentium supra indicata per differentias luculente perspicitur.

PRO-

80 DEMONSTRATIO THEOREMATIS

PROPOSITIO IV.

S E V

THEOREMA PRINCIPALE DEMONSTRANDVM.

Si haec scribendi formula f_n denotet summam omnium diuisorum numeri n , similique modo numerorum minorum, veluti $n-a$, designentur per $f(n-a)$, summa diuisorum numeri n , seu f_n , ita pendebit a summis diuisorum numerorum minorum, vt fit

$$f_n = f(n-1) + f(n-2) - f(n-5) - f(n-7) + f(n-12) + f(n-15) \\ - f(n-22) - f(n-26) + f(n-35) + f(n-40) - f(n-51) - f(n-57) \text{ etc.}$$

Vbi sequentia sunt notanda :

1°. Signa $+$ et $-$ geminata terminos huius progressionis alternatim afficere.

2°. Legem numerorum 1, 2, 5, 7, 12, 15, 22, 26, etc. ex eorum differentiis, quae sunt 1, 3, 2, 5, 3, 7, 4, etc. fieri manifestam; vnde colligitur hos numeros omnes in formula hac generali $\frac{zzz+z}{2}$ contineri.

3°. Quouis casu istius progressionis eos tantum terminos ab initio esse accipiendos, qui post signum f numeros affirmatiuos retineant; reliquos vero omnes, quibus signum f numeris negatiuis praefigitur, esse omittendos; ita si fit $n = 10$, erit $f_{10} = f_9 + f_8 - f_5 - f_3 = 13 + 15 - 6 - 4 = 18$.

4°. Quibus casibus occurrit terminus $f(n-n)$, quod euenit, si n fuerit numerus huius seriei 1, 2, 5, 7, 12, 15 etc. iis casibus pro valore huius termini $f(n-n)$, seu f_0 assumi oportere ipsum numerum propositum n ; sic si fit

DE SUMMIS DIVISORVM. 81

fit $n = 7$, erit $f_7 = f_6 + f_5 - f_2 - f_0 = 12 + 6 - 3 - 7 = 8$, et si fit $n = 12$, erit $f_{12} = f_{11} + f_{10} - f_7 - f_5 + f_0 = 12 + 18 - 8 - 6 + 12 = 28$.

DEMONSTRATIO.

Formetur series $z = x f_1 + x^2 f_2 + x^3 f_3 + x^4 f_4 + x^5 f_5 + \text{etc.}$ ubi quaelibet potestas ipsius x multiplicata fit per summam diuisorum exponentis eius potestatis. Quodsi iam singulae diuisorum summae resoluantur, manifestum est, hanc seriem transformari in hanc formam

$$z = 1(x + x^2 + x^3 + x^4 + x^5 + \text{etc.}) + x(x^2 + x^4 + x^6 + x^8 + x^{10} + \text{etc.}) + x^2(x^3 + x^6 + x^9 + x^{12} + x^{15} + \text{etc.}) + x^3(x^4 + x^8 + x^{12} + x^{16} + x^{20} + \text{etc.}) + x^4(x^5 + x^{10} + x^{15} + x^{20} + x^{25} + \text{etc.}) + x^5(x^6 + x^{12} + x^{18} + x^{24} + x^{30} + \text{etc.}) + \text{etc.}$$

quibus seriebus geometricis summatis fiet :

$$z = \frac{1x}{1-x} + \frac{2xx}{1-xx} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{6x^6}{1-x^6} + \text{etc.}$$

Multiplicetur haec forma per $-\frac{dx}{x}$, ac producti integrale erit

$$-\int \frac{zdx}{x} = l(1-x) + l(1-xx) + l(1-x^3) + l(1-x^4) + l(1-x^5) + \text{etc.}$$

$$\text{seu } -\int \frac{zdx}{x} = l(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

quae expressio post signum logarithmicum, cum sit eadem, quae in propositione praecedente vocata est $= s$,

erit $-\int \frac{zdx}{x} = l s$, ideoque alterum valorem pro s sumendo, erit quoque :

$$-\int \frac{zdx}{x} = l(1-x-x^2+x^5+x^7-x^{12}-x^{13}+x^{22}+x^{26} - \text{etc.})$$

Tom. V. Nou. Com.

L

cuius

etc. qui in illis formulis conspiciuntur, vicem tenere termini f_0 ; vnde concluditur, in casibus quibus pro f_n in progressionē illa reperta occurrit terminus $f(n - n)$, seu f_0 , valorem eius semper ipsi numero proposito n aequalem esse capiendum: sicque habetur plena ac perfecta demonstratio theorematis propositi, quae, cum praeter tractationem serierum infinitarum, per logarithmos et differentia procedat, minus quidem naturalis, sed ob hoc ipsum multo magis notabilis est aestimanda.