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Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae

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E 190

CONSIDERATIO QVARVMDAM SERIERVM, QVAE SINGVLARIBVS PROPRIETATIBVS SVNT PRAEDITAE.

AVCTORE
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§. 1.

Saepe numero *conſideratio ſerierum*, quae quaſi caſu ſe nobis offerunt, non contemnenda ſuppeditare ſolet artificia, quibus deinceps in vniuerſa ſerierum doctrina ſummo cum fructu vti licet. Cum igitur doctrina *de ſeriebus* ſit maximi momenti in Analyſi, huiusmodi ſpeculationes omnino dignae ſunt habendae, quae omni induſtria euoluantur. Hunc in finem ſequentem ſeriem offerre conſtitui, quae, tum ob ſingulares, quibus praedita deprehenditur proprietate, tum vero propter inſignes vſus, quos nobis exhibet, omni attentione digna videtur. Series autem ita ſe habet:

$$\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6}+\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}+\text{etc.}$$

Lex numeratorum ex ſola inſpectione eſt manifeſta, formantur enim ex multiplicatione terminorum huius ſeriei:

$1-x$; $a-x$; a^2-x; a^3-x; a^4-x; a^5-x; a^6-x; etc.

Denominatores omnes duobus conſtant terminis, qui ſunt poteſtates ipſius a, quarum exponentes ſunt numeri trigonales. Hinc terminus ordine n ſeriei propoſitae erit:

$$\frac{(1-x)(a-x)(a^2-x)(a^3-x)\;.\;.\;.\;.\;.\;(a^{n-1}-x)}{a^{n(n-1):2}-a^{n(n+1):2}}$$

§. 2.

§. 2. Primo quidem patet, ſi quantitas x poteſtati cuipiam ipſius a aequalis capiatur, tum ſeriem alicubi ita obrumpi, vt omnes ſequentes termini abeant in nihilum. Ponamus ergo in genere s pro ſumma ſeriei propoſitae, vt ſit:

$$s = \frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^3} + \frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} + \frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}} + \text{etc.}$$

ac ſtatuatur primo $x = 1$, ſeu $x = a^0$, eritque ob omnes terminos evaneſcentes $s = 0$. Sit porro $x = a$, vt ſolus primus terminus ſuperſit, eritque $s = 1$. Sit $x = a^2$, fietque $s = \frac{1-a^2}{1-a} + \frac{(1-a^2)(a-a^2)}{a-a^3}$ ſeu $s = 2$. Ponatur $x = a^3$, ac prodibit:

$$s = \frac{1-a^3}{1-a} + \frac{(1-a^3)(a-a^3)}{a-a^3} + \frac{(1-a^3)(a-a^3)(a^2-a^3)}{a^3-a^6}.$$

Horum terminorum primus dat $1 + a + aa$; ſecundus dat $1 - a^3$, et tertius dat $1 - a - aa + a^3$; quibus collectis fiet $s = 3$.

§. 3. Simili modo ſi ponatur $x = a^4$, operatione inſtituta reperietur $s = 4$; et poſito $x = a^5$, prodibit $s = 5$. Vnde ſatis tuto per inductionem concludi poſſe videtur, quoties x cuicunque poteſtati ipſius a, cuius exponens ſit $= n$, aequatio ſtatuatur, toties hunc ipſum exponentem praebiturum eſſe valorem ipſius s. At vero haec inductio tantum valet, ſi n ſit numerus integer affirmatiuus. Quod ſi enim pro quouis numero fracto valeret, tum foret $s =$ logarithmo ipſius x, ſumto a pro numero, cuius longarithmus ſit $= 1$. Sic ſi hoc verum eſſet, poſito $a = 10$, ſumma ſeriei s ſemper exprimere deberet logarithmum communem ipſius x, eſſetque:

$$s = -\frac{(1-x)}{9} - \frac{(1-x)(10-x)}{990} - \frac{(1-x)(10-x)(100-x)}{999000} - \frac{(1-x)(10-x)(100-x)(1000-x)}{9999000000} - \text{etc.} = lx.$$

Ex

Ex ſequentibus autem perſpicuum euadet, hanc aequalitatem non ſubſiſtere, niſi ſit x poteſtas ipſius a, exponentem habens integrum affirmatiuum.

§. 4. Quod autem, poſito $x = a^n$, non ſemper ſit $s = n$, niſi n ſit numerus integer affirmatiuus, ex caſu quo $x = 0$ facile colligitur. Hoc enim caſu, ſi ſuperior inductio ſe ad omnes omnino numeros extenderet, fieri deberet $s = -\infty$, cum $-\infty$ ſit perpetuo longarithmus cyphrae. Verum poſito $x = 0$, fiet:

$$s = \frac{1}{1-a} + \frac{1}{1-aa} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

quae ſeries etſi ſummari non poteſt, tamen quilibet facile perſpiciet, eius ſummam eſſe debere finitam, neque propterea logarithmum ipſius $x = 0$ exprimere poſſe. Simili modo, ſi poſito $a = 10$, atque x non poteſtati ipſius 10 aequale ponatur, per ſummationem valor inuenietur, plerumque ſatis notabiliter a $l\,x$ diſcrepans. Sit enim $x = 9$, poſito $a = 10$, eritque:

$$s = \frac{8}{9} + \frac{8 \cdot 1}{990} + \frac{8 \cdot 1 \cdot 91}{999000} + \frac{8 \cdot 1 \cdot 91 \cdot 991}{9999000000} + \frac{8 \cdot 1 \cdot 91 \cdot 991 \cdot 9991}{99999000000000} \text{ etc.}$$

qui termini ſi in fractionibus decimalibus exprimantur, prodibit:

$$\begin{array}{rr}
s = 0, & 88888888888888888888 \\
 & 808080808080808080 \\
 & 8008008008008008 \\
 & 80008000800080 \\
 & 800008000080 \\
 & 8000008000 \\
 & 80000008 \\
 & 800000 \\
 & 8000 \\
 & 80 \\
\hline
s = 0, & 89705058521067321224
\end{array}$$

qui

qui valor vtique minor eſt, quam logarithmus nouenarii.

§. 5. Series igitur noſtra ita eſt comparata, vt ſi pro x ſubſtituantur poteſtates ipſius a rationales, ſumma ſeriei aequalis fiat exponenti illius poteſtatis: ſcilicet

ſi ſit $x = a^0, a^1, a^2, a^3, a^4, a^5, a^6, a^7, a^8$, etc.
erit $s = 0, 1, 2, 3, 4, 5, 6, 7, 8$, etc.

quae etſi eſt proprietas logarithmorum, tamen non niſi exponentes ipſius a ſint numeri integri. Quod ſi ergo concipiatur linea curua, cuius abſciſſae ſint s, et applicatae $= x$, haec curua logarithmicam in punctis innumeris interſecabit, ſcilicet quoties abſciſſa s per numerum integrum exprimitur, toties applicata per interſectionem tranſibit. Vnde patet, curuam logarithmicam ne per infinita quidem puncta determinari; quod etiam in omnibus aliis lineis curuis vſu venit. Hinc itaque intelligitur, quam libet ſeriem, etſi omnes eius termini indicibus integris reſpondentes dentur, infinitis modis diuerſis interpolari poſſe, quod argumentum alia occaſione vberius pertractabo.

§. 6. Quo autem propius ad cognitionem noſtrae ſeriei perueniamus, eam in hanc formam transmutare licet:

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-b^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \frac{1}{1-a^4}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\left(1-\frac{x}{a^3}\right) \text{etc.}$$

quae propterea ſimplicior eſt praecedente, quod hic numeri trigonales abierint. Ponamus nunc ax in locum ipſius x, denotetque t ſummam ſeriei hinc reſultantis, erit:

$$t = \frac{1}{1-a}(1-ax) + \frac{1}{1-a^2}(1-ax)(1-x) + \frac{1}{1-a^3}(1-ax)(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^4}(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

ſubtrahatur prior ſeries a poſteriore, ac reperietur:

$$t-s = x+\frac{x}{a}(1-x)+\frac{x}{aa}(1-x)(1-\tfrac{x}{a})+\frac{x}{a^3}(1-x)(1-\tfrac{x}{a})(1-\tfrac{x}{aa})+\text{etc.}$$

ſubtrahatur haec ſeries ab vnitate, et cum reſiduum ſit per $1-x$ diuiſibile erit:

$$1+s-t = (1-x)\left(1-\frac{x}{a}-\frac{x}{aa}(1-\tfrac{x}{a})-\frac{x}{a^3}(1-\tfrac{x}{a})(1-\tfrac{x}{aa})-\text{etc.}\right)$$

Hic factor poſterior autem porro diuiſibilis eſt per $1-\frac{x}{a}$, vnde fit $1+s-t = (1-x)(1-\frac{x}{a})(1-\frac{x}{aa}-\frac{x}{a^3}(1-\frac{x}{aa})-\text{etc.})$ Hic denuo factor deprehenditur $1-\frac{x}{aa}$, hocque ſeorſim expreſſo, factor apparebit $1-\frac{x}{a^3}$, et ita porro, vnde tandem reperitur fore:

$$1+s-t = (1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3})(1-\frac{x}{a^4})(1-\frac{x}{a^5})\ \text{etc.}$$

§. 7. Hinc igitur patet, quoties x aequalis capiatur cuipiam poteſtati ipſius a, ob vnum factorem huius expreſſionis euaneſcentem fore $1+s-t=0$, ſeu $t=1+s$. Quare ſi poſito $x=a^n$, denotante n numerum integrum affirmatiuum, fuerit ſumma ſeriei propoſitae $s=n$, poſito $x=a^{n+1}$, erit ſumma ſeriei $t=s+1=n+1$. Cum igitur ſumto $n=0$, ſeu $x=1$, ſit ſumma ſeriei $s=0$, erit, poſito $x=a^1$, ſumma ſeriei $s=1$: hincque porro ſequitur, ſi ponatur $x=a^2$, fore $s=2$, et ſi $x=a^3$, fore $s=3$. Atque in genere nunc patet, quod ante per ſolam inductionem elicuimus, ſi fiat $x=a^n$, denotante n numerum integrum affirmatiuum, fore perpetuo $s=n$. Sin autem n non ſit numerus integer affirmatiuus, atque s deſignet ſummam ſeriei initio propoſitae, facto $x=a^n$, tum poſito $x=a^{n+1}$, ſumma ſeriei, quae ſit $=t$ non erit $=s+1$, fiet enim:

$$t = 1+s-(1-a^n)(1-a^{n-1})(1-a^{n-2})(1-a^{n-3})(1-a^{n-4})\ \text{etc.}$$

His

His ergo caſibus valor ſeriei manifeſte recedit a natura logarithmorum.

§. 8. Quemadmodum hic valores ipſius x per a multiplicando ex valore ipſius s elicuimus valorem ipſius t, ita viciſſim valores ipſius x per a diuidendo ex valore ipſius t obtinebimus valorem ipſius s; hincque ad valores negatiuos exponentis n deſcendere poterimus. Scilicet in ſerie initio propoſita, vel ad hanc formam perducta:

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)(1-\frac{x}{a}) + \frac{1}{1-a^3}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) + \text{etc.}$$

pro ſequentibus caſibus ſummam ſeriei ita indicemus:

ſi $x = 1$ ſit $s = A = 0$

$x = \frac{1}{a}$ - - - $s = B$

$x = \frac{1}{a^2}$ - - - $s = C$

$x = \frac{1}{a^3}$ - - - $s = D$

$x = \frac{1}{a^4}$ - - - $s = E$

etc.

Quod ſi iam ponatur $x = \frac{1}{a}$; fiet $s = B$, et $t = A = 0$, quia t oritur ex s, ſi loco x ſcribatur $a\,x$: ex praecedentibus oritur:

$$1 + B = (1-\frac{1}{a})(1-\frac{1}{a^2})(1-\frac{1}{a^3})(1-\frac{1}{a^4})(1-\frac{1}{a^5}) \text{ etc.}$$

ſeu $B = -1 + (1-\frac{1}{a})(1-\frac{1}{a^2})(1-\frac{1}{a^3})(1-\frac{1}{a^4})(1-\frac{1}{a^5})$ etc.

ſic ſi $a = 10$, fiet $B = -0,109989900000998$.

§. 9. Sit $x = \frac{1}{a^2}$, eritque $s = C$, et $t = B$; vnde habebitur;

$$1 + C - B = (1-\frac{1}{a^2})(1-\frac{1}{a^3}(1-\frac{1}{a^4})(1-\frac{1}{a^5}) \text{ etc.}$$

ad hanc addatur prior $1 + B$, eritque:

$$2 + C = (2-\frac{1}{a})(1-\frac{1}{a^2})(1-\frac{1}{a^3})(1-\frac{1}{a^4})(1-\frac{1}{a^5}) \text{ etc.}$$

et $C = -2 + (2 - \frac{1}{a})(1 - \frac{1}{a^2})(1 - \frac{1}{a^3})(1 - \frac{1}{a^4})(1 - \frac{1}{a^5})$ etc.
Vel ipſa ſerie eliminata erit:

$1 + B = (1 - \frac{1}{a})(1 + C - B)$, ſeu $C - 2B = \frac{1}{a}(1 + C - B)$.
Simili modo ſi ponatur $x = \frac{1}{a^2}$. erit $s = D$, et $t = C$, vnde fiet:

$1 + D - C = (1 - \frac{1}{a^3})(1 - \frac{1}{a^4})(1 - \frac{1}{a^5})(1 - \frac{1}{a^6})$ etc.
ad quam prior ſeries addita praebebit:

$3 + D = (3 - \frac{1}{a} - \frac{2}{a^2} + \frac{1}{a^3})(1 - \frac{1}{a^3})(1 - \frac{1}{a^4})(1 - \frac{1}{a^5})(1 - \frac{1}{a^6})$ etc.
Ac poſito $x = \frac{1}{a^4}$ cum fiat:
$1 + E - D = (1 - \frac{1}{a^4})(1 - \frac{1}{a^5})(1 - \frac{1}{a^6})(1 - \frac{1}{a^7})$ etc. erit
$4 + E = (4 - \frac{1}{a} - \frac{2}{a^2} - \frac{2}{a^3} + \frac{1}{a^4} + \frac{2}{a^5} - \frac{1}{a^6})(1 - \frac{1}{a^4})(1 - \frac{1}{a^5})(1 - \frac{1}{a^6})$ etc.
ſicque quousque libuerit, vlterius progredi licet.

§. 10. Poteſt autem inter ternos valores ſummae ſeriei s, pro ternis valoribus ipſius x ſucceſſiuis, relatio per expreſſionem finitam exhiberi. Manente enim pro valore x ſumma $= s$, ſit ſi loco x ponatur ax, ſumma ſeriei $= t$, et ſi loco x ponatur aax, ſit ſumma ſeriei $= u$. Cum igitur inter t et s hanc inuenerimus relationem:

$1 + s - t = (1 - x)(1 - \frac{x}{a})(1 - \frac{x}{a^2})(1 - \frac{x}{a^3})(1 - \frac{x}{a^4})$ etc.
ſi hic pro x ſcribamus ax, prodibit relatio ſimilis inter u et t:

$1 + t - u = (1 - ax)(1 - x)(1 - \frac{x}{a})(1 - \frac{x}{a^2})(1 - \frac{x}{a^3})$ etc.
Hinc ergo erit $1 + t - u = (1 - ax)(1 + s - t)$ ſiue

$$u = 2t - s + ax(1 + s - t)$$

vel $s = \frac{2t - u + ax(1 - t)}{1 - ax}$.

Atque

Atque hinc pro ſupra aſſumtis valoribus A, B, C, D, etc. ſequentes prodibunt relationes.

Si $x = \frac{1}{a^2}$; erit $A = 2B - C + \frac{1}{a}(1 + C - B)$

ſeu $C = \frac{1 + (2a-1)B - aA}{a-1} = B + \frac{1 + a(B-A)}{a-1}$

ſi $x = \frac{1}{a^3}$; erit $D = C + \frac{1 + a^2(C-B)}{a^2-1}$

ſi $x = \frac{1}{a^4}$; erit $E = D + \frac{1 + a^3(D-C)}{a^3-1}$

ſi $x = \frac{1}{a^5}$; erit $F = E + \frac{1 + a^4(E-D)}{a^4-1}$

etc.

Hae relationes autem ſequenti modo commodius exprimi poſſunt:

$$C = 2B - A + \frac{1+B-A}{a-1}$$
$$D = 2C - B + \frac{1+C-B}{a^2-1}$$
$$E = 2D - C + \frac{1+D-C}{a^3-1}$$
$$F = 2E - D + \frac{1+E-D}{a^4-1}$$

etc.

Cum ergo ſit $A = 0$, ſi ſolius litterae B valor fuerit repertus:

$B = -1 + (1-\frac{1}{a})(1-\frac{1}{a^2})(1-\frac{1}{a^3})(1-\frac{1}{a^4})$ etc.

hinc omnium ſequentium litterarum C, D, E, F, etc. valores exacto poterunt aſſignari.

§. 11. Cum autem denotante n numerum integrum affirmatiuum, ſi ponatur $x = a^n$, ſit $s = n$, ex noſtra aſſumta ſerie conſequemur hanc ſummabilem.

$$n = \frac{1-a^n}{1-a} + \frac{(1-a^n)(1-a^{n-1})}{1-a^2} + \frac{(1-a^n)(1-a^{n-1})(1-a^{n-2})}{1-a^3} + \text{etc.}$$

Tum vero hoc caſu, quia eſt $t = n + 1$, erit:

$1 = a^n + a^{n-1}(1-a^n) + a^{n-2}(1-a^n)(1-a^{n-1}) + a^{n-3}(1-a^n)(1-a^{n-1})(1 a^{n-2})$ etc.
cuius veritas omnibus terminis ad eandem partem coniectis eſt manifeſta, fiet enim:

$(1-a^n)(1-a^{n-1})(1-a^{n-2})(1-a^{n-3})(1-a^{n-4})$ etc. $= 0$.

Hinc anſam nanciſcimur generalius huiusmodi formas contemplandi. Sit enim A, B, C, D, E, F, etc. ſeries quantitatum quarumuis, ſitque:

$$(1-A)(1-B)(1-C)(1-D)(1-E) \text{ etc.} = S.$$

Atque hinc obtinebitur:

$1 - A - B(1-A) - C(1-A)(1-B) - D(1-A)(1-B)(1-C) - \text{etc.} = S$;

haec enim formula facillime reducitur ad illam. Hanc ob rem habebimus:

$A + B(1-A) + C(1-A)(1-B) + D(1-A)(1-B)(1-C) + \text{etc.} = S + 1$.

§. 12. Quod ſi ergo quaepiam harum quantitatum A, B, C, etc. vnitati fiat aequalis, erit $S = 0$, prodibitque ſeries, cuius ſumma $= 1$. Sumatur verbi gratia haec ſeries:

A	B	C	D	E	F	
$\frac{1}{2}$;	$\frac{2}{3}$;	$\frac{3}{4}$;	$\frac{4}{5}$;	$\frac{5}{6}$;	$\frac{6}{7}$;	etc.

quarum fractionum cum infinitiſſima ſit $= 1$, erit $S = 0$, et ſequens naſcetur ſeries:

$1 = \frac{1}{2} + \frac{2}{2.3} + \frac{3}{2.3.4} + \frac{4}{2.3.4.5} + \frac{5}{2.3.4.5.6} +$ etc.

cuius quidem veritas facile perſpicitur, oritur enim ea hoc modo:

ſit $z = 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} +$ etc.

erit $z - 1 = \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} +$ etc. hincq. per ſubtr. prodit

$1 =$

$1 = \frac{1}{2} + \frac{2}{2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{4}{2\cdot 3\cdot 4\cdot 5} + \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$

§. 13. Sit $A = \frac{1}{9}$; $B = \frac{1}{25}$; $C = \frac{1}{49}$; $D = \frac{1}{81}$; etc. erit $S = \frac{8}{9}\cdot\frac{24}{25}\cdot\frac{48}{49}\cdot\frac{80}{81}\cdot\frac{120}{121}\cdot \text{etc.} = \frac{\pi}{4}$ denotante π peripheriam circuli, cuius diameter est $= 1$. Hinc ergo orietur haec series pro quadratura circuli.

$\frac{\pi}{4} + 1 = \frac{1}{9} + \frac{8}{9\cdot 25} + \frac{8\cdot 24}{9\cdot 25\cdot 49} + \frac{8\cdot 24\cdot 48}{9\cdot 25\cdot 49\cdot 81} + \text{etc.}$

seu $\frac{9}{4}\pi + 8 = \frac{2\cdot 4}{5\cdot 5} + \frac{2\cdot 4\cdot 4\cdot 6}{5\cdot 5\cdot 7\cdot 7} + \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9} + \text{etc.}$

Cum ergo huiusmodi producta, quorum valor S exhiberi potest, innumerabilia habeantur: ex quolibet hoc modo series infinita, cuius summa assignari queat, deriuabitur. Amplissimus ergo hinc aperitur campus, series summabiles, quotquot libuerit, inueniendi.

§. 14. Reuertor autem ad seriem initio assumtam

$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)(1-\frac{x}{a}) + \frac{1}{1-a^3}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2}) + \text{etc.}$

quam in aliam formam, in qua termini secundum potestates ipsius x procedant, transfundere animus est. Hoc primo quidem per euolutionem singulorum terminorum fieri posset, at quia hoc pacto prodituri essent singuli coefficientes in seriebus infinitis, commodissime in hunc finem adhibebitur formula supra inuenta $u = 2t - s + ax(1 - t + s)$, seu $u - 2t + s = ax + ax(s - t)$, vbi ex s nascitur t, si loco x ponatur ax, parique modo ex t fit u, si loco x denuo ponatur ax. Quare si pro serie quaesita assumamus

$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$ erit:

$t = A + Bax + Ca^2x^2 + Da^3x^3 + Ea^4x^4 + Fa^5x^5 + \text{etc.}$ et

$u = A + Ba^2x + Ca^4x^2 + Da^6x^3 + Ea^8x^4 + Fa^{10}x^5 + \text{etc.}$

Ex

Ex his ergo conficietur:

$u-2t+s=B(1-a)^2x+C(1-aa)^2x^2+D(1-a^3)^2x^3+E(1-a^4)^2x^4+\text{etc.}$

$ax(1+s-t)=ax+Ba(1-a)x^2+Ca(1-aa)x^3+Da(1-a^3)x^4+\text{etc.}$

Ex quarum ſerierum aequalitate concluditur fore:

$B=\frac{a}{(1-a)^2};\ C=\frac{Ba(1-a)}{(1-aa)^2};\ D=\frac{Ca(1-aa)}{(1-a^3)^2};\ E=\frac{Da(1-a^3)}{(1-a^4)^2};\ \text{etc.}$

§. 15. Hinc ergo ſequentes coefficientium aſſumtorum valores obtinebuntur:

$$B=\frac{a}{(1-a)^2}$$
$$C=\frac{a^2}{(1-a)(1-aa^2)}$$
$$D=\frac{a^3}{(1-a)(1-aa)(1-a^3)^2}$$
$$E=\frac{a^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2}$$
$$F=\frac{a^5}{(1-a)(1-aa)(1-a^3)(1-a^4)(1-a^5)^2}$$
etc.

Primus autem terminus A hinc non definitur. At quia A praebet valorem ipſius s, ſi ponatur $x=0$, perſpicuum eſt fore:

$$A=\frac{1}{1-a}+\frac{1}{1-a^2}+\frac{1}{1-a^3}+\frac{1}{1-a^4}+\frac{1}{1-a^5}+\text{etc.}$$

His ergo valoribus definitis, ſeries initio propoſita:

$$s=\frac{1}{1-a}(1-x)+\frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right)+\frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)+\text{etc.}$$

transmutabitur in hanc formam:

$$s=\left\{\begin{array}{l}\frac{1}{1-a}+\frac{1}{1-a^2}+\frac{1}{1-a^3}+\frac{1}{1-a^4}+\frac{1}{1-a^5}+\text{etc.}\\ +\frac{ax}{(1-a)^2}+\frac{a^2x^2}{(1-a)(1-aa)^2}+\frac{a^3x^3}{(1-a)(1-aa)(1-a^3)^2}+\frac{a^4x^4}{(1-a)(1-a^2)(1-a^3)(1-a^4)^2}+\text{etc}\end{array}\right\}$$

§. 16. Cum igitur poſito $x=a^n$, denotante n numerum integrum affirmatiuum, fiat $s=n$, habebitur haec ſummatio:

$$n + \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.} =$$
$$\frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(aa-1)^2} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2} - \frac{a^{4n+4}}{(a-1)(a^2-1)(a^3-1)(a^4-1)^2} + \text{etc.}$$

Quod ſi ergo fuerit $n = 0$, erit:

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.} = \frac{a}{(a-1)^2} - \frac{a^2}{(a-1)(aa-1)^2} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)^2} + \text{etc.}$$

ac, ſi ponatur $n = 1$, erit:

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.} = \frac{a^2}{(a-1)^2} - \frac{a^4}{(a-1)(a^2-1)^2} + \frac{a^6}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - 1$$

Generaliter ergo erit:

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \text{etc.} = \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(a^2-1)^2} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2} - \text{etc.} - n$$

denotante n numerum integrum quemcunque affirmatiuum.

§. 17. Si loco n ponatur $n-1$, habebitur:

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \text{etc.} = \frac{a^n}{(a-1)^2} - \frac{a^{2n}}{(a-1)(a^2-1)^2} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)^2} - n + 1$$

a qua, ſi ſeries ſuperior auferatur, prouenIet:

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \frac{a^{4n}}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

Huius ergo ſeriei ſumma ſemper aequalis eſt vnitati, quicunque valor ipſi a tribuatur, et quicunque numerus integer affirmatiuus pro n ſubſtituatur. Caſu autem quo $n = 1$ haec ſummatio facile perſpicitur. Quod enim ſit:

$$1 = \frac{a}{a-1} - \frac{a^2}{(a-1)(a^2-1)} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

ſequitur luculenter ex conſideratione huius ſeriei:

$$z = 1 - \frac{1}{a-1} + \frac{1}{(a-1)(a^2-1)} - \frac{1}{(a-1)(a^2-1)(a^3-1)} + \text{etc.}$$ vnde fit:

$$1 - z = \frac{1}{a-1} - \frac{1}{(a-1)(a^2-1)} + \frac{1}{(a-1)(a^2-1)(a^3-1)} - \frac{1}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

quae inuicem additae dabunt:

 $1 =$

$$1 = \frac{a}{a-1} - \frac{aa}{(a-1)(a^2-1)} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)} - \frac{a^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

§. 18. Deinde autem veritas iſtius ſeriei pro reliquis ipſius n valoribus ſequentem in modum oſtendi poteſt. Si fuerit:

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

dico fore quoque:

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

Nam cum ſit per hypotheſin:

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc. erit quoque}$$

$$0 = a^n - \frac{a^{2n}}{a-1} + \frac{a^{3n}}{(a-1)(a^2-1)} - \text{etc.}$$

quae ſeries inuicem additae dabunt:

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

Quare cum haec ſeries:

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

vera ſit oſtenſa caſu $n = 1$, erit quoque vera caſu $n = 2$, hincque porro caſibus $n = 3$, $n = 4$, etc. ita vt quicunque numerus integer affirmatiuus pro n ſubſtituatur, ſumma ſeriei perpetuo futura ſit $= 1$.

§. 19. Quoniam ſeriem initio propoſitam $s = \frac{t}{1-a}(1-x)$ etc. ſecundum dimenſiones ipſius x hic diſpoſui, ope proprietatis ſupra demonſtratae $u - 2t + s = ax + ax(s-t)$; non incongruum erit eandem transmutationem immediate ex ex ipſa ſerie s deriuare; ſic enim ad ſummationem innumerabilium nouarum ſerierum pertingemus. Oportebit ergo ſingulos ſeriei s terminos per multiplicationem euoluere, quod vt expeditius fieri poſſit, conſiderabo terminum quemcunque: $\frac{1}{(1-a^m)}(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3})\dots(1-\frac{x}{a^{m-1}})$

Ponam ergo $P=(1-x)(1-\frac{x}{a})(1-\frac{x}{a^2})(1-\frac{x}{a^3})\dots(1-\frac{x}{a^{m-1}})$

eritque $lP=l(1-x)+l(1-\frac{x}{a})+l(1-\frac{x}{a^2})+\dots+l(1-\frac{x}{a^{m-1}})$

et differentiando fiet:

$$\frac{dP}{P}=\frac{-dx}{1-x}-\frac{dx}{a-x}-\frac{dx}{aa-x}-\dots-\frac{dx}{a^{m-1}-x}.$$ ſeu

$$\frac{dP}{P}=-dx\left\{\begin{array}{l} 1+x+x^2+x^3+x^4+x^5+\text{ etc. infin.}\\ \frac{1}{a}+\frac{x}{a^2}+\frac{x^2}{a^3}+\frac{x^3}{a^4}+\frac{x^4}{a^5}+\frac{x^5}{a^6}+\text{ etc.}\\ \frac{1}{a^2}+\frac{x}{a^4}+\frac{x^2}{a^6}+\frac{x^3}{a^8}+\frac{x^4}{a^{10}}+\frac{x^5}{a^{12}}+\text{ etc.}\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \frac{1}{a^{m-1}}+\frac{x}{a^{2m-2}}+\frac{x^2}{a^{3m-3}}+\frac{x^3}{a^{4m-4}}+\frac{x^4}{a^{5m-5}}+\frac{x^5}{a^{6m-6}}+\text{ etc} \end{array}\right\}$$

ſingulas nunc ſeries verticales ſummando orietur:

$$dP=-Pdx\left(\frac{a^m-1}{a^m-a^{m-1}}+\frac{a^{2m}-1}{a^{2m}-a^{2m-2}}x+\frac{a^{3m}-1}{a^{3m}-a^{3m-3}}x^2+\frac{a^{4m}-1}{a^{4m}-a^{4m-4}}x^3+\text{ etc.}\right)$$

§ 20, Fingatur nunc pro P haec ſeries:

$P=\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+$ etc. eritque:

$\frac{dP}{dx}=\beta+2\gamma x+3\delta x^2+4\varepsilon x^3+5\zeta x^4+$ etc.

Facta iam ſubſtitutione fiet:

$$\beta + \frac{a^m - 1}{a^m - a^{m-1}}\alpha = 0$$

$$2\gamma + \frac{a^m - 1}{a^m - a^{m-1}}\beta + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}}\alpha = 0$$

$$3\delta + \frac{a^m - 1}{a^m - a^{m-1}}\gamma + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}}\beta + \frac{a^{3m} - 1}{a^{3m} - a^{3m-3}}\alpha = 0$$

etc.

atque cum poſito $x = 0$, fiat $P = 1$, patet eſſe $\alpha = 1$. Erit ergo $\beta = \frac{-a^m + 1}{a^m - a^{m-1}}$ et $2\gamma - \frac{(a^m - 1)^2}{(a^m - a^{m-1})^2} + \frac{a^{2m} - 1}{a^{2m} - a^{2m-2}} = 0$ ſeu $2\gamma = \frac{a^m - 1}{a^m - a^{m-1}}\left(\frac{a^m - 1}{a^m - a^{m-1}} - \frac{a^m - 1}{a^m + a^{m-1}}\right) = \frac{2a^m(a^{m-1} - 1)(a^m - 1)}{(a^m - a^{m-1})(a^{2m} - a^{2m-2})}$, ideoque $\gamma = \frac{(a^m - 1)(a^{m-1} - 1)}{(a^m - a^{m-1})(a^m - a^{m-2})}$. Simili modo reliqui coefficientes, verum tamen non ſine ingenti labore eruentur, atque tandem ſatis concinne exprimi deprehendentur.

§. 21. Quo igitur hanc coefficientium determinationem commodius expediam, methodum hic iam aliquoties vſurpatam adhibebo. Scilicet in ſerie $P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 +$ etc. loco x pono $\frac{x}{a}$, ſerieique reſultantis ſumma ſit $= Q$, nempe;

$$Q = \alpha + \frac{\beta x}{a} + \frac{\gamma x^2}{a^2} + \frac{\delta x^3}{a^3} + \frac{\varepsilon x^4}{a^4} + \text{etc.}$$

Cum autem ſit $P = (1 - x)(1 - \frac{x}{a})(1 - \frac{x}{aa}) \ldots \ldots (1 - \frac{x}{a^{m-1}})$

erit $Q = (1 - \frac{x}{a})(1 - \frac{x}{a^2})(1 - \frac{x}{a^3}) \ldots \ldots (1 - \frac{x}{a^m})$, ideoque

P(1 =

$P(1-\frac{x}{a^m}=Q(1-x)$. seu $a^mP-Px-a^mQ+a^mQx=0$,

substituantur hic series pro P et Q assumtae, fietque

$$\left.\begin{array}{l} \alpha a^m+\beta a^m x+\gamma a^m x^2+\delta a^m x^3+\text{ etc.} \\ \quad -\alpha x \quad -\beta x^2 \quad -\gamma x^3 \quad -\text{ etc.} \\ -\alpha a^m-\beta a^{m-1}x-\gamma a^{m-2}x^2-\delta a^{m-3}x^3-\text{ etc.} \\ \quad +\alpha a^m x+\beta a^{m-1}x^2+\gamma a^{m-2}x^3+\text{ etc.} \end{array}\right\}=0.$$

Ex comparatione terminorum homogeneorum hinc invenitur:

$$\beta=\frac{-\alpha(a^m-1)}{a^{m-1}(a-1)}\;;\quad \delta=\frac{-\gamma(a^{m-2}-1)}{a^{m-3}(a^3-1)}$$

$$\gamma=\frac{-\beta(a^{m-1}-1)}{a^{m-2}(aa-1)}\;;\quad \varepsilon=\frac{-\delta(a^{m-3}-1)}{a^{m-4}(a^4-1)}$$

etc.

§. 22. Cum igitur sit $\alpha=1$, coefficientes ita se habebunt;

$$\alpha=1$$

$$\beta=\frac{-(a^m-1)}{a^{m-1}(a-1)}$$

$$\gamma=\frac{+(a^m-1)(a^{m-1}-1)}{a^{2m-3}(a-1)(aa-1)}$$

$$\delta=\frac{-(a^m-1)(a^{m-1}-1)(a^{m-2}-1)}{a^{3m-6}(a-1)(aa-1)(a^3-1)}$$

$$\varepsilon=\frac{+(a^m-1)(a^{m-1}-1)(a^{m-2}-1)(a^{m-3}-1)}{a^{4m-10}(a-1)(a^2-1)(a^3-1)(a^4-1)}$$

etc.

 Termi-

Terminus ergo feriei s, quicunque $\frac{1}{1-a^m}(1-x)(1-\frac{x}{a})(1-\frac{x}{aa}) \ldots (1-\frac{x}{a^{m-1}})$ evolutus, dabit hanc progreffionem:

$$\frac{1}{1-a^m} - \frac{1}{a^{m-1}(1-a)}x + \frac{(1-a^{m-1})x^2}{a^{2m-3}(1-a)(1-a^2)} - \frac{(1-a^{m-1})(1-a^{m-2})x^3}{a^{3m-6}(1-a)(1-a^2)(1-a^3)}.$$

Si igitur fucceffiue pro m numeri 1, 2, 3, 4, etc. fubftituantur, prodibunt fequentes formulae, feu termini feriei s.

Primus $= \frac{1}{1-a} - \frac{x}{1-a}$

Secund: $= \frac{1}{1-a^2} - \frac{x}{a(1-a)} + \frac{(1-a)x^2}{a(1-a)(1-a^2)}$

Tert: $= \frac{1}{1-a^3} - \frac{x}{a^2(1-a)} + \frac{(1-a^2)x^2}{a^3(1-a)(1-a^2)} - \frac{(1-a)(1-a^2)x^3}{a^3(1-a)(1-a^2)(1-a^3)}$

Quart: $= \frac{1}{1-a^4} - \frac{x}{a^3(1-a)} + \frac{(1-a^3)xx}{a^5(1-a)(1-a^2)} - \frac{(1-a^2)(1-a^3)x^3}{a^6(1-a)(1-a^2)(1-a^3)} + \frac{(1-a)(1-a^2)(1-a^4)x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}$

etc.

§. 23. Si ergo omnes ifti termini in vnam fummam colligantur, prodibit congeries infinitarum ferierum, quae fimul fumtae, feriei initio propofitae, erunt aequales. Scilicet cum fit:

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)(1-\frac{x}{a}) + \frac{1}{1-a^3}(1-x)(1-\frac{x}{a})(1-\frac{x}{aa}) + \text{etc. erit:}$$

$$s = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

$$\frac{-x}{1-a}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \text{etc.}\right)$$

$$\frac{+x^2}{a(1-a)(1-a^2)}\left(\frac{1-a}{1} + \frac{1-a^2}{a^2} + \frac{1-a^3}{a^4} + \frac{1-a^4}{a^5} + \text{etc.}\right)$$

$$\frac{-x^3}{a^3(1-a)(1-a^2)(1-a^3)}\left(\frac{(1-a)(1-a^2)}{1} + \frac{(1-a^2)(1-a^3)}{a^3} + \frac{(1-a^3)(1-a^4)}{a^6} + \text{etc.}\right)$$

$$\frac{+x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}\left(\frac{(1-a)(1-a^2)(1-a^3)}{1} + \frac{(1-a^2)(1-a^3)(1-a^4)}{a^4} + \text{etc.}\right)$$

etc.

Cum

Cum igitur haec ſeries congruere debeat cum ante inuenta, ex conſenſu ſingularum harum ſerierum ſumma reperientur.

$$1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \text{etc.} = \frac{-a}{1-a}$$

$$\frac{1-a}{1} + \frac{1-a^2}{a^2} + \frac{1-a^3}{a^4} + \frac{1-a^4}{a^6} + \text{etc.} = \frac{+a^3}{1-aa}$$

$$\frac{(1-a)(1-a^2)}{1} + \frac{(1-a^2)(1-a^3)}{a^3} + \frac{(1-a^3)(1-a^4)}{a^6} + \text{etc.} = \frac{-a^6}{1-a^3}$$

$$\frac{(1-a)(1-a^2)(1-a^3)}{1} + \frac{(1-a^2)(1-a^3)(1-a^4)}{a^4} + \text{etc.} = \frac{+a^{10}}{1-a^4}$$

$$\frac{(1-a)(1-a^2)(1-a^3)(1-a^4)}{1} + \frac{(1-a^2)(1-a^3)(1-a^4)(1-a^5)}{a^5} + \text{etc.} = \frac{-a^{15}}{1-a^5}$$

§. 24. Hae ſeries in ſequentes formas transfundi poſſunt, ex quibus lex progreſſionis clarius perſpicietur:

$$\frac{a}{a-1} = 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \text{etc.}$$

$$\frac{a^2}{a^2-1} = \left(1-\frac{1}{a}\right) + \frac{1}{a}\left(1-\frac{1}{a^2}\right) + \frac{1}{a^2}\left(1-\frac{1}{a^3}\right) + \frac{1}{a^3}\left(1-\frac{1}{a^4}\right) + \frac{1}{a^4}\left(1-\frac{1}{a^5}\right) + \text{etc.}$$

$$\frac{a^3}{a^3-1} = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right) + \frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right) + \frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right) + \text{etc.}$$

$$\frac{a^4}{a^4-1} = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right) + \frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right) + \text{etc.}$$

$$\frac{a^5}{a^5-1} = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right) + \frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right) + \text{etc.}$$

etc.

Vnde colligitur fore generaliter $\frac{a^{m+1}}{a^{m+1}-1} = \frac{1}{1-\frac{1}{a^{m+1}}}$

$$= \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\ldots\left(1-\frac{1}{a^m}\right) + \frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\ldots\left(1-\frac{1}{a^{m+1}}\right) +$$

$$\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\ldots\left(1-\frac{1}{a^{m+2}}\right) + \frac{1}{a^3}\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)\ldots\left(1-\frac{1}{a^{m+3}}\right) + \text{etc.}$$

§. 25. Summa huius ſeriei etiam hoc modo inueſtigari poteſt. Sit breuitatis gratia $\frac{1}{a} = b$, atque ponatur ſumma quaeſita:

$z =$

$$z = (1-b)(1-b^2)\ldots(1-b^m) + b(1-b^2)(1-b^3)\ldots(1-b^{m+1}) +$$
$$b^2(1-b^3)(1-b^4)\ldots(1-b^{m+2}) + b^3(1-b^4)(1-b^5)\ldots(1-b^{m+3}) + \text{etc.}$$

Multiplicetur vtrinque per $1-b^{m+1}$, atque prodibit:

$$(1-b^{m+1})z = (1-b)(1-b^2)\ldots(1-b^m)(1-b^{m+1}) + (1-b^2)(1-b^3)\ldots(1-b^{m+1})(b-b^{m+2})$$
$$+ (1-b^3)(1-b^4)\ldots(1-b^{m+2})(b^2-b^{m+3}) + \text{etc.}$$

At eſt $b-b^{m+2} = 1-b^{m+2}-(1-b)$; $b^2-b^{m+3} = 1-b^{m+3}-(1-bb)$ $b^3-b^{m+4} = 1-b^{m+4}-(1-b^3)$, etc. qui valores loco vltimorum factorum ſubſtituti dabunt:

$$(1-b^{m+1})z = (1-b)(1-b^2)\ldots(1-b^{m+1}) + (1-b^2)(1-b^3)\ldots(1-b^{m+2})$$
$$- (1-b)(1-b^2)\ldots(1-b^{m+1}) - (1-b^2)(1-b^3)\ldots(1-b^{m+2})$$
$$+ (1-b^3)(1-b^4)\ldots(1-b^{m+3}) + (1-b^4)(1-b^5)\ldots(1-b^{m+4}) + \text{etc.}$$
$$- (1-b^3)(1-b^4)\ldots(1-b^{m+3}) - \quad \text{etc.}$$

Cum ergo omnes termini deſtruantur, ſolus remanebit vltimus, $(1-b^{m+1})z = (1-b^\infty)(1-b^{\infty+1})\ldots(1-b^{m+\infty})$, vnde patet, ſi fuerit $b < 1$, hoc eſt $a > 1$, vti aſſumimus, fore $(1-b^{m+1})z = 1$, ideoque $z = \frac{1}{1-b^{m+1}} = \frac{a^{m+1}}{a^{m+1}-1}$, vti inueneramus.

§ 26. Ex iis, quae §. XXI. ſunt tradita, facile reperitur ſeries ſecundum dimenſiones ipſius x procedens, quae aequalis ſit huic producto infinitorum Factorum.

$$P = (1-x)(1-\tfrac{x}{a})(1-\tfrac{x}{a^2})(1-\tfrac{x}{a^3})(1-\tfrac{x}{a^4}) \text{ etc.}$$

Poſito enim $P = 1 - \alpha x + \mathfrak{G} x^2 - \gamma x^3 + \delta x^3 - \varepsilon x^5 + \text{etc.}$ ſcribatur ax loco x, et valor reſultans ſit $= Q$, erit:

$$Q = (1-ax)(1-x)(1-\tfrac{x}{a})(1-\tfrac{x}{aa})(1-\tfrac{x}{a^3}) \text{ etc.} = P - axP$$

et

et $Q = 1 - \alpha a x + \beta a^2 x^2 - \gamma a^3 x^3 + \delta a^4 x^4 - \varepsilon a^5 x^5 +$ etc.
sed $axP = ax - \alpha a x^2 + \beta a x^3 - \gamma a x^4 + \delta a x^5 -$ etc.
$-P = -1 + \alpha x - \beta x^2 + \gamma x^3 - \delta x^4 + \varepsilon x^5 -$ etc.
vnde fit $\alpha = \frac{a}{a-1}$; $\beta = \frac{\alpha a}{a^2-1}$; $\gamma = \frac{\beta a}{a^3-1}$; $\delta = \frac{\gamma a}{a^4-1}$ etc.
Quam ob rem productum infinitum $P = (1-x)(1-\frac{x}{a})(1-\frac{x}{aa})$ etc. resoluetur in hanc seriem infinitam:

$$P = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \frac{a^4x^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} \text{ etc.}$$

§. 27. Si igitur istud productum P nihilo aequale ponatur haec aequatio infinita:

$$0 = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \text{ etc.}$$

omnes suas radices x habebit reales, eruntque valores ipsius x terminis istius progressionis Geometricae:

$$1,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,\ a^7,\ \text{etc.}$$

vnde si ponatur $x = a^n$, denotante n numerum integrum affirmatiuum quemcunque, erit:

$$0 = 1 - \frac{a^{n+1}}{a-1} + \frac{a^{2n+2}}{(a-1)(a^2-1)} - \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} + \text{ etc.}$$

cuius veritas iam supra §. XVIII. est demonstrata.

§. 28. Praecipue autem est notatu digna series, cui supra innumerabiles aliae aequales sunt inuentae (§. XVI.), quae est

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{ etc.}$$

cuius summa, si $a > 1$, etsi est finita et per approximationes facile assignatur, tamen neque numeris rationalibus, neque irrationalibus exprimi potest. Quo circa ea imprimis digna videtur, vt Geometriae naturam illius quan-

 tita-

titatis transcendentis inuestigent, qua eius summa exprimatur.

§. 29. Monstrabo autem, quem ad modum summa huiusmodi serierum vero proxime expedite inueniri posset, et quidem hanc seriem in aliquanto latiori sensu considerabo. Sit:

$$s = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \frac{1}{a^5-z} + \text{etc.}$$

Conuertantur singuli termini in series Geometricas, eritque:

$$s = \tfrac{1}{a} + \tfrac{1}{a^2} + \tfrac{1}{a^3} + \tfrac{1}{a^4} + \tfrac{1}{a^5} + \text{etc.}$$
$$+ z\left(\tfrac{1}{a^2} + \tfrac{1}{a^4} + \tfrac{1}{a^6} + \tfrac{1}{a^8} + \tfrac{1}{a^{10}} + \text{etc.}\right)$$
$$+ z^2\left(\tfrac{1}{a^3} + \tfrac{1}{a^6} + \tfrac{1}{a^9} + \tfrac{1}{a^{12}} + \tfrac{1}{a^{15}} + \text{etc.}\right),$$
etc.

quae series denuo summatae dabunt:

$$s = \frac{1}{a-1} + \frac{z}{aa-1} + \frac{zz}{a^3-1} + \frac{z^3}{a^4-1} + \frac{z^4}{a^5-1} + \text{etc.}$$

Quod si ergo fuerit $z = 1$, hae ambae series in eandem recidunt, neque haec transmutatio vllum affert discrimen.

§. 30. Ad seriem hanc summandam ponamus prioris formae iam n terminos actu esse summatos, quorum summa sit $= A$, ita vt sit:

$$A = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \ldots\ldots + \frac{1}{a^n-z}$$

Erit ergo tota summa quaesita:

$$s = A + \frac{1}{a^{n+1}-z} + \frac{1}{a^{n+2}-z} + \frac{1}{a^{n+3}-z} + \frac{1}{a^{n+4}-z} + \text{etc.}$$

Iam istae fractiones in series Geometricas euoluantur, eritque:

$s = A$

$$s = A + \frac{1}{a^{n+1}} + \frac{1}{a^{n+2}} + \frac{1}{a^{n+3}} + \frac{1}{a^{n+4}} + \text{etc.}$$

$$+ z\left(\frac{1}{a^{2n+2}} + \frac{1}{a^{2n+4}} + \frac{1}{a^{2n+6}} + \frac{1}{a^{2n+8}} + \text{etc.}\right)$$

$$+ z^2\left(\frac{1}{a^{3n+3}} + \frac{1}{a^{3n+6}} + \frac{1}{a^{3n+9}} + \frac{1}{a^{3n+12}} + \text{etc.}\right)$$

etc.

quae ſeries denuo ſummatae dabunt:

$$s = A + \frac{1}{a^n(a-1)} + \frac{z}{a^{2n}(aa-1)} + \frac{zz}{a^{3n}(a^3-1)} + \frac{z^3}{a^{4n}(a^4-1)} + \text{etc.}$$

quae eo citius conuergit, quam prima, quo maior fuerit numerus n.

§. 31. Sit $a=2$, vt ſit $s = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \frac{1}{16-z} + \text{etc.}$ Si igitur fuerit $A = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \ldots\ldots + \frac{1}{2^n - z}$

erit: $s = A + \frac{1}{1.2^n} + \frac{z}{3.2^{2n}} + \frac{z^2}{7.2^{3n}} + \frac{z^3}{15.2^{4n}} + \frac{z^4}{31.2^{5n}} + \text{etc.}$

Ponamus autem $z = 1$, ita vt quaeratur ſumma huius ſeriei: $s = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{63} + \text{etc.}$

Addantur exempli cauſa quatuor termini initiales actu, vt ſit $n = 4$; erit:

$1 = 1,000000000000000$
$\frac{1}{3} = 0,333333333333333$
$\frac{1}{7} = 0,142857142857142$
$\frac{1}{15} = 0,066666666666666$

$A = 1,542857142857141$

Hinc erit $s = A + \frac{1}{16.1} + \frac{1}{16^2.3} + \frac{1}{16^3.7} + \frac{1}{16^4.15} + \text{etc.}$

 atque

atque isti termini in fractionibus decimalibus dabunt :

$$0,063838009558149$$

$$A = 1,542857142857142$$

Ergo $s = 1,606695152415291$

§. 32. Ceterum si seriei $s = \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} +$ etc. singuli termini in series Geometricas resoluantur, atque potestates similes ipsius a colligantur, reperietur haec forma :

$$s = \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \text{etc.}$$

quae series hanc habet proprietatem, vt cuiusuis fractionis numerator indicet, quot diuisores habeat exponens ipsius a in denominatore. Sic fractionis $\frac{4}{a^6}$ numerator est $= 4$, quia exponens 6 quatuor habet diuisores 1, 2, 3, 6. Vnde si exponens ipsius a in denominatore sit numerus primus, numerator perpetuo erit $= 2$: pro numeris autem non primis erit is binario maior. Hinc facile patet, si $a = 10$ fore :

$$s = 0,1223242434262445262644283444628.$$

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