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Methodus aequationes differentiales altiorum graduum integrandi ulterius promota

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METHODVS
AEQVATIONES DIFFERENTIALES ALTIORVM GRADVVM INTEGRANDI VLTERIVS PROMOTA

AVCTORE

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§. 1.

Tradidi in volumine septimo Miscellaneorum Berolinensium methodum facilem aequationes differentiales cuiusque gradus, in quibus altera variabilis vbique vnicam obtinet dimensionem, alterius vero tantum differentiale, quod constans assumitur, occurrit, integrandi, atque adeo aequationem finitam, quae differentialem propositam penitus exhauriat, inueniendi. Neque enim, si aequatio proposita differentialis primum gradum superet, pluribus repetitis integrationibus opus erat, sed vno quasi ictu cuiuscunque demum fuerit gradus aequatio proposita, methodus ibi exposita eandem suppeditat aequationem finitam, quae proditura esset, si successiue tot instituerentur integrationes, quot gradus differentialia in ea obtinent. Sic si aequatio proposita sit differentialis quarti gradus, more solito ea per vnam integrationem primo ad aequationem differentialem tertii gradus reduci, tum vero denuo integratio suscipi deberet, vt ad gradum secundum reuocetur: quo facto adhuc duae superessent integrationes instituendae, antequam ad aequationem quantitatibus finitis expressam perueniretur. Hanc igitur operationum plerumque difficillimarum multiplicitatem per methodum meam prorsus euito, dum vnica operatione statim veram aequationem integralem elicio.

§. 2. Quantopere autem modum integrandi vulgarem toties repetendum, quoties differentialitas in aequatione inest, secuti in molestissimos calculos incidamus, vnico exemplo ostendisse iuuabit. Sit ergo proposita haec aequatio differentialis tertii gradus $d^3y = ydx^3$, in qua elementum dx constans ponitur. Haec aequatio, etsi mea methodo facillime ter integratur, tamen ne quidem modus eam semel tantum integrandi perspicitur. Statim quidem, qui variabilis x ipsa deest, apparet eam ad gradum secundum deprimi posse. Si enim ponatur $dx = pdy$, ob dx constans erit $o = pddy + dpdy$ et denuo differentiando $o = pd^3y + 2dpddy + dyddp$. vnde fit $ddy = -\frac{dpdy}{p}$ et $d^3y = -\frac{2dpddy}{p} - \frac{dyddp}{p} = \frac{2dp^2dy}{pp} - \frac{dyddp}{p}$, qui valores in aequatione proposita $d^3y = ydx^3$ substituti dabunt:

$\frac{2dp^2dy}{pp} - \frac{dyddp}{p} = yp^3dy^3$ seu $yp^5dy^2 = 2dp^2 - pddp$. Quae cum neque dp neque dy sit constans, sed constantiae ratio ex aequatione $ddy = -\frac{dpdy}{p}$ definiatur, per methodos solitas vix vlterius tractari potest. Transmutari quidem aequatio potest in aliam formam, in qua nullum differentiale constans insit. Ponatur $dp = qdy$; erit $ddp = qddy + dqdy$ at $ddy = -\frac{dpdy}{p}$ dabit $ddy = -\frac{qdy^2}{p}$, vnde $ddp = -\frac{qqdy^2}{p} + dqdy$

sicque

ſicque aequatio inuenta hanc induet formam :

$$yp^5 dy = 2qqdy + qqdy - pdq = 3qqdy - pdq.$$

In qua pro lubitu differentiale conſtans aſſumere licet. Sit dy conſtans, ob $q = \frac{dp}{dy}$ erit $dq = \frac{ddp}{dy}$; habebiturque

$$yp^5 dy^2 = 3dp^2 - pddp.$$

At ſi ponatur $p = \frac{1}{r}$ fiet $ydy^2 = rdr^2 + rrddr$ quae aequatio cum ambae variabiles vbique totidem ſcilicet tres dimenſiones teneant, ope methodi meae in III. Tomo Comment. explicatae tractari poteſt. Ponatur ſcilicet $y = e^{\int zdu}$ et $r = e^{\int zdu} u$ denotante e numerum cuius logarithmus hyperbolicus $= 1$, erit $dy = e^{\int zdu} zdu$ et $ddy = 0 = e^{\int zdu}(zddu + dudz + zzdu^2)$. Deinde eſt $dr = e^{\int zdu}(du + zudu)$ et ob $r = uy$ erit $ddr = 2dudy + yddu = e^{\int zdu}(ddu + 2zdu^2)$. Sed $ddu = -\frac{dudz}{z} - zdu^2$ vnde $ddr = e^{\int zdu}(zdu^2 - \frac{dudz}{z})$. Qui valores in aequatione $ydy^2 = rdr^2 + rrddr$ ſubſtituti dabunt:

$$zzdu = u(1 + zu)^2 du + uuzdu - \frac{uudz}{z}$$

quae aequatio etſi eſt differentialis primi gradus, tamen multo difficilius tractatur, quam ipſa aequatio propoſita ſimplicior quidem aliquantum reddi poteſt ponendo $z = tu$, fiet enim. $\frac{dt}{t} = ttu^3 du + 3tudu - ttdu$

Quin potius cum aequatio propoſita ipſa facile conficiatur, inde integratio huius aequationis petenda videtur. Ponatur porro $t = \frac{1}{s}$, atque aequatio inuenta abibit in hanc

$$sds + 3sudu = du(1 - u^3)$$

quae aequatio immediate ex propoſita elicitur, ponendo $dx = \frac{du}{s}$ et $\frac{dy}{y} = \frac{udu}{s}$, fiet enim ob $\frac{du}{s}$ conſtans, $sddu$

 $= dsdu$

$= dsdu$ et $\frac{ddy}{y} = \frac{u^2du^2}{ss} + \frac{du^2}{s}$ et $\frac{d^3y}{y} = \frac{u^3du^3}{s^3} + \frac{3udu^3}{ss} + \frac{duddu}{s} = \frac{u^3du^3}{s^3} + \frac{3udu^3}{ss} + \frac{du^2ds}{ss}$, qui valores in aequatione $d^3y = ydx^3$ ſubſtituti praebebunt aequationem inuentam.

$$sds + 3sudu = du(1 - u^3).$$

§. 3. Totum ergo negotium ad integrationem huius aequationis reuocatur; quam integrabilem eſſe vel inde patet, quod aequatio differentialis tertii gradus, ex qua eſt nata, integrationem admittat. Quemadmodum autem hoc opus ſit abſoluendum in aequatione latius patente, quae per eandem ſubſtitutionem ex hac aequatione differentiali tertii gradus oritur,

$$Aydx^3 + Bdx^2dy + Cdxddy + Dd^3y = 0.$$

Prodibit autem ponendo $dx = \frac{du}{s}$ et $\frac{dy}{y} = \frac{udu}{s}$ haec aequatio differentialis primi gradus.

$$Dsds + sdu(C + 3Du) + du(A + Bu + Cuu + Du^3) = 0$$

quam primum obſeruo huiusmodi valorem pro $s = \alpha + \beta u + \gamma uu$ admittere. Erit enim $ds = \beta du + 2\gamma udu$. Vnde fit

$$\begin{aligned} \frac{Dsds}{du} &= D\alpha\beta + 2D\alpha\gamma u + 2D\beta\gamma u^2 + 2D\gamma^2u^3 \\ &\qquad + D\beta\beta u + D\beta\gamma u^2 \\ s(C + 3Du) &= C\alpha + C\beta u + C\gamma uu \\ &\qquad + 3D\alpha u + 3D\beta u^2 + 3D\gamma u^3 \\ A + Bu + Cu^2 + Du^3 &= A + Bu + Cu^2 + Du^3. \end{aligned}$$

Reddantur iam ſinguli termini homologi $= 0$, fietque primo $1 + 3\gamma + 2\gamma\gamma = 0$. Vnde fit vel $1 + \gamma = 0$ vel $1 + 2\gamma = 0$. Deinde eſt $3D\beta(\gamma + 1) + C(\gamma + 1) = 0$, cui aequationi quoque ſatisfacit $\gamma + 1 = 0$, ergo erit $\gamma = -1$.

Porro

Porro fiet $D\alpha = -B - C\beta - D\beta\beta$. Seu $\alpha = \frac{-B - C\beta - D\beta\beta}{D}$
Substituatur hic valor in aequatione $D\alpha\beta + C\alpha + A = 0$, seu $D^2\alpha\beta + CD\alpha + AD = 0$ eritque

$$\begin{array}{l} \quad -BD\beta - CD\beta^2 - DD\beta^3 = 0 \\ -BC - CC\beta - CD\beta^2 \\ \;\; AD \end{array}$$

Ad β ergo inueniendum hanc aequationem cubicam resoluere oportet. Sin autem α quaeratur erit:

$$D^2\alpha^3 + BD\alpha^2 + AC\alpha + A^3 = 0$$

Sit $\alpha = \frac{A\omega}{D}$, fiet $A\omega^3 + B\omega^2 + C\omega + D = 0$
Sit ergo ω radix huius aequationis cubicae, fiet

$$\alpha = \frac{A\omega}{D};\ \beta = -\frac{D - C\omega}{D\omega} \text{ et } \gamma = -1$$

atque $s = \frac{A\omega^2 - (D + C\omega)u - D\omega u^2}{D\omega}$ Porro fiet

$$x = \int \frac{du}{s} = \int \frac{D\omega du}{A\omega^2 - (D + C\omega)u - D\omega u^2} \text{ atque}$$

$$ly = \int \frac{u du}{s}\ \int \frac{D\omega u du}{A\omega^2 - (D + C\omega)u - D\omega u^2}.$$

Quamuis autem laborem has formulas integrandi susciperemus, tamen integrale tantum particulare obtineremus, neque adeo totum negotium etiam nunc esset confectum. Non enim valor ipsius s hic inuentus aequationem exhaurit, quia in eo nulla noua occurrit constans, quae in ipsa aequatione non insit. At vero cognito valore particulari ipsius s, ex eo valor completus sequenti modo eruetur. Ponatur valor iam inuentus $\frac{A\omega^2 - (D + C\omega)u - D\omega u^2}{D\omega} = V$ ac ponatur $s = V + z$; vt sit $ds = dV + dz$, atque prodibit

$$\left.\begin{array}{ll} DVdV & + DVdz + DzdV + Dzdz \\ + CVdu & + Czdu \\ + 3DVudu & + 3Duzdu \\ + (A + Bu & + Cuu + Du^3)du \end{array}\right\} = 0.$$

Cum

Cum vero ſit per hypotheſin:
$DVdV + Vdu(C+3Du) + du(A+Bu+Cu^2+Du^3) = 0$
erit $Dzdz + z(Cdu + 3Dudu + DdV) + DVdz = 0$
At ob $V = \frac{A\omega}{D} - \frac{u}{\omega} - \frac{Cu}{D} - uu$ erit $dV = -\frac{du}{\omega} - \frac{Cdu}{D} - 2udu$ atque $Dzdz + z\left(\frac{-Ddu}{\omega} + Dudu\right) + \frac{dz}{\omega}(A\omega^2 - (D+C\omega)u - D\omega u^2) = 0$ ſeu $zdz + zdu\left(u - \frac{1}{\omega}\right) + dz\left(\frac{A\omega}{D} - \frac{(D+C\omega)u}{D\omega} - uu\right) = 0$ quae aequatio niſi bene tractetur, difficulter ad ſeparationem variabilium perducitur. Interim tamen continetur in hac forma generali, quae ſeparationem admittit:

$$zdz + zdu(u+a) = dz(uu + 2bu + c).$$

Ad quam ſeparandam pono $dz = pdu$ fietque

$$z = \frac{(uu + 2bu + c)p}{p + u + a} \text{ et differentiando:}$$

$$dz = pdu = \frac{(u+a)(uu+2bu+c)dp + pdu(2p(u+b) + uu + 2au + 2ab - c)}{(p+u+a)^2}$$

ſeu $pdu(pp + 2ap - 2bp + aa - 2ab + c) = (u+a)(uu + 2bu + c)dp$ in qua variabiles ſponte a ſe inuicem ſeparantur: erit enim:

$$\frac{dp}{p(pp + 2(a-b)p + aa - 2b + c)} = \frac{du}{(u+a)(uu + 2bu + c)}$$

Opus autem foret ſumme taedioſum, ſi hanc aequationem integrare, atque exinde integrale aequationis differentialis tertii gradus eruere vellemus.

§. 4. Apparet hinc quanto labore tandem huiusmodi regulas ſequendo integrale aequationis differentialis tertii gradus erui poſſit, vnde vtilitas methodi meae in Vol. VII. Miſc: expoſitae non mediocriter perſpicitur. Eo magis autem eius vtilitas in oculos incurret, ſi loco aequationis differentialis tertii gradus alia, quae ſit quarti altioriſue gradus more vſitato tractetur, tum enim ſubſtitutiones

nes hic adhibitae aequationem differentialem non primi, ſed ſecundi altiorisue gradus praebebit, cuius integrale vix vllis artificiis obtineri poterit Et quamuis tandem etiam huius aequationis integrale inueniretur, tamen id plerumque tantum foret particulare, et poſt moleſtiſſimas demum ſubſtitutione ſuppeditat, et ipſius aequationis propoſitae integrale, et quidem particulare tantum: cum mea methodus fere ſine vllo labore ſtatim integrale completum praebeat. Quod vt clarius intelligatur vtamur ante tradita ſubſtitutione in hac aequatione differentiali quarti gradus:

$$Aydx^4 + Bdx^3dy + Cdx^2ddy + Ddxd^3y + Ed^4y = 0.$$

in qua dx ponitur conſtans. Sit igitur $dx = \frac{du}{s}$ ſeu $du = sdx$, et $\frac{dy}{y} = \frac{udu}{s} = udx$; erit ob dx conſtans: $\frac{ddy}{y} - \frac{dy^2}{y^2} = dx\,du = sdx^2$; ideoque $\frac{ddy}{y} = u^2dx^2 + sdx^2$. Hinc fiet porro $\frac{d^3y}{y} - \frac{dy\,ddy}{y^2} = 2usdx^3 + dsdx^2$ et $\frac{d^3y}{y} = u^3dx^3 + 3usdx^3 + dsdx^2$: iterumque differentiando prodibit $\frac{d^4y}{y} - \frac{dy\,d^3y}{yy} = 3uusdx^4 + 3udx^3ds + 3ssdx^4 + dx^2dds$, ideoque $\frac{d^4y}{y} = u^4dx^4 + 6uusdx^4 + 4udx^3ds + 3ssdx^4 + dx^2dds$. Quibus valoribus in aequatione hac ſubſtitutis.

$$Adx^2 + \frac{Bdxdy}{y} + \frac{Cddy}{y} + \frac{Dd^3y}{ydx} + \frac{Ed^4y}{ydx^2} = 0$$

proueniet haec aequatio:

$$Adx^2 + Budx^2 + Cu^2dx^2 + Csdx^2 + Du^3dx^2 + 3Dusdx^2 + Ddxds$$
$$+ Eu^4dx^2 + 6Euusdx^2 + 4Eudxds + 3Essdx^2 + Edds = 0$$

Cum autem ſit $dx = \frac{du}{s}$ erit

$$du^2(A + Bu + Cu^2 + Du^3 + Eu^4) + sdu^2(C + 3Du + 6Euu)$$
$$+ 3Essdu^2 + sduds(D + 4Eu) + Essdds = 0$$

 Appa-

Apparet quidem huic aequationi satisfieri, si sit $s = 0$ et u radix huius aequationis:

$$A + Bu + Cu^2 + Du^3 + Eu^4 = 0.$$

Sit ergo α vna ex radicibus huius aequationis, et sumendo $u = \alpha$, erit $\frac{dy}{y} = \alpha dx$ et $y = e^{\alpha x}$, qui valor quoque aequationi differentiali quarti gradus propositae conueniet. Erit autem tantum integrale maxime particulare; etiamsi autem quaternae aequationis $A + Bu + Cu^2 + Du^3 + Eu^4 = 0$ radices, quae sint α, β, γ, δ, suppedituro queant valorem

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x}$$

qui est integrale completum, tamen hinc non facile patet, qualis futurus sit valor ipsius y, si radicum α, β, γ, δ quaedam fuerint imaginariae vel inter se aequales. Contra vero potius ex valore ipsius y cognito integrale superioris aequationis differentio differentialis inter u et s assignabitur. Erit enim $u = \frac{dy}{y\,dx}$ et $s = \frac{du}{dx}$; ideoque

$$u = \frac{\mathfrak{A}\alpha e^{\alpha x} + \mathfrak{B}\beta e^{\beta x} + \mathfrak{C}\gamma e^{\gamma x} + \mathfrak{D}\delta e^{\delta x}}{\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x}} \text{ et}$$

$$s = \frac{\mathfrak{A}\mathfrak{B}(\alpha\delta\beta)^2 e^{(\alpha+\beta)x} + \mathfrak{A}\mathfrak{C}(\alpha-\gamma)^2 e^{(\alpha+\gamma)x} + \mathfrak{A}\mathfrak{D}(\alpha-\delta)^2 e^{(\alpha+\delta)x} + \mathfrak{B}\mathfrak{C}(\beta-\gamma)^2 e^{(\beta+\gamma)x} + \text{etc.}}{(\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x})^2}$$

Hinc concluditur fore:

$$s + uu = \frac{\begin{array}{l} + \mathfrak{A}^2\alpha^2 e^{2\alpha x} + \mathfrak{B}^2\beta^2 e^{2\beta x} + \mathfrak{C}^2\gamma^2 e^{2\gamma x} + \mathfrak{D}^2\delta^2 e^{2\delta x} \\ + \mathfrak{A}\mathfrak{B}(\alpha^2 + \beta^2) e^{(\alpha+\beta)x} + \mathfrak{A}\mathfrak{C}(\alpha^2 + \gamma^2) e^{(\alpha+\gamma)x} + \text{etc.} \end{array}}{(\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x})^2}$$

quae fractio deprimi potest, eritque

$s +$

$$s+uu=\frac{\mathfrak{A}\alpha^2e^{\alpha x}+\mathfrak{B}\beta^2e^{\beta x}+\mathfrak{C}\gamma^2e^{\gamma x}+\mathfrak{D}\delta^2e^{\delta x}}{\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}+\mathfrak{C}e^{\gamma x}+\mathfrak{D}e^{\delta x}}$$

Cum iam sit

$$u=\frac{\mathfrak{A}\alpha e^{\alpha x}+\mathfrak{B}\beta e^{\beta x}+\mathfrak{C}\gamma e^{\gamma x}+\mathfrak{D}\delta e^{\delta x}}{\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}+\mathfrak{C}e^{\gamma x}+\mathfrak{D}e^{\delta x}}$$

si hinc x, quod autem actu fieri nequit, eliminetur, prodibit aequatio inter s et u. Si quidem ponatur $\mathfrak{C}=0$ et $\mathfrak{D}=0$, prodibit aequatio integralis particularis haec

$$s+uu-(\alpha+\beta)u+\alpha\beta=0.$$

Quare si fuerint α et β duae radices huius aequationis

$$A+Bu+Cu^2+Du^3+Eu^4=0.$$

aequationi differentio differentiali inter s et u satisfaciet hic valor $s=-\alpha\beta+(\alpha+\beta)u-uu$. In aequatione autem illa non du sed $\frac{du}{s}$ positum est constans, quae consideratio exuetur ponendo $ds=qdu$: erit enim $\frac{ds}{qs}$ constans ideoque $qsdds=qds^2+sdsdq$, et $dds=\frac{ds^2}{s}+\frac{dsdq}{q}$, statuatur iam du constans, erit $dq=\frac{dds}{du}$ et $\frac{dq}{q}=\frac{dds}{ds}$, vnde fit $dds=\frac{ds^2}{s}+dds$. Prodibit ergo haec aequatio:

$$du^2(A+Bu+Cu^2+Du^3+Eu^4)+sdu^2(C+3Du+6Eu^2)$$
$$+3Essdu^2+sduds(D+4Eu)+Esds^2+Essdds=0$$

in qua differentiale du assumtum est constans. Quodsi iam formulae $A+Bu+Cu^2+Du^3+Eu^4$ factor trinomialis sit $L+Mu+Nu^2$ erit integrale particulare

$$L+Mu+Nu^2+Ns=0.$$

§. 5. Quoniam autem hic methodum meam integrandi aequationes differentiales altiorum graduum vlterius

 exten-

extendere conſtitui, regulam quam loco citato dedi paucis repetam. Patet vero methodus mea ad omnes aequationes in hac forma generali contentas:

$$o = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Ed^3y}{dx^3} + \frac{Fd^4y}{dx^4} + \frac{Gd^5y}{dx^5} + \text{etc.}$$

Vbi differentiale dx poſitum eſt conſtans. Ad huius aequationis integrale finitis terminis expreſſum inueniendum ex ea formetur ſequens forma Algebraica:

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \text{etc.}$$

cuius quaerantur omnes factores reales tam ſimplices quam trinomiales, inter quos, ſi qui fuerint inter ſe aequales, coniunctim repraeſententur. Ex quolibet autem factore naſcetur integralis pars, et, ſi omnes iſtae partes ex ſingulis factoribus oriundae in vnam ſummam coniiciantur, habebitur integrale completum aequationis propoſitae. Ex ſequenti autem tabella partes integralis ex ſingulis factoribus oriundae deſumentur.

Factores	Partes Integralis
$z-k$	αe^{kx}
$(z-k)^2$	$(\alpha + \beta x)e^{kx}$
$(z-k)^3$	$(\alpha + \beta x + \gamma x^2)e^{kx}$
$(z-k)^4$	$(\alpha + \beta x + \gamma x^2 + \delta x^3)e^{kx}$
etc.	etc.
$zz - 2kz \,\text{coſ.}\, \Phi + kk$	$\alpha e^{kx\,\text{coſ.}\,\Phi} \,\text{ſin.}\, kz \,\text{ſin.}\, \Phi + \mathfrak{A} e^{kx\,\text{coſ.}\,\Phi} \,\text{coſ.}\, kx \,\text{ſin.}\, \Phi$
$(zz - 2kz \,\text{coſ.}\, \Phi + kk)^2$	$(\alpha + \beta x)e^{kx\,\text{coſ.}\,\Phi} \,\text{ſin.}\, kx \,\text{ſin.}\, \Phi +$ $(\mathfrak{A} + \mathfrak{B}x)e^{kx\,\text{coſ.}\,\Phi} \,\text{coſ.}\, kx \,\text{ſin.}\, \Phi$
$(zz - 2kz \,\text{coſ.}\, \Phi + kk)^3$	$(\alpha + \beta x + \gamma x^2)e^{kx\,\text{coſ.}\,\Phi} \,\text{ſin.}\, kx \,\text{ſin.}\, \Phi +$ $(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2)e^{kx\,\text{coſ.}\,\Phi} \,\text{coſ.}\, kx \,\text{ſin.}\, \Phi$
$(zz - 2kx \,\text{coſ.}\, \Phi + kk)^4$	$(\alpha + \beta x + \gamma x^2 + \delta x^3)e^{kx\,\text{coſ.}\,\Phi} \,\text{ſin.}\, kx \,\text{ſin.}\, \Phi +$ $(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3)e^{kx\,\text{coſ.}\,\Phi} \,\text{coſ.}\, kx \,\text{ſin.}\, \Phi$
etc.	etc.

In his formulis litterae α, β, γ, δ, etc. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. denotant constantes quantitates arbitrarias. Hinc in partibus integralis colligendis cauendum est, ne eadem harum litterarum bis scribatur, quia alioquin extensio integralis restringeretur. Oportebit ergo has constantes continuo nouis litteris indicari, hocque modo in aequationem integralem tot ingredientur constantes arbitrariae, quoti gradus fuerit aequatio differentialis proposita: id quod certum est indicium integrale hoc modo inuentum esse completum, atque in aequatione differentiali nihil contineri, quod non simul in hac aequatione integrali contineatur. Ceterum in eo loco, vbi hanc methodum fusius exposui, pluribus eam exemplis illustraui, ita vt circa eius applicationem nulla difficultas locum habere queat.

§ 6. Aequatio autem generalior, cuius integrationem hic sum traditurus, denotante X functionem quamcunque ipsius x ita se habet:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.}$$

in qua iterum differentiale dx constans est assumtum. Hanc igitur aequationem quotumque constet terminis, seu ad quemcunque ea differentialium gradum ascendat, semper per quantitates finitas integrari posse affirmo, perinde atque aequationem ante memoratam, quae tanquam casus ex hac nascitur, si fuerit functio $X = 0$. Ac primo quidem patet, rem nulli difficultati fore subiectam, si X fuerit functio rationalis integra ipsius x, seu si habeat huiusmodi formam:

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \text{etc.}$$

Quodſi enim functio X ita ſit comparata, adhibeatur huiusmodi ſubſtitutio:

$$y = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \text{etc.} + v \text{ eritque}$$

$$\frac{dy}{dx} = \mathfrak{B} + 2\mathfrak{C}x + \mathfrak{D}x^2 + \text{etc.} + \frac{dv}{dx}$$

$$\frac{ddy}{dx^2} = 2\mathfrak{C} + 6\mathfrak{D}x + \text{etc.} + \frac{ddv}{dx^2}$$

$$\frac{d^3y}{dx^3} = 6\mathfrak{D} + \text{etc.} + \frac{d^3v}{dx^3}$$

$$\frac{d^4y}{dx^4} = \text{etc.} + \frac{d^4v}{dx^4}$$

$$\text{etc.}$$

Ponamus autem eſſe $X = \alpha + \beta x + \gamma x^2 + \delta x^3$, atque in valore ipſius y omnes termini poſt $\mathfrak{D}x^3$ euaneſcentes erunt ponendi. Facta ergo ſubſtitutione habebitur:

$$\begin{array}{l} \alpha + \beta x + \gamma x^2 + \delta x^3 = \\ \mathfrak{A}A + \mathfrak{B}Ax + \mathfrak{C}Ax^2 + \mathfrak{D}Ax^3 = Av + \frac{Bdv}{dx} + \frac{Cddv}{dx^2} + \frac{Dd^3v}{dx^3} + \frac{Ed^4v}{dx^4} + \text{etc.} \\ \mathfrak{B}B + 2\mathfrak{C}Bx + 3\mathfrak{D}Bx^2 \\ 2\mathfrak{C}C + 6\mathfrak{D}Cx \\ 6\mathfrak{D}D \end{array}$$

Hic iam coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita definiri poterunt, vt omnes termini, in quibus non ineſt v eiusue differentialia, euaneſcant, fiet enim:

$$\mathfrak{D} = \frac{\delta}{A}$$

$$\mathfrak{C} = \frac{\gamma}{A} - \frac{3\mathfrak{D}B}{A} = \frac{\gamma}{A} - \frac{3\delta B}{AA}$$

$$\mathfrak{B} = \frac{\beta}{A} - \frac{2\mathfrak{C}B}{A} - \frac{6\mathfrak{D}C}{A} = \frac{\beta}{A} - \frac{2\gamma B}{A^2} + \frac{6\delta B^2}{A^3} - \frac{6\delta C}{AA}$$

$$\mathfrak{A} = \frac{\alpha}{A} - \frac{\mathfrak{B}B}{A} - \frac{2\mathfrak{C}C}{A} - \frac{6\mathfrak{D}D}{A} = \frac{\alpha}{A} - \frac{\beta B}{A^2} + \frac{2\gamma B^2}{A^3} + \frac{18\delta BD}{A^3} - \frac{6\delta B^5}{A^4} - \frac{2\gamma C}{A^2} - \frac{6\delta D}{A^2}$$

His ergo valoribus pro $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ aſſumtis erit

$o =$

$$0 = Av + \frac{B\,dv}{dx} + \frac{C\,ddv}{dx^2} + \frac{D\,d^3v}{dx^3} + \frac{E\,d^4v}{dx^4} + \text{etc.}$$

quae aequatio ope superioris methodi integrabitur.

§. 7. Quo autem facilius aequationis propositae, qualiscunque X fuerit functio ipsius x integrale eruamus, a casibus simplicioribus inchoemus, ac primo quidem sit aequatio tantum differentialis primi gradus,

$$X = A.y + \frac{B\,dy}{dx}.$$

quam patet integrabilem reddi posse, si multiplicetur per huiusmodi formam $e^{\alpha x}dx$ denotante e numerum cuius logarithmus hyperbolicus $= 1$. Fiet enim

$$e^{\alpha x}X\,dx = Ae^{\alpha x}y\,dx + Be^{\alpha x}dy.$$

Atque α ita comparatum esse oportet, vt pars posterior sit differentiale cuiuspiam quantitatis finitae: quae ex termino vltimo alia esse nequit nisi $Be^{\alpha x}y$, cuius differentiale cum sit $= Be^{\alpha x}dy + \alpha Be^{\alpha x}y\,dx$ necesse est vt sit $A = \alpha B$ et $\alpha = \frac{A}{B}$. Hoc ergo valore pro α sumto erit

$$\int e^{\alpha x}X\,dx = Be^{\alpha x}y \text{ et } y = \frac{\alpha}{A}e^{-\alpha x}\int e^{\alpha x}X\,dx.$$

§. 8. Sit aequatio proposita differentialis secundi gradus:

$$X = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2}.$$

Multiplicetur ea per $e^{\alpha x}dx$ ac definiatur α ita, vt integratio succedat. Habebitur ergo

$$e^{\alpha x}X\,dx = A\,e^{\alpha x}y\,dx + Be^{\alpha x}dy + \frac{Ce^{\alpha x}ddy}{dx},$$

cuius integrale sit:

$$\int e^{\alpha x}X\,dx = e^{\alpha x}\left(A'y + \frac{B'\,dy}{dx}\right)$$

Quo

Quo differentiato habebitur:

$$e^{\alpha x}X\,dx = e^{\alpha x}(\alpha A'y\,dx + A'dy + \frac{B'ddy}{dx}) + \alpha B'dy$$

Comparatione ergo facta fiet $B' = C$: $A' = B - \alpha C$ et $A = \alpha B - \alpha^2 C$, debet ergo esse α radix huius aequationis $0 = A - \alpha B + \alpha^2 C$, quae cum habeat duas radices vtramlibet assumere licet; eritque $A' = B - \alpha C$ et $B' = C$. Peruentum est ergo ad hanc aequationem differentialem primi gradus:

$$e^{-\alpha x}\int e^{\alpha x}X\,dx = A'y + \frac{B'dy}{dx}.$$

Ad quam denuo integrandam multiplicetur per $e^{\beta x}dx$ vt habeatur.

$$e^{(\beta-\alpha)x}dx\int e^{\alpha x}X\,dx = A'e^{\beta x}y\,dx + B'e^{\beta x}dy$$

quae vt sit integrabilis, debet esse $\beta = \frac{A'}{B'} = \frac{B-\alpha C}{C}$ seu $\alpha + \beta = \frac{B}{C}$, vnde patet β esse alteram radicem aequationis $0 = A - \alpha B + \alpha^2 C$, eritque integrale:

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}X\,dx = B'e^{\beta x}y = Ce^{\beta x}y.$$

Est vero $\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}X\,dx = \frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x}X\,dx - \frac{1}{\beta-\alpha}\int e^{\beta x}X\,dx$

Ergo $Cy = \frac{e^{-\alpha x}}{\beta-\alpha}\int e^{\alpha x}X\,dx + \frac{e^{-\beta x}}{\alpha-\beta}\int e^{\beta x}X\,dx.$

In hac aequatione integrali ambae radices α et β aequationis quadraticae $0 = A - Bz + Czz$ aequaliter insunt, et hanc ob rem si istius aequationis radices sint cognitae ex iis statim aequatio integralis formatur. Ista autem aequatio $0 = A - Bz + Czz$ ex ipsa aequatione proposita

$$X = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2}$$

facilli-

facillime formatur: ſimili ſcilicet modo, quo in caſu $X = 0$ ſumus vſi. Ponatur enim 1 pro y; z pro $\frac{dy}{dx}$; et z^2 pro $\frac{ddy}{dx^2}$, vt prodeat iſta expreſſio $A + Bz + Czz$; cuius factores ſi fuerint $C(z + \alpha)(z + \beta)$, erunt α et β eae ipſae litterae, quae ad aequationem integralem formandam requiruntur.

§. 9. His praemiſſis aditus ad integrationem aequationis integralis non adeo erit difficilis. Sit ergo propoſita haec aequatio:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.}$$

cuius vltimus terminus ſit $\frac{\Delta d^n y}{dx^n}$. Formetur hinc iſta expreſſio modo ante indicato:

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots + \Delta z^n = P.$$

quae in factores ſimplices reſoluta ſit:

$$P = D(z + \alpha)(z + \beta)(z + \gamma)(z + \delta) \text{ etc.}$$

Dico iam ſi aequatio differentialis propoſita per $e^{\alpha x} dx$ multiplicetur eam euadere integrabilem. Erit enim

$$e^{\alpha x} X dx = e^{\alpha x} dx \left(Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots + \frac{\Delta d^n y}{dx^n} \right)$$

cuius integrale ponamus eſſe:

$$\int e^{\alpha x} X dx = e^{\alpha x} \left(A'y + \frac{B'dy}{dx} + \frac{C'ddy}{dx^2} + \frac{D'd^3y}{dx^3} + \ldots + \frac{\Delta d^{n-1} y}{dx^n} \right)$$

Sumto autem differentiali habebitur

$$e^{\alpha x} X dx = e^{\alpha x} dx \left(\alpha A'y + \frac{A'dy}{dx} + \frac{B'ddy}{dx^2} + \frac{C'd^3y}{dx^3} + \ldots + \frac{\Delta d^n y}{dy^n} \right)$$
$$+ \frac{\alpha B'dy}{dx} + \frac{\alpha C'ddy}{dx^2}$$

 quae

quae ſi cum propoſita conferatur erit:

$$A' = \frac{A}{\alpha};$$

$$B' = \frac{B}{\alpha} - \frac{A}{\alpha^2}$$

$$C' = \frac{C}{\alpha} - \frac{B}{\alpha^2} + \frac{A}{\alpha^3}$$

$$D' = \frac{D}{\alpha} - \frac{C}{\alpha^2} + \frac{B}{\alpha^3} - \frac{A}{\alpha^4}$$

quibus valoribus vsque ad vltimum continuatis, peruenietur ad hanc aequationem:

$$A - B\alpha + C\alpha^2 - D\alpha^3 + E\alpha^4 - \ldots\ldots \pm \Delta\alpha^n = 0$$

cum igitur α ſit radix huius aequationis erit $z + \alpha$ factor iſtius expreſſionis

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots + \Delta z^n.$$

exiſtente $P = \Delta(z+\alpha)(z+\beta)(z+\gamma)(z+\delta)$ etc.

§. 10. Prima ergo integratione abſoluta erit

$$e^{-\alpha x}\int e^{\alpha x} X\,dx = A'y + \frac{B'dy}{dx} + \frac{C'ddy}{dx^2} + \frac{D'd^3y}{dx^3} + \ldots \frac{\Delta d^{n-1}y}{dx^{n-1}})$$

Formetur hinc iterum modo ante expoſito haec expreſſio:

$$P' = A' + B'z + C'z^2 + D'z^3 + \ldots + \Delta z^{n-1}$$

Cum iam ſit:

$$A = \alpha A'$$

$$B = \alpha B' + A'$$

$$C = \alpha C' + B'$$

$$D = \alpha D' + C'$$

etc.

manifeſtum eſt fore $P = (\alpha + z)P'$, ideoque $P' = \frac{P}{z+\alpha}$.

et

et $P' = \Delta(z+\beta)(z+\gamma)(z+\delta)(z+\varepsilon)$ etc.
Simili ergo modo, quo supra vsi sumus, euincetur hanc aequatione denuo reddi integrabilem, si multiplicetur per $e^{\beta x}dx$.
Sit igitur aequatio integralis hinc oriunda.

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = e^{\beta x}\left(A''y + \frac{B''dy}{dx} + \frac{C''ddy}{dx^2} + \ldots + \frac{\Delta d^{n-2}y}{dx^{n-2}}\right)$$

fietque comparatione instituta

$$A' = \beta A''$$
$$B' = \beta B'' + A''$$
$$C' = \beta C'' + B''$$
$$D' = \beta D'' = C''$$
etc.

Ergo si ponatur

$$P'' = A'' + B''z + C''z^2 + D''z^3 + \ldots + \Delta z^{n-2}$$

erit $P' = (\beta + z)P''$, et $P'' = \frac{P'}{z+\beta} = \frac{P}{(z+\alpha)(z+\beta)}$ vnde fit $P'' = \Delta(z+\gamma)(z+\delta)(z+\varepsilon)$ etc. scilicet hinc duo iam factores $z+\alpha$ et $z+\beta$ sunt egressi. Est autem:

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = \frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x}Xdx - \frac{1}{\beta-\alpha}\int e^{\beta x}Xdx$$

vnde aequatio bis integrata reducitur ad hanc formam

$$\frac{e^{-\alpha x}}{\beta-\alpha}\int e^{\alpha x}Xdx + \frac{e^{-\beta x}}{\alpha-\beta}\int e^{\beta x}Xdx = A''y + \frac{B''dy}{dx} + \frac{C''ddy}{dx^2} + \frac{D''d^3y}{dx^3} + \ldots \frac{\Delta d^{n-2}y}{dx^{n-2}}$$

§. 11. Cum porro hinc posito 1 pro y et z pro $\frac{dy}{dx}$ etc. prodeat haec expressio,

 $P'' =$

$$P'' = A'' + B''z + C''z^2 + \ldots\ldots + \Delta z^{n-2}$$

fitque $P'' = \Delta(z+\gamma)(z+\delta)(z+\varepsilon)$ etc.

manifeftum eft aequationem vltimo inuentam denuo reddi integrabilem fi multiplicetur per $e^{\gamma x}dx$, fit aequatio integralis hinc oriunda haec:

$$\int\frac{e^{(\gamma-\alpha)x}dx}{\beta-\alpha}\int e^{\alpha x}X\,dx + \int\frac{e^{(\gamma-\beta)x}dx}{\alpha-\beta}\int e^{\beta x}X\,dx = e^{\gamma x}\left(A'''y + \frac{B'''dy}{dx}\right.$$

$$\left. + \frac{C'''ddy}{dx^2} + \ldots\ldots + \frac{\Delta d^{n-3}y}{dx^{n-3}}\right)$$

fietque ex comparatione terminorum homogeneorum:

$$A'' = \gamma A'''$$
$$B'' = \gamma B''' + A'''$$
$$C'' = \gamma C''' + B'''$$
$$D'' = \gamma D''' + C'''$$

etc.

Quare fi ponatur:

$$P''' = A''' + B'''z + C'''z^2 + D'''z^3 + \ldots\ldots + \Delta z^{n-3}$$

erit $P'' = (\gamma + z)P'''$ et $P''' = \frac{P''}{z+\gamma} = \frac{P}{(z+\alpha)(z+\beta)(z+\gamma)}$

vnde fequitur fore:

$$P''' = \Delta(z+\delta)(z+\varepsilon)(z+\zeta) \text{ etc.}$$

Cum autem fit generaliter $\int e^{(\mu-\nu)x}dx\int e^{\nu x}X\,dx = \frac{e^{(\mu-\nu)x}}{\mu-\nu}\int e^{\nu x}X\,dx + \frac{1}{\nu-\mu}\int e^{\mu x}X\,dx$, fi hinc integralia reducantur reperietur.

$$\frac{e^{-\alpha x}}{(\beta-\alpha)(\gamma-\alpha)}\int e^{\alpha x}X\,dx + \frac{e^{-\beta x}}{(\alpha-\beta)(\gamma-\beta)}\int e^{\beta x}X\,dx + \frac{e^{-\gamma x}}{(\alpha-\gamma)(\beta-\gamma)}\int e^{\gamma x}X\,dx$$

$$= A'''y + \frac{B'''dy}{dx} + \frac{C'''ddy}{dx^2} + \frac{D'''d^3y}{dx^3} + \frac{\Delta d^{n-3}y}{dx^{n-3}}.$$

§. 12. Si hoc modo eo vsque progrediamur, quoad nulla amplius differentialia ipsius y supersint, tum ex altera parte aequationis habebitur vnicus terminus $\frac{\Delta d^0 y}{dx^0} = \Delta y$; id quod eueniet, si integratio toties fuerit instituta quot maximus exponens n continet vnitates. Ad hoc ergo vltimum integrale commode exprimendum, cum sit

$A + Bz + Cz^2 + Dz^3 + \ldots + \Delta z^n = \Delta(z+\alpha)(z+\beta)(z+\gamma)$ etc.

formentur ex radicibus α, β, γ, δ, etc. sequentes valores

$\mathfrak{A} = \Delta(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)$ etc.

$\mathfrak{B} = \Delta(\alpha-\beta)(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)$ etc.

$\mathfrak{C} = \Delta(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)(\varepsilon-\gamma)$ etc.

$\mathfrak{D} = \Delta(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(\varepsilon-\delta)$ etc.

$\mathfrak{E} = \Delta(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon)$ etc.

etc.

quibus inuentis erit integralis aequatio vltima quae sita:

$$y = \frac{e^{-\alpha x}}{\mathfrak{A}} \int e^{\alpha x} X\, dx + \frac{e^{-\beta x}}{\mathfrak{B}} \int e^{\beta x} X\, dx + \frac{e^{-\gamma x}}{\mathfrak{C}} \int e^{\gamma x} X\, dx + \text{etc.}$$

quae cum tot contineat terminos, quoti gradus fuerit aequatio differentialis proposita.

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + \frac{\Delta d^n y}{dx^n}$$

totidem inuoluet constantes arbitrarias, ideoque erit integralis completa.

§. 13. Alio autem modo valores quantitatum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. exprimi possunt, qui plerumque multo commodius negotium conficit. Dico enim fore $\mathfrak{A} = \frac{dP}{dz}$,

 si

si vbique pro z substituatur $-\alpha$, seu si ponatur $z+\alpha = 0$. Cum enim sit

$$P = \Delta(z+\alpha)(z+\beta)(z+\gamma)(z+\delta) \text{ etc.}$$

erit differentiando:

$$\frac{dP}{dz} = \Delta(z+\beta)(z+\gamma)(z+\delta)\text{etc.} + \frac{\Delta(z+\alpha)d.}{dz}(z+\beta)(z+\gamma)(z+\delta)\text{etc.}$$

Si iam ponatur $z = -\alpha$ posterius membrum euanescet, et prius dabit:

$$\frac{dP}{dz} = \Delta(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha) \text{ etc.} = \mathfrak{A}.$$

Cum autem sit $P = A + Bz + Cz^2 + Dz^3 + \ldots + \Delta z^n$ erit:

$$\frac{dP}{dz} = B + 2Cz + 3Dz^2 + 4Ez^3 + \ldots + n\Delta z^{n-1}$$

ponatur ergo $z = -\alpha$, seu fiat $z+\alpha = 0$, erit

$$\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 + \text{etc.} \ldots \pm n\Delta\alpha^{n-1}$$

simili modo reperietur fore

$$\mathfrak{B} = B - 2C\beta + 3D\beta^2 - 4E\beta^3 + \ldots \pm n\Delta\beta^{n-1}$$

$$\mathfrak{C} = B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 + \ldots \pm n\Delta\gamma^{n-1}$$

etc.

§. 14. Si ergo huiusmodi proponatur aequatio:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.}$$

quam integrari oporteat, ante omnia ex ea formetur haec expressio Algebraica

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

cuius quaerantur omnes factores simplices, cuiusmodi vnus sit $z+\alpha$, atque quilibet factor dabit partem integralis ita vt omnes partes, quae hoc modo ex singulis factoribus eruuntur, iunctim sumtae exhibeant completum ipsius fius y valorem finitum. Scilicet fi factor fimplex fuerit inuentur $z + \alpha$, tum quaeratur quantitas $\mathfrak{A}$ vt fit

$$\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 + \text{etc.}$$

qua inuenta erit pars integralis ex hoc factore $z + \alpha$ oriunda haec

$$\frac{e^{-\alpha x}}{\mathfrak{A}} \int e^{\alpha x} X\, dx.$$

Hinc perfpicitur fi factor fimplex formae P fuerit $z - \alpha$; tum fore

$$\mathfrak{A} = B + 2C\alpha + 3D\alpha^2 + 4E\alpha^3 + \text{etc.}$$

atque integralis partem hinc oriundam effe

$$+ \frac{e^{\alpha x}}{\mathfrak{A}} \int e^{-\alpha x} X\, dx.$$

§. 15. Supereft autem vt oftendamus, quomodo iftae integralis partes fint comparatae, fi factorum fimplicium aliquot fuerint vel inter fe aequales vel imaginariae. Ex fuperioribus enim liquet vtroque cafu partes integralis fingulari modo adornari debere, vt formam finitam et realem obtineant. Sint igitur primo duo factores $z - \alpha$ et $z - \beta$ inter fe aequales feu $\beta = \alpha$, eritque tam $\mathfrak{A} = 0$ quam $\mathfrak{B} = 0$; et vtraque pars integralis euadet infinita, altera quidem affirmatiue altera negatiue, ita vt differentia fit finita. Ad quam inueniendam ponamus $\beta = \alpha + \omega$, denotante ω quantitatem euanefcentem. Cum ergo fit

$$\mathfrak{A} = \Delta(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \varepsilon) \text{ etc. et}$$

$$\mathfrak{B} = \Delta(\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \varepsilon) \text{ etc.}$$

fumtis

ſumtis litteris α, β, γ, δ, etc. negatiuis, erit,

$$\mathfrak{A} = -\Delta\omega(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon) \text{ etc. et}$$

$$\mathfrak{B} = \Delta\omega(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon) \text{ etc.}$$

Tum vero erit $e^{\beta x} = e^{\alpha x + \omega x} = e^{\alpha x}(1+\omega x)$ et $e^{-\beta x} = e^{-\alpha x}(1-\omega x)$. Hinc pars integralis ex factoribus binis aequalibus $z-\alpha$ et $z-\beta$ oriunda erit

$$\frac{e^{\alpha x}}{\mathfrak{A}}\int e^{-\alpha x}X\,dx + \frac{e^{\alpha x}(1+\omega x)}{\mathfrak{B}}\int e^{-\alpha x}(1-\omega x)X\,dx$$

Ponatur:

$$\mathfrak{A}' = \Delta(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon) \text{ etc.}$$

erit $\mathfrak{A} = -\mathfrak{A}'\omega$ et $\mathfrak{B} = \mathfrak{A}'\omega$, vnde fiet iſta pars $=$

$$\frac{e^{\alpha x}}{\mathfrak{A}'\omega}\left((1+\omega x)\int e^{-\alpha x}(1-\omega x)X\,dx - \int e^{-\alpha x}X\,dx\right) =$$

$$\frac{e^{\alpha x}}{\mathfrak{A}'\omega}\left(\omega x\int e^{-\alpha x}X\,dx - \omega\int e^{-\alpha x}X\,x\,dx\right) =$$

$$\frac{e^{\alpha x}}{\mathfrak{A}'}\left(x\int e^{-\alpha x}X\,dx - \int e^{-\alpha x}X\,x\,dx\right) = \frac{e^{\alpha x}}{\mathfrak{A}'}\int dx\int e^{-\alpha x}X\,dx.$$

quae eſt pars integralis ex factore expreſſionis P quadrato $(z-\alpha)^2$ oriunda.

§. 16. Valor autem ipſius $\mathfrak{A}'$ ſequenti modo commodius exhiberi poterit. Ob $\beta = \alpha$, cum ſit

$P = \Delta(z-\alpha)^2(z-\gamma)(z-\delta)(z-\varepsilon)\text{etc.} = A + Bz + Cz^2 + Dz^3 + \text{ etc.}$ ponatur $\Delta(z-\gamma)(z-\delta)(z-\varepsilon)$ etc. $= Q$, ita vt valor ipſius Q praebeat $\mathfrak{A}'$ ſi loco z ponatur α. Erit ergo $P = (z-\alpha)^2 Q$, et differentiando $\frac{dP}{dz} = (z-\alpha)^2\frac{dQ}{dz} + 2(z-\alpha)Q$ atque $\frac{ddP}{dz^2} = (z-\alpha)^2\frac{ddQ}{dz^2} + 4(z-\alpha)\frac{dQ}{dz} + 2Q$; poſito nunc $z = \alpha$ fiet $Q = \frac{ddP}{2dz^2} = \mathfrak{A}'$, orieturque $\mathfrak{A}'$ ſi in $\frac{ddP}{2dz^2}$ ponatur $z = \alpha$. Eſt vero

$$\frac{ddP}{2dz^2} = C + 3Dz + 6Ez^2 + 10Fz^3 + 15Gz^4 + \text{etc.}$$

vnde fit

$$\mathfrak{A}' = C + 3D\alpha + 6E\alpha^2 + 10F\alpha^3 + 15G\alpha^4 + \text{etc.}$$

Quare si propofita hac aequatione:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.}$$

expreffio hinc formata

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

habeat factorem quadratum $(z-\alpha)^2$, fumatur

$$\mathfrak{A}' = C + 3D\alpha + 6E\alpha^2 + 10F\alpha^3 + 15G\alpha^4 + \text{etc.}$$

eritque pars integralis inde oriunda:

$$\frac{e^{\alpha x}}{\mathfrak{A}'}\int dx \int e^{-\alpha x} X\, dx.$$

Sin autem reliqui factores formulae P fuerint cogniti, nempe

$$P = \Delta(z-\alpha)^2(z-\gamma)(z-\delta)(z-\varepsilon) \text{ etc.}$$

erit $\mathfrak{A}' = \Delta(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)$ etc.

§. 17. Ponamus iam tres factores inter fe effe aequales, feu fit infuper $\gamma = \alpha$, at ob rationes fupra expofitas ponamus

$\gamma = \alpha + \omega$, erit $\mathfrak{A}' = -\Delta\omega(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)$ etc.

et $\mathfrak{C} = \Delta(\gamma-\alpha)^2(\gamma-\delta)(\gamma-\varepsilon)(\gamma-\zeta)$ etc.

feu $\mathfrak{C} = \Delta\omega^2(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)$ etc.

fit $\mathfrak{A}'' = \Delta(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)$ etc.

erit $\mathfrak{A}' = -\mathfrak{A}''\omega$ et $\mathfrak{C} = \mathfrak{A}''\omega^2$. Factisque his fubftitutionibus tandem reperietur pars integralis ex factore cubico $(z-\alpha)^3$ oriunda haec,

$$\frac{e^{\alpha x}}{\mathfrak{A}''}\int dx\int dx\int e^{-\alpha x}X\,dx$$

existente:

$$\mathfrak{A}'' = D + 4E\alpha + 10F\alpha^2 + 20G\alpha^3 + \text{etc.}$$

Facilius autem hoc immediate ex aequalitate trium factorum ostenditur. Sint enim, tres factores quicunque $(z-\alpha)(z-\beta)(z-\gamma)$ ac positos

$$\mathfrak{A} = \Delta(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)\ \text{etc.}$$
$$\mathfrak{B} = \Delta(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\varepsilon)\ \text{etc.}$$
$$\mathfrak{C} = \Delta(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\varepsilon)\ \text{etc.}$$

erunt integralis partes hinc oriundae.

$$\frac{e^{\alpha x}}{\mathfrak{A}}\int e^{-\alpha x}X\,dx + \frac{e^{\beta x}}{\mathfrak{B}}\int e^{-\beta x}X\,dx + \frac{e^{\gamma x}}{\mathfrak{C}}\int e^{-\gamma x}X\,dx.$$

Ponatur iam $\beta = \alpha + \omega$ et $\gamma = \alpha + \phi$, existentibus ω et ϕ quantitatibus euanescentibus, ac posito

$$\mathfrak{A}'' = \Delta(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)\ \text{etc.}$$

erit $\mathfrak{A} = \mathfrak{A}''\omega\phi$; $\mathfrak{B} = \mathfrak{A}''\omega(\omega-\phi)$, et $\mathfrak{C} = \mathfrak{A}''\phi(\phi-\omega)$. tum vero erit $e^{\beta x} = e^{\alpha x}(1+\omega x+\frac{1}{2}\omega^2x^2)$, $e^{-\beta x} = e^{-\alpha x}(1-\omega x+\frac{1}{2}\omega^2x^2)$ et $e^{\gamma x} = e^{\alpha x}(1+\phi x+\frac{1}{2}\phi^2x^2)$, $e^{-\gamma x} = e^{-\alpha x}(1-\phi x+\frac{1}{2}\phi^2x^2)$. Quibus substitutis ternae integralis partes abeunt in:

$$\frac{e^{\alpha x}}{\mathfrak{A}''\omega\phi(\omega-\phi)}\left\{\begin{array}{l}\int e^{-\alpha x}X\,dx(\omega-\phi+\phi+\omega\phi x+\frac{1}{2}\omega^2\phi x^2-\omega-\omega\phi x+\frac{1}{2}\omega\phi^2x^2)\\ \int e^{-\alpha x}Xx\,dx(-\omega\phi-\omega\omega\phi x+\omega\phi+\omega\phi\phi x)\\ \int e^{-\alpha x}Xx^2\,dx(\frac{1}{2}\omega\omega\phi-\frac{1}{2}\omega\phi\phi)\end{array}\right.$$

sublatis nunc per diuisionem litteris euanescentibus ω et ϕ factor cubicus $(z-\alpha)^3$ dabit hanc integralis partem

$\frac{e^{\alpha x}}{\mathfrak{A}''}$

$$\frac{e^{\alpha x}}{\mathfrak{A}'''}\left(\tfrac{1}{2}xx\int e^{-\alpha x}\mathrm{X}dx - x\int e^{-\alpha x}\mathrm{X}xdx + \tfrac{1}{2}\int e^{-\alpha x}\mathrm{X}xxdx\right)$$

quae reducitur ad hanc formam simpliciorem:

$$\frac{e^{\alpha x}}{\mathfrak{A}'''}\int dx\int dx\int e^{-\alpha x}\mathrm{X}dx.$$

existente $\mathfrak{A}''' = \mathrm{D} + 4\mathrm{E}\alpha + 10\mathrm{F}\alpha^2 + 20\mathrm{G}\alpha^3 +$ etc. scilicet valor ipsius $\mathfrak{A}'''$ oritur ex formula $\frac{d^3\mathrm{P}}{6dz^3}$ posito $z = \alpha$.

§. 18. Simili modo vlterius procedendo patebit quaternos factores inter se aequales seu formulae $\mathrm{P} = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 +$ etc. factorem $(z-\alpha)^4$ praebiturum fore hanc integralis partem:

$$\frac{e^{\alpha x}\int dx\int dx\int dx\int e^{-\alpha x}\mathrm{X}dx}{\mathrm{E} + 5\mathrm{F}\alpha + 15\mathrm{G}\alpha^2 + 35\mathrm{H}\alpha^3 + \text{etc.}}$$

qui denominator ex formula $\frac{d^4\mathrm{P}}{24dz^4}$ nascitur ponendo $z = \alpha$. Superfluum foret pro pluribus factoribus simplicibus inter se aequalibus partes integralis, quae ex ipsis conflantur hic exhibere, cum lex, qua hae partes formantur, per se sit manifesta. Ceterum complicatio plurium signorum integralium in his formulis nullam inuoluit difficultatem, cum facillime ad simplicia integralia reuocentur. Est enim

$$\int dx\int e^{-\alpha x}\mathrm{X}dx = \frac{x\int e^{-\alpha x}\mathrm{X}dx - \int e^{-\alpha x}\mathrm{X}xdx}{1.}$$

$$\int dx\int dx\int e^{-\alpha x}\mathrm{X}dx = \frac{x^2\int e^{-\alpha x}\mathrm{X}dx - 2x\int e^{-\alpha x}\mathrm{X}xdx + \int e^{-\alpha x}\mathrm{X}xxdx}{1.\ 2.}$$

$$\int dx\int dx\int dx\int e^{-\alpha x}\mathrm{X}dx = \frac{x^3\int e^{-\alpha x}\mathrm{X}dx - 3x^2\int e^{-\alpha x}\mathrm{X}xdx + 3x\int e^{-\alpha x}\mathrm{X}xxdx - \int e^{-\alpha x}\mathrm{X}x^3dx}{1.\ 2.\ 3.}$$

etc.

 §. 19.

§. 19. Expeditis factoribus aequalibus pergo ad factores imaginarios. Sint ergo formulae $P = \Delta(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)(z-\varepsilon)$ etc. $= A + Bz + Cz^2 + Dz^3 + Ez^4$ etc. bini factores $z-\alpha$ et $z-\beta$ imaginarii, qui hoc non obstante multiplicato praebeant productum reale

$$zz - 2kz \operatorname{cos}. \Phi + kk$$

erit ergo $\alpha = k \operatorname{cos}. \Phi + k\sqrt{-1}. \operatorname{sin}. \Phi$
et $\beta = k \operatorname{cos}. \Phi - k\sqrt{-1}. \operatorname{sin}. \Phi$

harumque litterarum potestates quaecunque ita se habebunt.

$$\alpha^n = k^n \operatorname{cos}. n\Phi + k^n\sqrt{-1}. \operatorname{sin}. n\Phi$$
$$\beta^n = k^n \operatorname{cos}. n\Phi - k^n\sqrt{-1}. \operatorname{sin}. n\Phi$$

Iam primo erit:

$$e^{\alpha x} = e^{kx \operatorname{cos}. \Phi}\left(1 + \frac{k\sqrt{-1}}{1} x \operatorname{sin}. \Phi - \frac{kk}{1.2} x^2 \operatorname{sin}. \Phi^2 - \frac{k^3\sqrt{-1}}{1.2.3} x^3 \operatorname{sin}. \Phi^3 + \frac{k^4}{1.2.3.4} x^4 \operatorname{sin}. \Phi^4 - \text{etc.}\right)$$

ideoque

$$e^{\alpha x} = e^{kx \operatorname{cos}. \Phi}(\operatorname{cos}. kx \operatorname{sin}. \Phi + \sqrt{-1}. \operatorname{sin}. kx \operatorname{sin}. \Phi)$$
$$e^{\beta x} = e^{kx \operatorname{cos}. \Phi}(\operatorname{cos}. kx \operatorname{sin}. \Phi - \sqrt{-1}. \operatorname{sin}. kx \operatorname{sin}. \Phi)$$
$$e^{-\alpha x} = e^{-kx \operatorname{cos}. \Phi}(\operatorname{cos}. kx \operatorname{sin}. \Phi - \sqrt{-1}. \operatorname{sin}. kx \operatorname{sin}. \Phi)$$
$$e^{-\beta x} = e^{-kx \operatorname{cos}. \Phi}(\operatorname{cos}. kx \operatorname{sin}. \Phi + \sqrt{-1}. \operatorname{sin}. kx \operatorname{sin}. \Phi)$$

Deinde cum sit:

$$\mathfrak{A} = B + 2C\alpha + 3D\alpha^2 + 4E\alpha^3 + 5F\alpha^4 + \text{etc. et}$$
$$\mathfrak{B} = B + 2C\beta + 3D\beta^2 + 4E\beta^3 + 5F\beta^4 + \text{etc.}$$

superioribus valoribus pro α et β substitutis habebitur

$$\mathfrak{A} = \begin{array}{l} B + 2Ck \operatorname{cos}. \Phi + 3Dk^2 \operatorname{cos}. 2\Phi + 4Ek^3 \operatorname{cos}. 3\Phi + \text{etc.} \\ + (2Ck \operatorname{sin}. \Phi + 3Dk^2 \operatorname{sin}. 2\Phi + 4Ek^3 \operatorname{sin}. 3\Phi + \text{etc.})\sqrt{-1} \end{array}$$

$$\mathfrak{B} = \begin{array}{l} B + 2Ck \operatorname{cos}. \Phi + 3Dk^2 \operatorname{cos}. 2\Phi + 4Ek^3 \operatorname{cos}. 3\Phi + \text{etc.} \\ - (2Ck \operatorname{sin}. \Phi + 3Dk^2 \operatorname{sin}. 2\Phi + 4Ek^3 \operatorname{sin}. 3\Phi + \text{etc.})\sqrt{-1}. \end{array}$$

§. 20.

§. 20. Cum autem $z-\alpha$ et $z-\varepsilon$ ſint factores formulae $P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$ erit

$$A + Bk\,\text{coſ.}\,\Phi + Ck^2\,\text{coſ.}\,2\Phi + Dk^3\,\text{coſ.}\,3\Phi + Ek^4\,\text{coſ.}\,4\Phi + \text{etc.} = 0$$

et $Bk\,\text{ſin.}\,\Phi + Ck^2\,\text{ſin.}\,2\Phi + Dk^3\,\text{ſin.}\,3\Phi + Ek^4\,\text{ſin.}\,4\Phi + \text{etc.} = 0$

Ponatur nunc:

$$\mathfrak{M} = B + 2Ck\,\text{coſ.}\,\Phi + 3Dk^2\,\text{coſ.}\,2\Phi + 4Ek^3\,\text{coſ.}\,3\Phi + \text{etc.}$$

$$\mathfrak{N} = \qquad 2Ck\,\text{ſin.}\,\Phi + 3Dk^2\,\text{ſin.}\,2\Phi + 4Ek^3\,\text{ſin.}\,3\Phi + \text{etc.}$$

atque fiet:

$$\mathfrak{A} = \mathfrak{M} + \mathfrak{N}\sqrt{-1} \quad \text{et} \quad \mathfrak{B} = \mathfrak{M} - \mathfrak{N}\sqrt{-1}$$

ſicque imaginaria a realibus erunt ſeparata. Cum nunc ex ambobus factoribus $z-\alpha$ et $z-\varepsilon$ naſcantur iſtae integralis partes

$$\frac{e^{\alpha x}}{\mathfrak{A}}\int e^{-\alpha x}X\,dx + \frac{e^{\varepsilon x}}{\mathfrak{B}}\int e^{-\varepsilon x}X\,dx$$

hae abibunt in hanc formam:

$$\frac{(\mathfrak{M} - \mathfrak{N}\sqrt{-1})\,e^{\alpha x}\int e^{-\alpha x}X\,dx + (\mathfrak{M} + \mathfrak{N}\sqrt{-1})\,e^{\varepsilon x}\int e^{-\varepsilon x}X\,dx}{\mathfrak{M}^2 + \mathfrak{N}^2}$$

At eſt:

$$e^{\alpha x}\int e^{-\alpha x}X\,dx = \begin{cases} + e^{kx\,\text{coſ.}\,\Phi}\,\text{coſ.}\,kx\,\text{ſin.}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{coſ.}\,kx\,\text{ſin.}\,\Phi \\ -\sqrt{-1}\,.\,e^{kx\,\text{coſ.}\,\Phi}\,\text{coſ.}\,kx\,\text{ſin.}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{ſin.}\,kx\,\text{ſin.}\,\Phi \\ +\sqrt{-1}\,.\,e^{kx\,\text{coſ.}\,\Phi}\,\text{ſin.}\,kx\,\text{ſin.}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{coſ.}\,kx\,\text{ſin.}\,\Phi \\ + e^{kx\,\text{coſ.}\,\Phi}\,\text{ſin.}\,kx\,\text{ſin.}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{ſin.}\,kx\,\text{ſin.}\,\Phi \end{cases}$$

$$e^{\varepsilon x}\int e^{-\varepsilon x}X\,dx = \begin{cases} + e^{kx\,\text{coſ.}\,\Phi}\,\text{coſ.}\,kx\,\text{ſin.}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{coſ.}\,kx\,\text{ſin.}\,\Phi \\ +\sqrt{-1}\,.\,e^{kx\,\text{coſ.}\,\Phi}\,\text{coſ.}\,kx\,\text{ſin}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{ſin}\,kx\,\text{ſin.}\,\Phi \\ -\sqrt{-1}\,.\,e^{kx\,\text{coſ.}\,\Phi}\,\text{ſin.}\,kx\,\text{ſin}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{coſ}\,kx\,\text{ſin}\,\Phi \\ + e^{kx\,\text{coſ.}\,\Phi}\,\text{ſin.}\,kx\,\text{ſin.}\,\Phi\int e^{-kx\,\text{coſ.}\,\Phi}X\,dx\,\text{ſin.}\,kx\,\text{ſin.}\,\Phi \end{cases}$$

Partes ergo ambae integrales tranſibunt, imaginariis ſe mutuo ſublatis, in hanc formam,

$$\frac{2\mathfrak{M}e^{kx\cos.\Phi}}{\mathfrak{M}^2+\mathfrak{N}^2}(\cos.kx\sin.\Phi\int e^{-kx\cos.\Phi}Xdx\cos.kx\sin.\Phi+\sin.kx\sin.\Phi\int e^{-kx\cos.\Phi}Xdx\sin.kx\sin.\Phi)$$

$$+\frac{2\mathfrak{N}e^{kx\cos.\Phi}}{\mathfrak{M}^2+\mathfrak{N}^2}(\sin.kx\sin.\Phi\int e^{-kx\cos.\Phi}Xdx\cos.kx\sin.\Phi-\cos.kx\sin.\Phi\int e^{-kx\cos.\Phi}Xdx\sin.kx\sin.\Phi)$$

quae etiam hoc modo exprimi poteſt:

$$\frac{2\,e^{kx\cos.\Phi}}{\mathfrak{M}^2+\mathfrak{N}^2}\left\{\begin{array}{l}(\mathfrak{M}\cos.kx\sin.\Phi+\mathfrak{N}\sin.kx\sin.\Phi)\int e^{-kx\cos.\Phi}Xdx\cos.kx\sin.\Phi+\\(\mathfrak{M}\sin.kx\sin.\Phi-\mathfrak{N}\cos.kx\sin.\Phi)\int e^{-kx\cos.\Phi}Xdx\sin.kx\sin.\Phi\end{array}\right\}$$

Haec ergo pars integralis oritur ex formulae
$P=A+Bz+Cz^2+Dz^3+$ etc. factore trinomiali
$zz-2kz\cos.\Phi+kk$.

§. 21. Simili modo ſi bini huiusmodi factores trinomiales fuerint inter ſe aequales, ſeu ſi formula

$$P=A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}$$

factorem habuerit $(zz-2kz\cos.\Phi+kk)^2$, pars integralis hinc oriunda reperietur ex formulis pro binis factoribus ſimplicibus aequalibus ſupra inuentis reperietur. Ponatur nempe

$$\mathfrak{M}'=C+3Dk\cos.\Phi+6Ek^2\cos.2\Phi+10Fk^3\cos.3\Phi+\text{etc.}$$

$$\mathfrak{N}'=\quad 3Dk\sin.\Phi+6Ek^2\sin.2\Phi+10Fk^3\cos.4\Phi+\text{etc.}$$

eritque integralis pars hinc oriunda,

$$\frac{2\,e^{kx\cos.\Phi}}{\mathfrak{M}'\mathfrak{M}'+\mathfrak{N}'\mathfrak{N}'}\left\{\begin{array}{l}(\mathfrak{M}'\cos.kx\sin.\Phi+\mathfrak{N}'\sin.kx\sin\Phi)\int dx\int e^{-kx\cos.\Phi}Xdx\cos.kx\sin.\Phi+\\(\mathfrak{M}'\sin.kx\sin.\Phi-\mathfrak{N}'\cos.kx\sin.\Phi)\int dx\int e^{-kx\cos.\Phi}Xdx\sin.kx\sin.\Phi\end{array}\right\}$$

Sin autem tres factores trinomiales radices imaginarias continentes fuerint inter ſe aequales, ſeu ſi formulae
$P=A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+$ etc.
factor fuerit $(zz-2kz\cos.\Phi+kk)^3$ ſtatuatur

$\mathfrak{M}''=$

$$\mathfrak{M}''=D+4Ek\cos.\Phi+10Fk^2\cos.2\Phi+20Gk^3\cos.3\Phi+\text{etc.}$$
$$\mathfrak{N}''=\quad 4Ek\sin.\Phi+10Fk^2\sin.2\Phi+20Gk^3\sin.3\Phi+\text{etc.}$$

atque pars integralis ex hoc factore oriunda erit

$$\frac{2e^{kx\cos.\Phi}}{\mathfrak{M}''\mathfrak{M}''+\mathfrak{N}''\mathfrak{N}''}\left\{\begin{array}{l}(\mathfrak{M}''\cos.kx\sin.\Phi+\mathfrak{N}''\sin.kx\sin.\Phi)\int dx\int dx\int e^{-kx\cos.\Phi}X\,dx\cos.kx\sin.\Phi+\\ (\mathfrak{M}''\sin.kx\sin.\Phi-\mathfrak{N}''\cos.kx\sin.\Phi)\int dx\int dx\int e^{-kx\cos.\Phi}X\,dx\sin.kx\sin.\Phi\end{array}\right\}$$

Hinc igitur iam lex perſpicitur, ſecundum quam iſtae integralis partes formari debent, ſi maior poteſtas formulae $zz-2kz\cos.\Phi+kk$ fuerit factor ipſius P: ideoque omnes caſus, qui vnquam occurrere poſſunt hinc conficientur.

§. 22. Ex his ergo ſequenti modo reſolui poterit hoc

Problema.

Inuenire valorem ipſius y in quantitatibus finitis expreſſum, qui ipſi conuenit ex hac aequatione differentiali cuiuscunque gradus:

$$X=Ay+\frac{Bdy}{dx}+\frac{Cddy}{dx^2}+\frac{Dd^3y}{dx^3}+\frac{Ed^4y}{dx^4}+\frac{Fd^5y}{dx^5}\ \text{etc.}$$

vbi differentiale dx ponitur conſtans, atque X denotat functionem quamcunque ipſius x.

Solutio.

Ex aequatione propoſita formetur ſequens formula Algebraica:

$$P=A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.}$$

cuius quaerantur omnes factores reales tam ſimplices, quam trinomiales, quippe qui factorum ſimplicium imaginariorum vices ſuſtinent; et ſi qui horum factorum inter

ſe

ſe fuerint aequales, ii coniunctim repraeſententur. Quo facto pro ſingulis factoribus qnaerantur conuenientes integralis partes, atque omnes iſtae partes ex cunctis factoribus oriundae, ſi in vnam ſummam colligantur, dabunt valorem ipſius y quaeſitum, qui erit integrale completum aequationis propoſitae. Sequenti autem modo ex factoribus formulae P integralis partes reperientur.

I. *Si formulae* P *factor ſit* z − k

Ponatur $\mathfrak{K} = B + 2Ck + 3Dk^2 + 4Ek^3 + 5Fk^4 +$ etc. eritque integralis pars huic factori $z-k$ reſpondens:

$$\frac{e^{kx}}{\mathfrak{K}} \int e^{-kx} X\,dx.$$

II. *Si formulae* P *factor ſit* $(z-k)^2$

Ponatur $\mathfrak{K} = C + 3Dk + 6Ek^2 + 10Fk^3 + 15Gk^4 +$ etc. eritque integralis pars factori $(z-k)^2$ reſpondens:

$$\frac{e^{kx}}{\mathfrak{K}} \int dx \int e^{-kx} X\,dx.$$

III. *Si formulae* P *factor ſit* $(z-k)^3$

Panatur $\mathfrak{K} = D + 4Ek + 10Fk^2 + 20Gk^3 + 35Hk^4 +$ etc. eritque integralis pars factori $(z-k)^3$ reſpondens:

$$\frac{e^{kx}}{\mathfrak{K}} \int dx \int dx \int e^{-kx} X\,dx.$$

IV. *Si formulae* P *factor ſit* $(z-k)^4$

Ponatur $\mathfrak{K} = E + 5Fk + 15Gk^2 + 35Hk^3 + 70Ik^4 +$ etc. eritque integralis pars factori $(z-k)^4$ reſpondens:

$$\frac{e^{kx}}{\mathfrak{K}} \int dx \int dx \int dx \int e^{-kx} X\,dx.$$

V. *Si*

V. *Si formulae* P *factor sit* $zz - 2kz \operatorname{cof.} \Phi + kk$

Ponatur:

$$\mathfrak{M} = B + 2Ck \operatorname{cof.} \Phi + 3Dk^2 \operatorname{cof.} 2\Phi + 4Ek^3 \operatorname{cof.} 3\Phi + \text{etc.}$$
$$\mathfrak{N} = \quad 2Ck \operatorname{fin.} \Phi + 3Dk^2 \operatorname{fin.} 2\Phi + 4Ek^3 \operatorname{fin.} 3\Phi + \text{etc.}$$

erit pars integralis factori $zz - 2kz \operatorname{cof.} \Phi + kk$ refpondens:

$$\frac{2e^{kx \operatorname{cof.} \Phi}}{\mathfrak{M}^2 + \mathfrak{N}^2} \left\{ \begin{array}{l} (\mathfrak{M} \operatorname{cof.} kx \operatorname{fin.} \Phi + \mathfrak{N} \operatorname{fin} kx \operatorname{fin.} \Phi) \int e^{-kx \operatorname{cof.} \Phi} X dx \operatorname{cof.} kx \operatorname{fin.} \Phi + \\ (\mathfrak{M} \operatorname{fin} kx \operatorname{fin.} \Phi - \mathfrak{N} \operatorname{cof.} kx \operatorname{fin.} \Phi) \int e^{-kx \operatorname{cof.} \Phi} X dx \operatorname{fin} kx \operatorname{fin.} \Phi \end{array} \right\}$$

VI. *Si formulae* P *factor sit* $(zz - 2kz \operatorname{cof.} \Phi + kk)^2$

Ponatur:

$$\mathfrak{M} = C + 3Dk \operatorname{cof.} \Phi + 6Ek^2 \operatorname{cof.} 2\Phi + 10Fk^3 \operatorname{cof.} 3\Phi + \text{etc.}$$
$$\mathfrak{N} = \quad 3Dk \operatorname{fin.} \Phi + 6Ek^2 \operatorname{fin.} 2\Phi + 10Fk^3 \operatorname{fin.} 3\Phi + \text{etc.}$$

erit pars integralis factori $(zz - 2kz \operatorname{cof.} \Phi + kk)^2$ refpondens:

$$\frac{2e^{kx \operatorname{cof.} \Phi}}{\mathfrak{M}^2 + \mathfrak{N}^2} \left\{ \begin{array}{l} (\mathfrak{M} \operatorname{cof.} kx \operatorname{fin.} \Phi + \mathfrak{N} \operatorname{fin.} kx \operatorname{fin.} \Phi) \int dx \int e^{-kx \operatorname{cof.} \Phi} X dx \operatorname{cof.} kx \operatorname{fin.} \Phi + \\ (\mathfrak{M} \operatorname{fin.} kx \operatorname{fin.} \Phi - \mathfrak{N} \operatorname{cof.} kx \operatorname{fin.} \Phi) \int dx \int e^{-kx \operatorname{cof.} \Phi} X dx \operatorname{fin.} kx \operatorname{fin.} \Phi \end{array} \right\}$$

VII. *Si formulae* P *factor sit* $(zz - 2kz \operatorname{cof.} \Phi + kk)^3$

Ponatur:

$$\mathfrak{M} = D + 4Ek \operatorname{cof.} \Phi + 10Fk^2 \operatorname{cof.} 2\Phi + 20Gk^3 \operatorname{cof.} 3\Phi + \text{etc.}$$
$$\mathfrak{N} = \quad 4Ek \operatorname{fin.} \Phi + 10Fk^2 \operatorname{fin.} 2\Phi + 20Gk^3 \operatorname{fin.} 3\Phi + \text{etc.}$$

erit pars integralis factori $(zz - 2kz \operatorname{cof.} \Phi + kk)^3$ refpondens:

$$\frac{2e^{kx \operatorname{cof.} \Phi}}{\mathfrak{M}^2 + \mathfrak{N}^2} \left\{ \begin{array}{l} (\mathfrak{M} \operatorname{cof.} kx \operatorname{fin.} \Phi + \mathfrak{N} \operatorname{fin.} kx \operatorname{fin.} \Phi) \int dx \int dx \int e^{-kx \operatorname{cof.} \Phi} X dx \operatorname{cof.} kx \operatorname{fin.} \Phi + \\ (\mathfrak{M} \operatorname{fin.} kx \operatorname{fin.} \Phi - \mathfrak{N} \operatorname{cof.} kx \operatorname{fin.} \Phi) \int dx \int dx \int e^{-kx \operatorname{cof.} \Phi} X dx \operatorname{fin.} kx \operatorname{fin.} \Phi \end{array} \right\}$$

etc.

Omnes igitur iftae partes fingulis factoribus formulae P refpondentes in vnam fummam collectae dabunt valorem ipfius y quaefitum. Q. E. I.

§. 23 Explicata hac regula, cuius ope omnes aequationes differentiales in forma generali contentae integrari

grari poſſunt, aliquot exempla adiungam, ex quibus regulae huius vſus facilius perſpicietur.

Exempl. I. Propoſita ſit haec aequatio differentialis ſecundi gradus.

$$X = y - \frac{ddy}{dx^2}$$

Hinc igitur formula Algebraica P erit $= 1 - zz$ cuius factores ſunt $z + 1$ et $z - 1$. et ex formula prima erit $\mathfrak{R} = \frac{dP}{dz} = -2z$. pro factore ergo $z + 1$ ob $k = -1$ erit $\mathfrak{R} = 2$ et pars integralis $= \frac{e^{-x}}{2} \int e^x X dx$. Pro altero factore eſt $k = 1$ et $\mathfrak{R} = -2$, cui reſpondet pars integralis $-\frac{e^x}{2} \int e^{-x} X dx$, quibus partibus collectis erit integrale quaeſitum.

$$y = \tfrac{1}{2} e^{-x} \int e^x X dx - \tfrac{1}{2} e^x \int e^{-x} X dx.$$

Exempl. 2. Propoſita ſit haec aequatio:

$$X = y - \frac{3a\,dy}{dx} + \frac{3aa\,ddy}{dx^2} - \frac{a^3 d^3 y}{dx^3}$$

Erit ergo $P = 1 - 3az + 3aazz - a^3 z^3 = (1 - az)^3$. Sumenda ergo eſt formula tertia, eritque $k = \frac{1}{a}$, et $\mathfrak{R} = \frac{d^3 P}{6dz^3} = -a^3$, vnde prodit integrale quaeſitum

$$y = -\tfrac{1}{a^3} e^{x:a} \int dx \int dx \int e^{-x:a} X dx \text{ ſeu}$$

$$y = -\tfrac{1}{a^3} e^{x:a} (x \int dx \int e^{-x:a} X dx - \int x\, dx \int e^{-x:a} X dx) \text{ ſeu}$$

$$y = -\tfrac{1}{a^3} e^{x:a} (\tfrac{1}{2} xx \int e^{-x:a} X dx - x \int e^{-x:a} X x dx + \tfrac{1}{2} \int e^{-x:a} X xx dx)$$

Exempl. 3. Propoſita ſit haec aequatio:

$$X = y + \frac{aa\,ddy}{dx^2}$$

Erit ergo $P = 1 + aazz$, quae ad formulam V pertinet. Erit nempe coſ. $\Phi = 0$ ſin. $\Phi = 1$, et $k = \frac{1}{a}$. Porro ob

A =

$A=1$, $B=0$ et $C=aa$, erit $\mathfrak{M}=0$, et $\mathfrak{N}=2a$, vnde erit integrale:

$$y=\tfrac{1}{a}\text{fin.}\tfrac{x}{a}\int X dx \text{ cof.}\tfrac{x}{a}-\tfrac{1}{a}\text{cof.}\tfrac{x}{a}\int X dx \text{ fin.}\tfrac{x}{a}.$$

Exempl. 4. Propofita fit haec aequatio:

$$X=y+\frac{a^3 d^3 y}{dx^3}$$

Erit ergo $P=1+a^3z^3$, cuius duo funt factores $1+az$ et $1-az+aazz$, Prior ad formam $z-k$ reductus, dat $k=-\tfrac{1}{a}$; et ob $A=1$, $B=0$, $C=0$, et $D=a^3$, erit ex formula prima $\mathfrak{K}=3a$, et pars integralis:

$$\tfrac{1}{3a}e^{-x:a}\int e^{x:a}X dx.$$

Alter factor $1-az+aazz$ feu $zz-\tfrac{z}{a}+\tfrac{1}{aa}$ cum formula V comparatus, dat $k=\tfrac{1}{a}$; cof $\Phi=\tfrac{1}{2}$ et fin $\Phi=\tfrac{\sqrt{3}}{2}$ atque $\Phi=60^\circ$. Deinde eft $\mathfrak{M}=3a\text{ cof. }120^\circ=-\tfrac{3}{2}a$, et $\mathfrak{N}=3a\text{ fin. }120^\circ=\tfrac{3a\sqrt{3}}{2}$, vnde $\mathfrak{M}^2+\mathfrak{N}^2=9aa$, atque $\frac{2\mathfrak{M}}{\mathfrak{M}^2+\mathfrak{N}^2}=-\frac{1}{3a}$ et $\frac{2\mathfrak{N}}{\mathfrak{M}^2+\mathfrak{N}^2}=\frac{\sqrt{3}}{3a}$. Pars integralis ergo hinc oriunda eft:

$$\tfrac{1}{3a}e^{x:2a}\left(-\text{cof}\tfrac{x\sqrt{3}}{2a}+\sqrt{3}\cdot\text{fin.}\tfrac{x\sqrt{3}}{2a}\right)\int e^{-x:2a}X dx \text{ cof.}\tfrac{x\sqrt{3}}{2a}$$

$$+\tfrac{1}{3a}e^{x:2a}\left(-\text{fin.}\tfrac{x\sqrt{3}}{2a}-\sqrt{3}.\text{cof.}\tfrac{x\sqrt{3}}{2a}\right)\int e^{-x:2a}X dx \text{ fin.}\tfrac{x\sqrt{3}}{2a}$$

$$\text{feu }\tfrac{-2}{3a}e^{x:2a}\text{cof.}\left(\tfrac{x\sqrt{3}}{2a}+60^\circ\right)\int e^{-x:2a}X dx \text{ cof.}\tfrac{x\sqrt{3}}{2a}$$

$$-\tfrac{2}{3a}e^{x:2a}\text{fin.}\left(\tfrac{x\sqrt{3}}{2a}+60^\circ\right)\int e^{-x:2a}X dx \text{ fin.}\tfrac{x\sqrt{3}}{2a}.$$

Hinc igitur integrale quaefitum erit:

$$y=\tfrac{1}{3a}e^{-x:a}\int e^{x:a}X dx-\tfrac{2}{3a}e^{x:2a}\text{cof.}\left(\tfrac{x\sqrt{3}}{2a}+60^\circ\right)\int e^{-x:2a}X dx \text{ cof.}\tfrac{x\sqrt{3}}{2a}$$
$$-\tfrac{2}{3a}e^{x:2a}\text{fin}\left(\tfrac{x\sqrt{3}}{2a}+60^\circ\right)\int e^{-x:2a}X dx \text{ fin.}\tfrac{x\sqrt{3}}{3a}$$

Haec ergo exempla fufficiunt ad regulam pro quouis cafu oblato accommodandam.