1760

Demonstratio theorematis circa ordinem in summis divisorum observatum

Leonhard Euler

DEMONSTRATIO
THEOREMATIS CIRCA ORDINEM IN SVMMIS DIVISORVM OBSERVATVM.

AVCT. *L. EVLERO.*

Iam ab aliquo tempore incidi in theorema, quo natura numerorum non mediocriter illuſtrari eſt viſa, cum in eo ordo contineatur, quem ſummae diuiſorum, ex numeris ſerie naturali procedentibus ortae, inter ſe tenent. Oſtendi enim, ſi ſingulorum numerorum naturalium 1, 2, 3, 4, 5, 6, 7, 8, etc. omnes diuiſores in vnam ſummam colligantur, haeque diuiſorum ſummae in ſeriem diſponantur, quae erit

1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18 etc.

hanc ſeriem eſſe recurrentem, eiusque ſingulos terminos ex praecedentibus ſecundum quandam ſcalam relationis determinari. Atque hic ordo non ſolum ideo maxime notatu dignus eſt viſus, quod vix quisquam ſuſpicatus fuerit, hanc ſeriem certae cuipiam legi eſſe adſtrictam, ſed etiam, quod iſtius ordinis nullam demonſtrationem firmam mihi quidem tum temporis reperire licuerit, etiamſi pluribus modis rem tentauerim. Perductus quidem fui ad huius ordinis obſeruationem, dum ſequentem formulam in infinitum productam ſum contemplatus:

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7) \text{ etc.}$$

ex cuius euolutione per inductionem concluſi, fore

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{ etc.}$$

 vbi

vbi exponentium ipſius x ordo eorum differentiis ſumendis ſit manifeſtus; erit enim ſeries differentiarum

1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, 13, 7, 15, 8, etc.

Excerptis enim terminis alternis patet, hanc ſeriem eſſe permixtam ex ſerie numerorum imparium, et ex ſerie numerorum omnium integrorum. Verum quod ſit ſecundum hanc legem: $s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} +$ etc. ſiquidem fuerit $s = (1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ etc. per inductionem tantum collegi, neque aequalitatem harum duarum formularum ſolida demonſtratione euincere potui. Quam ob cauſam etiam ordinem illum, quem in ſummis diuiſorum hinc elicui, firmiter demonſtrare non valui; ſed eius demonſtrationem iam tum inniti declaraui demonſtrationi aequalitatis inter binas illas formulas infinitas modo exhibitas. Cum igitur nunc iſtam demonſtrationem ſim adeptus, ordinem quoque illum in ſummis diuiſorum detectum non amplius illis veritatibus, quae agnoſcuntur, neque tamen demonſtrari poſſunt, accenſeri conueniet, quemadmodum tum temporis ſum arbitratus, ſed iam merito ipſi locus inter veritates rigide demonſtratas aſſignari poterit. Cuius rei ne vllum dubium relinquatur, ſingulas propoſitiones, quibus demonſtratio huius veritatis innititur, hic ordine apponam atque demonſtrabo:

PROPOSITIO I.

Si ſit $s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)(1+\zeta)(1+\eta)$ etc. productum hoc, ex infinitis factoribus conſtans, in ſeriem ſequentem conuertitur:

$$s = (1+\alpha) + \beta(1+\alpha) + \gamma(1+\alpha)(1+\beta) + \delta(1+\alpha)(1+\beta)(1+\gamma)$$
$$+ \varepsilon(1+\alpha)(1+\beta)(1+\gamma)(1+\delta) + \zeta(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon) + \text{etc.}$$

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DEMONSTRATIO.

Cum enim ſeriei primus terminus ſit $(1+\alpha)$ et ſecundus $=\beta(1+\alpha)$, erit ſumma primi et ſecundi $=(1+\alpha)(1+\beta)$: ſi iam addatur tertius terminus $\gamma(1+\alpha)(1+\beta)$, prodibit $(1+\alpha)(1+\beta)(1+\gamma)$: addatur inſuper terminus quartus, qui eſt $\delta(1+\alpha)(1+\beta)(1+\gamma)$, erit ſumma $=(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)$. Atque ſic in infinitum procedendo, ſumma totius ſeriei, ſeu omnium eius terminorum, perducetur ad hoc productum: $(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)(1+\zeta)$ etc. Vnde manifeſtum eſt, ſi fuerit

$$s=(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)(1+\zeta)\text{ etc.}$$

fore viciſſim:

$$s=(1+\alpha)+\beta(1+\alpha)+\gamma(1+\alpha)(1+\beta)+\delta(1+\alpha)(1+\beta)(1+\gamma)+\text{ etc.}$$

PROPOSITIO II.

Si fuerit $s=(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. productum hoc, ex infinitis factoribus conſtans, reducetur ad hanc ſeriem:

$$s=1-x-xx(1-x)-x^3(1-x)(1-x^2)-x^4(1-x)(1-x^2)(1-x^3)\text{ etc.}$$

DEMONSTRATIO.

Si haec forma $s=(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ etc. cum forma praecedente $s=(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)$ etc. comparetur, manifeſtum eſt fore: $\alpha=-x$; $\beta=-x^2$; $\gamma=-x^3$; $\delta=-x^4$; $\varepsilon=-x^5$; etc. His igitur valoribus in ſerie ibi data, quae producto s aequalis eſt inuenta, rite ſubſtitutis, patebit propoſitionis veritas, ſcilicet eſſe:

$$s=1-x-xx(1-x)-x^3(1-x)(1-x^2)-x^4(1-x)(1-x^2)(1-x^3)-\text{ etc.}$$

PROPOSITIO III.

Si fuerit $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$ etc. erit hoc productum infinitum per multiplicationem euoluendo, terminosque secundum potestates ipsius x disponendo:

$$s = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - etc.$$

cuius seriei ratio est ea ipsa, quae supra est exposita.

DEMONSTRATIO.

Cum sit $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$ etc. erit $s = 1 - x - xx(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) + $ etc.

Ponatur $s = 1 - x - Axx$, erit:

$$A = 1 - x + x(1-x)(1-xx) + x^2(1-x)(1-x^2)(1-x^3) + etc.$$

Euoluantur singuli termini tantum secundum factorem $1-x$, ac sequenti modo disponantur:

$$A = \begin{array}{l} \quad\ \ -x \qquad\qquad\quad -x^2(1-xx) \qquad\qquad\qquad -x^3(1-x^2)(1-x^3) \qquad\qquad - etc. \\ +1 + x(1-xx) + x^2(1-x^2)(1-x^3) + x^3(1-x^2)(1-x^3)(1+x^4) + etc. \end{array}$$

eritque terminis subscriptis colligendis:

$$A = 1 - x^3 - x^5(1-x^2) - x^7(1-x^2)(1-x^3) - x^9(1-x^2)(1-x^3)(1-x^4) - etc.$$

Ponatur $A = 1 - x^2 - Bx^5$, erit

$$B = 1 - x^2 + x^2(1-x^2)(1-x^3) + x^4(1-x^2)(1-x^3)(1-x^4) + etc.$$

in quibus terminis subscriptis $1 - x^2$ tantum euoluatur, ac fiet

$$B = \begin{array}{l} \quad -x^2 \qquad\qquad\quad -x^4(1-x^3) \qquad\qquad\quad -x^6(1-x^3)(1-x^4) \\ 1 + x^2(1-x^3) + x^4(1-x^3)(1-x^4) + x^6(1-x^3)(1-x^4)(1-x^5) \end{array}$$

denuoque terminis subscriptis colligendis habebitur:

$$B = 1 - x^5 - x^8(1-x^3) - x^{11}(1-x^3)(1-x^4) - x^{14}(1-x^3)(1-x^4)(1-x^5) - etc.$$

Ponatur $B = 1 - x^5 - Cx^8$, erit

$$C = 1 - x^3 + x^3(1-x^3)(1-x^4) + x^6(1-x^3)(1-x^4)(1-x^5) + etc.$$

vbi in singulis terminis factor $1 - x^3$ euoluatur, vt fiat scribendo vt supra:

C =

$$C = \frac{-x^5 - x^6(1-x^4) - x^9(1-x^4)(1-x^5) - \text{etc.}}{1 + x^5(1+x^4) + x^6(1-x^4)(1-x^5) + x^9(1-x^4)(1-x^5)(1-x^6) + \text{etc.}}$$

vnde colligetur:

$$C = 1 - x^7 - x^{11}(1-x^4) - x^{15}(1-x^4)(1-x^5) - x^{19}(1-x^4)(1-x^5)(1-x^6) - \text{etc.}$$

Ponatur $C = 1 - x^7 - D\,x^{11}$, erit

$$D = 1 - x^4 + x^4(1-x^4)(1-x^5) + x^8(1-x^4)(1-x^5)(1-x^6) + \text{etc.}$$

quae abit in hanc formam:

$$D = \frac{-x^4 - x^8(1-x^5) - x^{12}(1-x^5)(1-x^6) - \text{etc.}}{1 + x^4(1+x^5) + x^8(1-x^5)(1-x^6) + x^{12}(1-x^5)(1-x^6)(1-x^7) + \text{etc.}}$$

ſicque erit

$$D = 1 - x^9 - x^{14}(1-x^5) - x^{19}(1-x^5)(1-x^6) - x^{24}(1-x^5)(1-x^6)(1-x^7) - \text{etc.}$$

Quodſi porro ponatur $D = 1 - x^9 - Ex^{14}$, reperietur ſimili modo:

$E = 1 - x^{11} - Fx^{17}$; hincque vltra:

$F = 1 - x^{17} - Gx^{20}$; $G = 1 - x^{15} - Hx^{23}$; $H = 1 - x^{17} - Ix^{26}$;

etc.

Reſtituamus iam ſucceſſiue hos valores, eritque:

$$\begin{aligned} s &= 1 - x - Axx \\ Ax^2 &= x^2(1 - x^3) - Bx^7 \\ Bx^7 &= x^7(1 - x^5) - Cx^{15} \\ Cx^{15} &= x^{15}(1 - x^7) - Dx^{26} \\ Dx^{26} &= x^{26}(1 - x^9) - Ex^{40} \end{aligned}$$

etc.

Quamobrem habebimus:

$$s = 1 - x - x^2(1-x^3) + x^7(1-x^5) - x^{15}(1-x^7) + x^{26}(1-x^9) - x^{40}(1-x^{11}) + \text{etc.}$$

ſiue id ipſum, quod demonſtrari oportet:

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.}$$

vnde ſimul lex exponentium ſupra indicata per differentias luculente perſpicitur.

PRO-

PROPOSITIO IV.

SEV

THEOREMA PRINCIPALE DEMONSTRANDVM.

Si haec ſcribendi formula $\int n$ denotet ſummam omnium diuiſorum numeri n, ſimilique modo numerorum minorum, veluti $n-\alpha$, deſignentur per $\int(n-\alpha)$, ſumma diuiſorum numeri n, ſeu $\int n$, ita pendebit a ſummis diuiſorum numerorum minorum, vt ſit

$$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15) - \int(n-22) - \int(n-26) + \int(n-35) + \int(n-40) - \int(n-51) - \int(n-57) \text{ etc.}$$

Vbi ſequentia ſunt notanda:

1°. Signa $+$ et $-$ geminata terminos huius progreſſionis alternatim afficere.

2°. Legem numerorum 1, 2, 5, 7, 12, 15, 22, 26, etc. ex eorum differentiis, quae ſunt 1, 3, 2, 5, 3, 7, 4, etc. fieri manifeſtam; vnde colligitur hos numeros omnes in formula hac generali $\frac{3zz \pm z}{2}$ contineri.

3°. Quouis caſu iſtius progreſſionis eos tantum terminos ab initio eſſe accipiendos, qui poſt ſignum $\int$ numeros affirmatiuos retineant; reliquos vero omnes, quibus ſignum $\int$ numeris negatiuis praefigitur, eſſe omittendos; ita ſi ſit $n = 10$, erit $\int 10 = \int 9 + \int 8 - \int 5 - \int 3 = 13 + 15 - 6 - 4 = 18$.

4°. Quibus caſibus occurrit terminus $\int(n-n)$, quod euenit, ſi n fuerit numerus huius ſeriei 1, 2, 5, 7, 12, 15 etc. iis caſibus pro valore huius termini $\int(n-n)$, ſeu $\int 0$ aſſumi oportere ipſum numerum propoſitum n; ſic ſi ſit

sit $n = 7$, erit $\int 7 = \int 6 + \int 5 - \int 2 - \int 0 = 12 + 6 - 3 - 7 = 8$, et si sit $n = 12$, erit $\int 12 = \int 11 + \int 10 - \int 7 - \int 5 + \int 0 = 12 + 18 - 8 - 6 + 12 = 28$.

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Formetur series $z = x\int 1 + x^2\int 2 + x^3\int 3 + x^4\int 4 + x^5\int 5 +$ etc. vbi quaelibet potestas ipsius x multiplicata sit per summam diuisorum exponentis eius potestatis. Quodsi iam singulae diuisorum summae resolvantur, manifestum est, hanc seriem transformari in hanc formam

$$z = 1(x + x^2 + x^3 + x^4 + x^5 + etc.) + 2(x^2 + x^4 + x^6 + x^8 + x^{10} + etc.)$$
$$+ 3(x^3 + x^6 + x^9 + x^{12} + x^{15} + etc.) + 4(x^4 + x^8 + x^{12} + x^{16} + x^{20} + etc).$$
$$+ 5(x^5 + x^{10} + x^{15} + x^{20} + x^{25} + etc. + 6(x^6 + x^{12} + x^{18} + x^{24} + x^{30} + etc.)$$

etc.

quibus seriebus geometricis summatis fiet:

$$z = \frac{1x}{1-x} + \frac{2xx}{1-xx} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{6x^6}{1-x^6} + \text{etc.}$$

Multiplicetur haec forma per $-\frac{dx}{x}$, ac producti integrale erit

$$-\int\frac{zdx}{x} = l(1-x) + l(1-xx) + l(1-x^3) + l(1-x^4) + l(1-x^5) + \text{etc.}$$

seu $-\int\frac{zdx}{x} = l(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc.

quae expressio post signum logarithmicum, cum sit eadem, quae in propositione praecedente vocata est $= s$, erit $-\int\frac{zdx}{x} = l\,s$, ideoque alterum valorem pro s sumendo, erit quoque:

$$-\int\frac{zdx}{x} = l(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.})$$

cuius differentiale per $\frac{-dx}{x}$ diuiſum, dabit alium valorem pro z, nempe

$$z = \frac{1x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - etc.}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + etc.}$$

qui valor ſi aequalis ponatur aſſumto, et vtrinque per denominatorem $1 - x - x^2 + x^5 + x^7 - x^{12}$ etc. multiplicetur, reperietur terminis ſecundum poteſtates ipſius x diſponendis, omnibusque ad eandem partem collocandis:

$$\begin{array}{rllllllllll}
0 = & x\int 1 & + x^2\int 2 & + x^3\int 3 & + x^4\int 4 & + x^5\int 5 & + x^6\int 6 & + x^7\int 7 & + x^8\int 8 & + x^9\int 9 & + x^{10}\int 10 \; etc. \\
 & \cdot & - \int 1 & - \int 2 & - \int 3 & - \int 4 & - \int 5 & - \int 6 & - \int 7 & - \int 8 & - \int 9 \\
 & \cdot & \cdot & - \int 1 & - \int 2 & - \int 3 & - \int 4 & - \int 5 & - \int 6 & - \int 7 & - \int 8 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & + \int 1 & + \int 2 & + \int 3 & + \int 4 & + \int 5 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \int 1 & + \int 2 & + \int 3 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & -1 & -2 & * & * & +5 & * & +7 & * & * & *
\end{array}$$

vnde ſingularum poteſtatum ipſius x coefficientibus nihilo aequatis, ſequitur fore

$$\begin{array}{ll}
\int 1 = 1 & \int 6 = \int 5 + \int 4 - \int 1 \\
\int 2 = \int 1 + 2 & \int 7 = \int 6 + \int 5 - \int 2 - 7 \\
\int 3 = \int 2 + \int 1 & \int 8 = \int 7 + \int 6 - \int 3 - \int 1 \\
\int 4 = \int 3 + \int 2 & \int 9 = \int 8 + \int 7 - \int 4 - \int 2 \\
\int 5 = \int 4 + \int 3 - 5 & \int 10 = \int 9 + \int 8 - \int 5 - \int 3
\end{array}$$

atque indolem illius aequationis vel leuiter attendenti patebit, eſſe generatim:

$$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15) \text{ etc.}$$

hac progreſſione quouis caſu eousque continuata, donec perueniatur ad ſummas numerorum negatiuorum. Deinde per ſe eſt perſpicuum, numeros abſolutos 1, 2, 5, 7, etc.

etc. qui in illis formulis conſpiciuntur, vicem tenere termini $\int o$; vnde concluditur, in caſibus quibus pro $\int n$ in progreſſione illa reperta occurrit terminus $\int(n-n)$, ſeu $\int o$, valorem eius ſemper ipſi numero propoſito n aequalem eſſe capiendum: ſicque habetur plena ac perfecta demonſtratio theorematis propoſiti, quae, cum praeter tractationem ſerierum infinitarum, per logarithmos et differentialia procedat, minus quidem naturalis, ſed ob hoc ipſum multo magis notabilis eſt aeſtimanda.