1751

Nova methodus inveniendi traiextorias reciprocas algebraicas

Leonhard Euler

NOVA METHODUS
Inveniendi Trajectorias reciprocas
Algebraicas.

1.

Tabula. III. Viginti abhinc annis & quod excurrit, hoc problema de trajectoriis reciprocis primum a Nicolao Bernoullio Johannis Filio non solum est propositum; sed etiam tam variæ & elegantes solutiones jam eo tempore sunt exhibitæ, ut hoc problema jam penitus exhaustum videri possit. Cum enim præcipua difficultas in inveniendis curvis algebraicis huic quæstioni satisfacientibus versaretur, tradideram equidem in secundo Comment. Petropol. Tomo methodum ex quolibet linearum curvarum ordine unam ad minimum inveniendi, quæ præscripta proprietate sit affecta. Tanto præterea studio hoc problema illo tempore a pluribus Geometris fuit pertractatum, ut etiamsi ad solam Geometriam pertineret, tamen inde universa Analysis tam eximia acceperit augmenta, ut pluribus aliis quæstionibus majoris momenti enodandis apta sit reddita, quæ sine his subsidiis intactæ essent relictæ.

2. Hanc igitur quæstionem maxime famosam denuo aggredior, non quo aliorum solutiones minus idoneas vel insufficientes censeam: sed quoniam tum temporis curvæ algebraicæ, quibus præcipua problematis vis continetur, non levi labore ac per operosas integrationes sunt erutæ, neque omnes in formulis generalibus comprehendi potuerunt; explicabo hic methodum singularem, cujus ope non

solum

solum curvæ algebraicæ, quibus problema solvitur, facili negotio, & quidem quod maxime paradoxon videatur, sine ulla integratione inveniri, sed etiam omnes simul finitis formulis contentæ repræsentari queant. Quanquam autem hac methodo jam sæpius in aliis quæstionibus solvendis sum usus, tamen eam nusquam adhuc exposui; ejusque applicatio ad præsens negotium peculiare requirit artificium, quod in aliis casibus haud parum utilitatis afferre poterit.

3. Problema autem hoc sequenti modo proponi est solitum:

„*Circa datum axem* ACB *describere ejusmodi lineam* „*curvam* ECF, *quæ circa axem in situ inverso* eCf *constituta, ac* „*secundum directionem axis motu sibi parallelo promota, in quovis* „*situ* c′e′f′ *priorem curvam* ECF *sub dato angulo in* M *interfecet*. Fig. 1.

Solutio vero sequenti modo a Celeb. Joh. Bernoullio ad Analysin est perducta. Cum angulus EMe′ ubique debeat esse datæ magnitudinis, erit is æqualis angulo ECe, ideoque duplus anguli ECA. Per M ducatur recta MP axi AB parallela; eritque EMP + e′MP = ECe; at ob motum parallelum est angulus e′MP = enP; & ob situm inversum si ducatur QN axi AB parallela, ab eoque æquidistans ac recta PM, erit ang: ENQ = enP. Quare requiritur, ut ductis binis quibusque rectis MP & NQ axi parallelis ab eoque æquidistantibus, summa angulorum EMP + ENQ sit ubique eadem atque æqualis duplo anguli ECA.

4. Cum igitur natura quæstionis ad unam lineam curvam sit revocata, ducatur ad axem AB recta GH, quæ cum eo faciat angulum CAH æqualem duplo angulo ECA Fig. 2.

seu

seu ipsi angulo intersectionis proposito æqualem. In hacque recta capiantur utrinque abscissæ AP, AQ, quæ ob æquales applicatarum PM & QN ab axe AB distantias erunt æquales; eritque EMP + ENQ = CAH. Ductis autem utrinque applicatis infinite propinquis pm & qn, rectæque GH parallelis $M\mu$ & $n\nu$; ob angulum $M\mu m = N\nu n =$ GAC erit: $Mm\mu + mM\mu = nN\nu + Nn\nu =$ CAH = EMP + ENQ. At $Mm\mu =$ EMP & $nN\nu =$ ENQ, unde sequitur fore $mM\mu = nN\nu$ & $Nn\nu = Mm\mu$. Erunt ergo triangula $Mm\mu$ & $Nn\nu$ æquiangula ac propterea similia; ex quo habebitur hæc proportio $m\mu : M\mu = n\nu : N\nu$, ideoque hæc æqualitas $m\mu . N\nu = M\mu . n\nu$, qua natura problematis continetur.

5. Vocemus abscissam $AP = x$, eique respondentem applicatam $PM = y$; erit abscissa ex altera parte sumta $AQ = -x$, cui respondens applicata ponatur $QN = z$; quæ talis erit functio ipsius $-x$, qualis y est ipsius $+x$; seu ex valore ipsius y prodibit valor ipsius z, si loco x ubique scribatur $-x$. His positis erit $Pp = M\mu = dx$; $\mu m = dy$; $Qq = n\nu = -dx$, & $N\nu = -dz$; atque æquatio modo inventa $m\mu . N\nu = M\mu . n\nu$ dabit hanc formulam

$$-dydz = -dx^2 \text{ seu } \frac{dy}{dx} \cdot \frac{dz}{dx} = 1.$$

Ponatur $\frac{dy}{dx} = M$, & $\frac{dz}{dx} = N$, eritque N talis functio ipsius $-x$, qualis M est ipsius $+x$, seu ex functione M proveniet functio N, si loco x ponatur $-x$. Quocirca ad problema resolvendum ejusmodi functiones pro M investigari oportet, ut fiat $MN = 1$: hocque facto erit $dy = Mdx$, quæ æquatio naturam curvæ exprimet.

6. Quæ-

6. Quaestio itaque huc est perducta, ut pro M ejusmodi investigetur functio ipsius x, quae, si loco x ponatur $-x$, abeat in N, ita ut sit $MN = 1$. Manifestum autem est huic conditioni satisfacere hujusmodi valores $M = a^x$; $M = a^{x^3}$, similesque alios; sed cum curvas algebraicas requiramus, hujusmodi valores exponentiales excludi oportet. Sit igitur P functio par ipsius x, quae scilicet eundem valorem retineat, posito $-x$ loco $+x$; deinde sit Q functio impar ipsius x, quae abeat in $-Q$, si loco x ponatur $-x$: hincque evidens est conditionem problematis impleri, si ponatur $M = \frac{P+Q}{P-Q}$, fiet enim $N = \frac{P-Q}{P+Q}$ ideoque $MN = 1$. Ponatur $\frac{Q}{P} = u$, ut sit u functio quaecunque ipsius x impar; ex quo erit $M = \frac{1+u}{1-u}$ & $dy = \frac{1+u}{1-u}\,dx$, quae aequatio solutionem problematis in latissimo sensu complectitur, dummodo sub littera u omnes functiones ipsius x comprehendantur.

7. Quamquam haec aequatio jam est generalis omnesque solutiones includit, tamen ex ea aliae formari possunt, quae latius patere videntur; & quae in inventione curvarum algebraicarum usum commodiorem praestant. Hujusmodi est ista formula $M = \left(\frac{1+u}{1-u}\right)^n$; quicunque enim numerus pro exponente n assumatur, erit semper $N = \left(\frac{1-u}{1+u}\right)^n$,

 ideo-

ideoque MN = 1. Quare si u sumatur pro functione quacunque impari ipsius x, natura curvae trajectoriae reciprocae cujuscunque hac exprimetur aequatione: $dy = \left(\frac{1+u}{1-u}\right)^n dx$. Manifestum autem est, si pro n sumantur numeri fracti, facile ejusmodi curvas obtineri, quae ex priori forma difficulter erui queant, etiamsi revera in ea contineantur.

8. Tametsi functiones irrationales ob ambiguitatem neque functionibus paribus neque imparibus proprie annumerari queant: tamen in hoc negotio hujusmodi expressiones $\sqrt{(1+uu)}$ pro functionibus paribus haberi possunt, dummodo uu sit functio par neque ex $(1+uu)$ radix quadrata actu extrahi queat. At si u sit functio ipsius x impar, erit uu ac propterea $\sqrt{(1+uu)}$ ejusdem x functio par. Quo notato facile patet, hunc valorem $M = \sqrt{(1+uu)} + u$ quaesito satisfacere debere; fiet enim inde $N = \sqrt{(1+uu)} - u$, ideoque MN = 1. Idem evenit, si statuatur $M = (\sqrt{(1+uu)}+u)^n$, quia fit $N = (\sqrt{(1+uu)}-u)^n$ & MN = 1. Hinc ergo duae novae aequationes generales pro trajectoriis reciprocis oriuntur:

$$dy = (\sqrt{(1+uu)}+u)\,dx \quad \&$$

$$dy = (\sqrt{(1+uu)}+u)^n\,dx$$

9. Potest etiam nova quaedam variabilis t introduci, a qua x ita pendeat, ut posito $-t$ loco t, abscissa x abeat in $-x$; seu sit x functio impar ipsius t. Ponatur $dx = v\,dt$, eritque v functio par ipsius t; statuatur autem ut ante u

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functio impar ipsius t. His positis, si fiat $M = \frac{dy}{dx} = \frac{1+u}{1-u}$, evadet denuo $N = \frac{1-u}{1+u}$, ideoque $MN = 1$. Hancobrem problemati satisfiet, si sumatur:

$$dx = vdt \;\; \& \;\; dy = \frac{1+u}{1-u}\, vdt$$

Simili modo problema solvetur his formulis generalibus

$$dx = vdt \;\; \& \;\; dy = \left(\frac{1+u}{1-u}\right)^{n} vdt$$

itemque his ex irrationalibus ortis:

$$dx = vdt \;\; \& \;\; dy = (\sqrt{(1+uu)}+u)\, vdt \;\; \text{atque}$$

$$dx = vdt \;\; \& \;\; dy = (\sqrt{(1+uu)}+u)^{n}\, vdt.$$

Quæcunque autem formulæ ex his assumantur, necesse est ut inde solutio problematis generalis obtineatur.

10. Si jam curvæ algebraicæ desiderentur, totum negotium huc redit, ut qualitas functionis u, & in his posterioribus binarum functionum u & v determinetur, qua hæ formulæ integrabiles reddentur. Plures autem imo infinitæ hujusmodi functiones, cum a Celeb. Bernoullio, tum a me sunt notatæ, quæ curvas algebraicas præbeant; sed hic modus maxime est particularis, neque omnes curvas algebraicas satisfacientes in se complectitur. Deinde assumtis illiusmodi functionibus idoneis, integratio harum formularum demum actu institui debet; sicque pro quovis curvarum genere peculiari operatione est opus, quæ sæpe non sine molesto calculo absolvitur. Cui incommodo ita

 occur-

occurram, ut non ſolum formulas generales pro omnibus curvis algebraicis ſim exhibiturus; ſed etiam quæ ſine prævia integratione ſolutionem ſuppeditent. Quin etiam has ipſas formulas algebraicas ex ſuperioribus differentialibus ſine actuali integratione ſum derivaturus, id quod plerisque maxime paradoxon videbitur. Methodum autem meam ad ſingulas formulas differentiales ante inventas ſeorſim accommodabo.

I. Modus inveniendi trajectorias reciprocas algebraicas ex formula

$$dy = \frac{1+u}{1-u}\,dx$$

11. Quæritur ergo hic, non ſolum qualis functio ipſius x debeat eſſe u, ut formula $\frac{1+u}{1-u}dx$ integrationem admittat, ſed etiam quænam ipſa ſit futura integralis forma. Cum autem u ſit functio impar ipſius x, erit viciſſim x functio impar ipſius u: hincque inveſtigabo, qualis functio quantitas x eſſe debeat ipſius u, ut quoque y per functionem algebraicam ipſius u exprimi queat. Quam inveſtigationem ita inſtituo: quia eſt $y=\int\frac{1+u}{1-u}dx$, erit per notam integralium reductionem:

$$y = \frac{1+u}{1-u}x - 2\int\frac{x\,du}{(1-u)^2}.$$

Supereſt ergo, ut formula $\int\frac{x\,du}{(1-u)^2}$ reddatur integrabi-

lis,

lis, in quo nulla foret difficultas, nisi x deberet esse functio impar ipsius u.

12. Denotet p functionem quamcunque parem, & q functionem imparem ipsius u, statuaturque:

$$2\int\frac{x\,du}{(1-u)^2}=\frac{(p+q)(1+u)}{1-u}$$

duas scilicet novas quantitates p & q introduco, ut non solum integrabilitas procuretur, sed etiam functioni x praescripta proprietas inducatur. Sumtis ergo differentialibus, erit:

$$\frac{2\,x\,du}{(1-u)^2}=\frac{(dp+dq)(1+u)}{1-u}+\frac{2(p+q)\,du}{(1-u)^2}$$

multiplicationeque per $(1-u)^2$ instituta orietur:

$$2\,x\,du=(1-uu)\,dp+(1-uu)\,dq+2\,p\,du+2\,q\,du$$

cujus aequationis alii termini pares ipsius u continebunt dimensiones, alii impares: quamobrem necesse est ut termini tam parium quam imparium dimensionum seorsim inter se aequentur.

13. Quia vero p est functio par ipsius u; reliquae vero quantitates q & x functiones impares, earumque differentialia eandem naturam sequuntur, erit $2\,x\,du$ functio par; $(1-uu)\,dp$ par; $(1-uu)\,dq$ impar; $2\,p\,du$ impar; & $2\,q\,du$ par. Aequatis ergo paribus & imparibus seorsim sequentes duae orientur aequationes:

$$2\,x\,du=(1-uu)\,dp+2\,q\,du:$$

$$\&\quad 2\,p\,du+(1-uu)\,dq=0.$$

Posterior aequatio statim dedit $p = -\frac{(1-uu)\,dq}{2\,du}$;
hinc enim ob q functionem imparem ipsius u fiet p functio par. Cognito jam valore functionis p, ex priore aequatione deducitur:

$$x = \frac{(1-uu)\,dp}{2\,du} + q: \text{ cui abscissae respondebit}$$

applicata $y = \frac{1+u}{1-u}x - \frac{(p+q)(1+u)}{1-u} = \frac{1+u}{1-u}(x-q-p)$

14. Si hic pro x valor ante inventus substituatur, invenietur:

$$y = \frac{(1+u)^2\,dp}{2\,du} - \frac{p(1+u)}{1-u} \text{ seu ob } p = -\frac{(1-uu)\,dq}{2\,du}$$

$$y = \frac{(1+u)^2\,dp + (1+u)^2\,dq}{2\,du} = \frac{(dp+dq)(1+u)^2}{2\,du}$$

Quocirca hinc nanciscimur

Primam regulam generalem pro inveniendis trajectoriis reciprocis algebraicis:

Sumatur q *functio quaecunque imparium dimensionum ipsius* u; *indeque quaeratur quantitas* $p = -\frac{(1-uu)\,dq}{2\,du}$, *qua inventa erit curvae quaesitae:*

abscissa $AP = x = q + \frac{(1-uu)\,dp}{2\,du}$

appli-

applicata PM =
qui valores semp
braica ipsius u
15. Si
que capiatur p
$x = q' + ($
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$x = X$
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$x = X$
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$\alpha X + \beta X' +$
que porro.

16. Pon
existente λ nu
fractio, cujus ta
eritque
$\frac{dq}{du} = \lambda u^{\lambda-1}$
porro

applicata $PM = y = \frac{(dp+dq)(1+u)^2}{2du} = \frac{1+u}{1-u}(x-p-q)$ *qui valores semper sunt algebraici, si quidem* q *fuerit functio algebraica ipsius* u

15. Si ipsius u alia sumatur functio impar q', ex eaque capiatur $p' = -\frac{(1-uu)dq'}{2du}$, habebitur simili modo:

$$x = q' + \frac{(1-uu)dp'}{2du}, \ \& \ y = \frac{(dp'+dq')(1+u)^2}{2du}$$

Atque cum hinc aeque fiat $\frac{dy}{dx} = \frac{1+u}{1-u}$: manifestum est si ex quapiam ipsius q hypothesi inventum fuerit:

$$x = X \ \& \ y = Y$$

ex alia autem hypothesi prodierit $x = X'$ & $y = Y'$, problemati quoque satisfieri his valoribus:

$$x = X + X' \ \& \ y = Y + Y'$$

atque generalius etiam, si ponatur:

$$x = \alpha X + \beta X' \ \& \ y = \alpha Y + \beta X'$$

sicque ex duabus curvis algebraicis jam inventis innumerabiles novae inveniri poterunt. Sin autem praeterea tertia fuerit inventa $x = X''$ & $y = Y''$, erit quoque $x = \alpha X + \beta X' + \gamma X''$ & $y = \alpha Y + \beta Y' + \gamma X''$ sicque porro.

16. Ponatur ut ad exempla descendamus $q = u^{\lambda}$ existente λ numero quocunque impari, sive integro sive fracto, cujus tam numerator quam denominator sit impar: eritque

$$\frac{dq}{du} = \lambda u^{\lambda-1} \ \& \ p = -\tfrac{1}{2}\lambda u^{\lambda-1} + \tfrac{1}{2}\lambda u^{\lambda+1}$$ hincque porro

dp

$\frac{dp}{du} = -\frac{1}{2}\lambda(\lambda-1)u^{\lambda-2} + \frac{1}{2}\lambda(\lambda+1)u^{\lambda}$. Unde habebitur:

$$x = u^{\lambda} - \frac{1}{2}\lambda(\lambda-1)u^{\lambda-2} + \frac{1}{4}\lambda(\lambda+1)u^{\lambda} + \frac{1}{4}\lambda(\lambda-1)u^{\lambda} - \frac{1}{4}\lambda(\lambda+1)u^{\lambda+2}$$

seu $x = -\frac{1}{4}\lambda(\lambda-1)u^{\lambda-2} + \frac{1}{2}(\lambda\lambda+2)u^{\lambda} - \frac{1}{4}\lambda(\lambda+1)u^{\lambda+2}$

& $y = \frac{1}{2}\lambda(u^{\lambda-1} - \frac{1}{2}(\lambda-1)u^{\lambda-2} + \frac{1}{2}(\lambda+1)u^{\lambda})(1+u)^{2}$ seu

$y = \frac{1}{4}\lambda u^{\lambda-2}(1+u)^{4} - \frac{1}{4}\lambda\lambda u^{\lambda-2}(1-u)(1+u)^{3}$ vel etiam

$$y = \frac{1}{4}\lambda u^{\lambda-2}(1-\lambda+u+\lambda u)(1+u)^{3}$$

Hinc sequitur fore:

si $\lambda = 1$: $x = \frac{1}{2}u - \frac{1}{2}u^{3}$; $y = \frac{1}{4}(1+u)^{3}$ pro parabola cubicali secunda

si $\lambda = 3$; $x = -\frac{3}{4}u(1-3uu)(3-2uu)$: &

$$y = -\frac{3}{2}u(1-2u)(1+u)^{3}$$

si $\lambda = 5$; $x = -\frac{5}{4}u^{3}(10-27u^{2}+15u^{4})$: &

$$y = -\frac{5}{2}u^{3}(2-3u)(1+u)^{3}$$

si $\lambda = 7$; $x = -\frac{7}{4}u^{5}(21-51u^{2}+28u^{4})$: &

$$y = -\frac{7}{2}u^{5}(3-4u)(1+u)^{3}$$

&c.

II. Se-

Inve

17. functio im... plicata y p... tur:

$y = \int$ ...

Sumtisque ... statuatur:

ex qua diffe...

$\frac{x\,du}{1-uu} =$ ...

$x\,du = (1$ ...

18. ... altera conti... dimensionu...

& o...

II. Secundus Modus Inveniendi Trajectorias reciprocas Algebraicas.

Ex formula $dy = \left(\frac{1+u}{1-u}\right)^n dx$

17. Quia ut ante ostensum est, abscissa x debet esse functio impar ipsius u, quaeratur qualis esse debeat, ut applicata y prodeat algebraice expressa. In hunc finem ponatur:

$$y = \int \left(\frac{1+u}{1-u}\right)^n dx = \left(\frac{1+u}{1-u}\right)^n x - 2n\int \frac{x\,du}{1-uu}\left(\frac{1+u}{1-u}\right)^n$$

Sumtisque p functione pari & q functione impari ipsius u statuatur:

$$\int \left(\frac{1+u}{1-u}\right)^n \frac{x\,du}{1-uu} = (p+q)\left(\frac{1+u}{1-u}\right)^n$$

ex qua differentiatione instituta habebitur:

$$\frac{x\,du}{1-uu} = dp + dq + \frac{2n(p+q)\,du}{1-uu} \quad \text{seu}$$

$$x\,du = (1-uu)\,dp + (1-uu)\,dq + 2np\,du + 2nq\,du$$

18. Discerpatur haec in duas aequationes, quarum altera contineat functiones parium, altera vero imparium dimensionum, sicque fiet:

$$x\,du = (1-uu)\,dp + 2nq\,du$$

$$\& \; 0 = (1-uu)\,dq + 2np\,du$$

 qua-

quarum posterior dat $p = -\frac{(1-uu)\,dq}{2n\,du}$, quo valore ipsius p invento erit ex priori:

$$x = 2nq + \frac{(1-uu)\,dp}{du}$$

Praeterea autem ex supra facta hypothesi oritur

$$y = \left(\frac{1+u}{1-u}\right)^n (r - 2np - 2nq) = \left(\frac{1+u}{1-u}\right)^n \frac{(1-uu)(dp+dq)}{du}$$

seu $y = \frac{(dp+dq)(1+u)^{n+1}}{du(1-u)^{n-1}}$. Sumta ergo pro q functione quacunque imparium dimensionum ipsius u, hae formulae praebebunt curvas algebraicas, quae erunt trajectoriae reciprocae.

19. Hinc ergo adepti sumus:

Secundam regulam generalem pro inveniendis trajectoriis reciprocis algebraicis.

Sumatur q functio quaecunque imparium dimensionum ipsius u, denotanteque n numerum quemcunque, quaeratur inde quantitas $p = -\frac{(1-uu)\,dq}{2n\,du}$; *hinc erit curvae quaesitae;*

Abscissa $AP = x = 2nq + \frac{(1-uu)\,dp}{du}$

Appli-

Applicata $PM = y = \dfrac{(dp+dq)(1+u)^{n+1}}{du(1-u)^{n-1}}$

Ubi iterum notandum est, si pro x & y jam aliquot inventi fuerint valores X & Y, X' & Y', X'' & Y'', &c. problemati quoque satisfieri, si capiatur

$$x = \alpha X + \beta X' + \gamma X'' \text{ \&c. } \quad \& \quad y = \alpha Y + \beta Y' + \gamma Y'' \text{ \&c.}$$

20. Sit λ numerus quicunque impar ac statuatur: $q = 2nu^{\lambda}$, erit $\dfrac{dq}{du} = 2n\lambda u^{\lambda-1}$ ideoque $p = -\lambda u^{\lambda-1} + \lambda u^{\lambda+1}$; unde fit porro $\dfrac{dp}{du} = -\lambda(\lambda-1)u^{\lambda-2} + \lambda(\lambda+1)u^{\lambda}$. Ex his valoribus colligitur:

$$x = 4nn\,u^{\lambda} - \lambda(\lambda-1)u^{\lambda-2} + 2\lambda\lambda u^{\lambda} - \lambda(\lambda+1)u^{\lambda+2} \quad \&$$

$$y = -(\lambda(\lambda-1)u^{\lambda-2} - 2n\lambda u^{\lambda-1} - \lambda(\lambda+1)u^{\lambda})\,\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

seu mutatis signis, sumtisque semissibus erit

$$x = u^{\lambda-2}\left(\frac{\lambda(\lambda-1)}{2} - (\lambda\lambda+2nn)u^2 + \frac{\lambda(\lambda+1)}{2}u^4\right);$$

$$y = \lambda u^{\lambda-2}\left(\frac{\lambda-1}{2} - nu - \frac{(\lambda+1)uu}{2}\right)\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

Hinc ergo erit:

si $\lambda = 1$: $x = u^3 - (1+2nn)u$; $y = -(n+u)\dfrac{(1+u)^{n+1}}{(1-u)^{n-1}}$

si $\lambda = 3$: $x = u(3-(9+2nn)u^2+6u^4)$; &

$$y = 3u(1-nu-2uu)\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

si $\lambda = 5$; $x = u^3(10-(15+2nn)u^2+15u^4)$; &

$$y = 5u^3(2-nu-3uu)\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

&c.

21. Sit porro m numerus quicunque, ac ponatur $q = 2nu^{\lambda}(1-uu)^{m}$, erit $\dfrac{dq}{du} = 2n\lambda u^{\lambda-1}(1-uu)^{m}$ $-4mnu^{\lambda+1}(1-uu)^{m-1}$ ac propterea $p = -\lambda u^{\lambda-1}(1-uu)^{m+1} + 2mu^{\lambda+1}(1-uu)^{m}$. Fiet itaque

$$\frac{dp}{du} = -\lambda(\lambda-1)u^{\lambda-2}(1-uu)^{m+1} + 2\lambda(m+1)u^{\lambda}(1-uu)^{m} + 2m(\lambda+1)u^{\lambda}(1-uu)^{m} - 4mmu^{\lambda+2}(1-uu)^{m-1}$$

ex quibus colligitur:

$x =$

$$x = 4mmu^{\lambda}(1-uu)^{m} - \lambda(\lambda-1)u^{\lambda-2}(1-uu)^{m+2}$$
$$+2(2m\lambda+m+\lambda)u^{\lambda}(1-uu)^{m+1}$$
$$-4mmu^{\lambda+2}(1-uu)^{m}$$

seu

$$x = u^{\lambda-2}(1-uu)^{m}\big(-\lambda(\lambda-1)+2(2mm+\lambda\lambda+2m\lambda+m)u^{2}$$
$$-(4mm+4m\lambda+\lambda\lambda+2m+\lambda)u^{4}\big)$$

sive hoc modo brevius:

$$x = -u^{\lambda-2}(1-uu)^{m}\big(\lambda(\lambda-1)-2(2mm+\lambda\lambda+2m\lambda+m)u^{2}$$
$$+(2m+\lambda)(2m+\lambda+1)u^{4}\big)$$

Deinde erit:

$$y = \frac{(1+u)^{n+1}}{(1-u)^{n-1}}\big(2n\lambda u(1-uu)-4mnu^{3}-\lambda(\lambda-1)(1-uu)^{2}$$
$$-4mmu^{4}+2(2m\lambda+m+\lambda)u^{2}(1-uu)\big)u^{\lambda-2}(1-uu)^{m-1}$$

seu $y = -\left(\frac{1+u}{1-u}\right)^{n} u^{\lambda-2}(1-uu)^{m}\big(\lambda(\lambda-1)-2(\lambda\lambda+$

$$2m\lambda+m)u^{2}+(2m+\lambda)(2m+\lambda+1)u^{4}$$
$$-2n\lambda u+2n(2m+\lambda)u^{3}\big)$$

mutatis ergo signis erit:

$$x = u^{\lambda-2}(1-uu)^m\big(\lambda(\lambda-1)-2(2nn+\lambda\lambda+2m\lambda+m)u^2$$
$$+(2m+\lambda)(2m+\lambda+1)u^4\big)$$

$$y = u^{\lambda-2}(1-uu)^m\left.\begin{array}{l}\big(\lambda(\lambda-1)-2(\lambda\lambda+2m\lambda+m)u^2\\ +(2m+\lambda)(2m+\lambda+1)u^4\\ -2n\lambda u+2n(2m+\lambda)u^3\end{array}\right\}\left(\frac{1+u}{1-u}\right)^n$$

22. Hinc duæ prodeunt ſolutiones præ ceteris ſimpliciores, quarum altera prodit, ſi $\lambda = 1$ & $m = -1$: tum enim fiet:

$$x = -\frac{4(nn-1)u}{1-uu} \;\&\; y = -\frac{2(n-2u+nuu)}{1-uu}\left(\frac{1+u}{1-u}\right)^n$$

ſeu ſi utrinque per conſtantem $\frac{-a}{2(nn-1)}$ multiplicetur erit

$$x = \frac{2au}{1-uu} \;\&\; y = \frac{a(n-2u+nuu)}{(nn-1)(2-uu)}\left(\frac{1+u}{1-u}\right)^n$$

unde non difficulter eliminatur variabilis u; eſt enim ex priori $u = \frac{-a+\sqrt{(aa+xx)}}{x}$: quo valore in altera ſubſtituto, irrationalibusque ex denominatore ſublatis pervenietur tandem ad hanc æquationem:

$$(nn-1)a^n y = (n\sqrt{(aa+xx)}-x)(\sqrt{(aa+xx)}+x)^n$$

Ad

Ad quam redu

unde reperiet

$(nn-1)a^n y =$

quæ abrumpitur

ſitque exempli

$n = 1$ erit

$n = 2;\; 3y =$

Ad quam reducendam notetur esse: $(\sqrt{(aa+xx)}+x)^n =$

$$a^n + \frac{nn}{1.2}a^{n-4}xx + \frac{nn(nn-4)}{1.2.3.4}a^{n-4}x^4 + \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}a^{n-6}x^6 + \&c.$$

$$+ \left(na^{n-2}x + \frac{n(nn-4)}{1.2.3.}a^{n-4}x^3 + \frac{n(nn-4)(nn-16)}{1.2.3.4.5}a^{n-6}x^5 + \&c.\right)\sqrt{(aa+xx)}$$

unde reperietur fore:

$$(nn-1)a^n y = \begin{cases} \left(na^n + \frac{n(nn-2)}{1.2}a^{n-2}x^2 + \frac{n(nn-4)(nn-4)}{1.2.3.4}a^{n-4}x^4 + \frac{n(nn-6)(nn-4)(nn-16)}{1.2.3.4.5.6}a^{n-6}x^6 + \&c.\right)\sqrt{(aa+xx)} \\ +(nn-1)a^n x + \frac{nn(nn-1)}{1.2.3}a^{n-2}x^3 + \frac{nn(nn-1)(nn-4)}{1.2.3.4.5}a^{n-4}x^5 + \frac{nn(nn-1)(nn-4)(nn-16)}{1.2.3.4.5.6.7}a^{n-6}x^7 \ \&c. \end{cases}$$

quae abrumpitur, quoties est n numerus par. Ponatur $a = 1$ sitque exempli gr.

$n = 1$ erit $y = x$

$n = 2$; $3y = (1+2xx)\sqrt{(1+xx)} + 3x + 2x^3$

$n =$

$$n = 4;\quad 15y = (4 + 18xx + 24x^4)\sqrt{(1+xx)} + 15x + 40x^3 + 24x^5$$

$$n = 6;\quad 35y = (6 + 102xx + 256x^4 + 160x^6)\sqrt{(1+xx)} + 35x + 210x^3 + 336x^5 + 160x^7$$

&c.

23. Sin autem n sit numerus impar, erit $(\sqrt{(aa+xx)}$

$$+x)^n = na^{n-1}x + \frac{n(nn-1)}{1.2.3}a^{n-3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}a^{n-5}x^5 + \&c. +$$

$$\left(a^{n-1} + \frac{(nn-1)}{1.2.}a^{n-3}x^2 + \frac{(nn-1)(nn-9)}{1.2.3.4}a^{n-5}x^4 + \frac{(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6}a^{n-7}x^6 + \&c.\right)\sqrt{(aa+xx)}$$

unde sequitur fore:

$$(nn-1)a^n y =$$

$$na^{n+1} + \frac{n(nn-1)}{1.2}a^{n-1}x^2 + \frac{n(nn-1)(nn-1)}{1.2.3.4}a^{n-3}x^4 + \frac{n(nn-1)(nn-1)(nn-9)}{1.2.3.4.5.6}a^{n-5}x^6 + \&c.$$

$$+ \left((nn-1)a^{n-1}x + \frac{(nn-3)(nn-1)}{1.2.3}a^{n-3}x^3 + \frac{(nn-5)(nn-1)(nn-9)}{1.2.3.4.5}a^{n-5}x^5 + \&c.\right)\sqrt{(aa+xx)}$$

sit

Sit $a = 1$, ac ponatur successive $n = 1, 3, 5,$ &c.

$n = 1;\quad 0\,y = 1$

$n = 3;\quad 8y = 3 + 12xx + 8x^4 + (8x + 8x^3)\sqrt{(1+xx)}$

$n = 5;\quad 24y = 5 + 60x^2 + 120x^4 + 64x^6 + (24x + 88x^3 + 64x^5)\sqrt{(1+xx)}$

Generaliter autem posito $a = 1$ erit: $(nn-1)\,y =$

$$(n-1)\,2^{n-1}x^{n+1} + \frac{(n+1)n}{1}\,2^{n-3}x^{n-1} + \frac{(n+1)n(n-1)}{1.2}\,2^{n-5}x^{n-3} + \frac{(n+1)n(n-1)(n-4)}{1.\,2.\,3}\,2^{n-7}x^{n-5} +$$

$$\frac{n(nn-1)(n-5)(n-6)}{1.\,2.\,3.\,4}\,2^{n-9}x^{n-7} + \frac{n(nn-1)(n-6)(n-7)(n-8)}{1.\,2.\,3.\,4.\,5}\,2^{n-11}x^{n-9} + \&c. +$$

$$\Big((n-1)\,2^{n-1}x^{n} + \frac{nn-n+2}{1}\,2^{n-3}x^{n-2} + \frac{(nn-n+4)(n-3)}{1.\,2}\,2^{n-5}x^{n-4} + \frac{(nn-n+6)(n-4)(n-5)}{1.\,2.\,3}\,2^{n-7}x^{n-5}$$

$$+ \frac{(nn-n+8)(n-5)(n-6)(n-7)}{1.\,2.\,3.\,4}\,2^{n-9}x^{n-8} + \&c.\Big)\sqrt{(1+xx)}$$

24. Si n fuerit numerus *fractus* puta $= \frac{n}{m}$, habebitur pro trajectoria reciproca hæc æquatio:

$$(nn-mm)y = m(n\sqrt{(1+xx)}-mx)(\sqrt{(1+xx)}+x)^{n:m}$$

seu

$$\left(\frac{nn-mm}{m}\right)^m y^m = (n\sqrt{(1+xx)}-mx)^m(\sqrt{(1+xx)}+x)^n$$

quæ maxime fœcunda est in curvis simplicioribus suppeditandis: exceptis enim ordinibus 2 & 3, ex quovis ordine adminimum unam largitur trajectoriam reciprocam. Convenit autem hæc æquatio cum ea, quam in Tomo Comment. II. exhibueram. Scilicet si $m = 1$, curva erit ordinis $n+2$: æquatio enim ab irrationalitate liberata non plures dimensiones obtinet quam $n+2$: sin autem denominator m fuerit numerus quicunque, æquatio ad rationalitatem reducta assurget ad $n+2m$ dimensiones. Desideratur autem adhuc methodus has æquationes rationales sine operosa elevatione ad potestates ex traditis formulis statim eliciendi; cujusmodi methodus sine dubio in Analysi datur, cum in sublatione surditatis plurimi termini se mutuo destruant. Æquatio vero rationalis ita erit comparata, ut sit $y^{2m} + Py^m + Q = 0$, ubi sint P & Q functiones rationales ipsius x.

25. Per inductionem autem ex pluribus casibus collegi, si sit $m = 1$, fore

(nn-

$$(nn-1)y^2-2(nn-1)y\left\{\begin{array}{l}(n-1)2^{n-1}x^{n+1}+\frac{(n+1)n}{1}2^{n-3}x^{n-1}\\ +\frac{n(nn-1)}{1.2}2^{n-5}x^{n-3}+\\ \frac{n(nn-1)(n-4)}{1.\ 2.\ 3}2^{n-7}x^{n-5}+\\ \frac{n(nn-1)(n-5)(n-6)}{1.\ 2.\ 3.\ 4}2^{n-9}x^{n-7}+\\ \&c.\end{array}\right\}$$

$$=\pm nn\pm(nn-1)xx$$

ubi signorum ambiguorum valet superius, si n sit numerus par, inferius si sit impar. Hinc ergo sequentes oriuntur aequationes simpliciores pro trajectoriis reciprocis;

$$9y^2 = 6y(2x^3+3x)+3xx+4$$

$$64y^2 = 16y(8x^4+12xx+3)-8xx-9$$

$$225y^2 = 30y(24x^5+40x^3+15x)+15xx+16$$

$$576y^2 = 48y(64x^6+120x^4+60x^2+5)-24xx-25$$

$$1225y^2 = 70y(16x^7+336x^5+210x^3+35x)+35xx+36$$

$$2304y^2 = 96y(384x^8+896x^6+672x^4+168x^2+7)-48xx-49$$

$$3969y^2 = 126y(896x^9+2304x^7+2016x^5+672x^3+63x)+63xx+64$$

quae ergo aequationum series, quousque libuerit facile continuabitur.

26. Supra duarum solutionum simpliciorum, quae ex formulis (21) sequuntur, mentionem fecimus, alteramque hic fusius sumus prosecuti. Altera igitur solutio expendenda est, in qua $\lambda = 1$ & $m = -\frac{1}{2}$. Hinc ergo oritur:

$$x = \frac{1}{u\sqrt{(1-uu)}}\left(-(4nn-1)uu\right) = -\frac{(4nn-1)u}{\sqrt{(1-uu)}}$$

$$y = \frac{1}{u\sqrt{(1-uu)}}\left(uu-2nu\right)\left(\frac{1+u}{1-u}\right)^n = \frac{u-2n}{\sqrt{(1-uu)}}\left(\frac{1+u}{1-u}\right)^n$$

Multiplicentur hae formulae per $\frac{-a}{4nn-1}$, habebiturque

$$x = \frac{au}{\sqrt{(1-uu)}}; \ \& \ y = \frac{a(2n-u)}{(4nn-1)\sqrt{(1-uu)}}\left(\frac{1+u}{1-u}\right)^n$$

unde fit $\frac{(4nn-1)y}{x} = \frac{2n-u}{u}\left(\frac{1+u}{1-u}\right)^n$. Ex priore autem valore oritur $u = \frac{x}{\sqrt{(aa+xx)}}$, qui in hac substitutus dat:

$$(4nn-1)y = \left(2n\sqrt{(aa+xx)}-x\right)\left(\frac{\sqrt{(aa+xx)}+x}{\sqrt{(aa+xx)}-x}\right)^n$$

sive:

$$(4nn-1)a^{2n}y = \left(2n\sqrt{(aa+xx)}-x\right)\left(\sqrt{(aa+xx)}+x\right)^{2n},$$

quae aequatio non differt ab ea, quam § 22. nacti sumus, nisi quod hic sit $2n$, quod ibi erat n

III.

III. Tertius Modus Inveniendi Trajectorias reciprocas Algebraicas ex formula

$$dy = (\sqrt{(1+uu)}+u)^n dx$$

27. Ob $d.(\sqrt{(1+uu)}+u)^n = \frac{n\,du}{\sqrt{(1+uu)}}(\sqrt{(1+uu)}+u)^n$ erit:

$$y = x(\sqrt{(1+uu)}+u)^n - n\int\frac{x\,du}{\sqrt{(1+uu)}}(\sqrt{(1+uu)}+u)^n$$

ubi notandum est, ut ante, esse oportere x functionem imparem ipsius u. Sit p functio quaecunque par ipsius u & q functio impar, statuaturque

$$\int\frac{x\,du}{\sqrt{(1+uu)}}(\sqrt{(1+uu)}+u)^n = (p+q)(\sqrt{(1+uu)}+u)^n$$

erit sumtis differentialibus, divisioneque per $(\sqrt{(1+uu)}+u)^n$ peracta,

$$\frac{x\,du}{\sqrt{(1+uu)}} = dp + dq + \frac{np\,du + nq\,du}{\sqrt{(1+uu)}}$$

Jam hinc formentur duae aequationes, quarum altera functiones pares, altera impares complectatur:

$$\frac{x\,du}{\sqrt{(1+uu)}} = dp + \frac{nq\,du}{\sqrt{(1+uu)}} \quad \& \quad dq + \frac{np\,du}{\sqrt{(1+uu)}} = 0$$

quarum posterior sponte dat: $p = -\frac{dq\sqrt{(1+uu)}}{n\,du}$

Inventa autem functione p aequatio prior praebet:

$$x = nq + \frac{dp\sqrt{(1+uu)}}{du}$$ & ex substitutione habebimus

$$y = (x - np - nq)(\sqrt{(1+uu)}+u)^n = \frac{(dp+dq)\sqrt{(1+uu)}}{du}(\sqrt{(1+uu)}+u)^n$$

28. Hinc itaque adipiscimur

Tertiam regulam pro inveniendis trajectoriis reciprocis algebraicis.

Sumatur q *functio quaecunque imparium dimensionum ipsius* u, *indeque formetur functio* $p = -\frac{dq\sqrt{(1+uu)}}{ndu}$; *qua inventa erit curvae quaesitae*

Abscissa $AP = x = nq + \frac{dp\sqrt{(1+uu)}}{du}$

Applicata $PM = y = \frac{(dp+dq)(1+uu)}{du}(\sqrt{(1+uu)}+u)^n$

Vel cum sumto elemento du constante fit $dp = -\frac{ddq\sqrt{(1+uu)}}{ndu} - \frac{udq}{n\sqrt{(1+uu)}}$, erit curvae quaesitae:

Abscissa $AP = x = nq - \frac{udq}{ndu} - \frac{ddq(1+uu)}{ndu^2}$

Applicata $PM = y = \left(\frac{dq\sqrt{(1+uu)}}{du} - \frac{udq}{ndu} - \frac{ddq(1+uu)}{ndu^2}\right)(\sqrt{(1+uu)}+u)^n$

Ubi

Ubi iterum
factis, pro x
Y'; X'' &
$x =$
$y =$
quo pacto
rum augetu
29.
u, cujus ex
erit $\frac{dq}{du} =$
$x = nu^{\lambda}$
$y = (\lambda u^{\lambda-1}$
$(\sqrt{}$
seu $y = -$
$(\sqrt{}$
Unde expon
postquam hi

Ubi iterum notandum eſt, ſi ex aliquot hypotheſibus pro q factis, pro x & y jam inventi fuerint valores X & Y; X′ & Y′; X″ & Y″, &c. quæſtioni quoque ſatisfieri his valoribus

$$x = \alpha X + \beta X' + \gamma X'' + \delta X''' \ \&c.$$
$$y = \alpha Y + \beta Y' + \gamma Y'' + \delta Y''' \ \&c.$$

quo pacto numerus curvarum inventarum facile in infinitum augetur.

29. Ponamus pro q poteſtatem quamcunque ipſius u, cujus exponens λ ſit numerus impar: ſit ſcilicet $q = u^{\lambda}$ erit $\frac{dq}{du} = \lambda u^{\lambda-1}$ & $\frac{ddq}{du^2} = \lambda(\lambda-1)u^{\lambda-2}$; hinc ergo fiet

$$x = nu^{\lambda} - \frac{\lambda u^{\lambda}}{n} - \frac{\lambda(\lambda-1)}{n}(u^{\lambda-2} + u^{\lambda}) = \frac{(nn-\lambda\lambda)}{n}u^{\lambda} - \frac{\lambda(\lambda-1)}{n}u^{\lambda-2}$$

$$y = \left(\lambda u^{\lambda-1}\sqrt{(1+uu)} - \frac{\lambda\lambda}{n}u^{\lambda} - \frac{\lambda(\lambda-1)}{n}u^{\lambda-2}\right)\left(\sqrt{(1+uu)}+u\right)^{n}$$

$$\text{ſeu } y = -\frac{\lambda u^{\lambda-2}}{n}\left(\lambda - 1 - nu\sqrt{(1+uu)} + \lambda u^{2}\right)\left(\sqrt{(1+uu)}+u\right)^{n}$$

Unde exponenti λ variis valoribus tribuendis reperietur, poſtquam hi valores per n fuerint multiplicati.

$\lambda =$

$$\lambda = 1; \begin{cases} x = (nn-1)u \\ y = (n\sqrt{(1+uu)}-u)(\sqrt{(1+uu)}+u)^n \end{cases}$$

$$\lambda = 3; \begin{cases} x = (nn-9)u^3 - 6u \\ y = 3u(nn\sqrt{(1+uu)}-2-3uu)(\sqrt{(1+uu)}+u)^n \end{cases}$$

$$\lambda = 5; \begin{cases} x = (nn-25)u^5 - 20u^3 \\ y = 5u^3(nn\sqrt{(1+uu)}-4-5uu)(\sqrt{(1+uu)}+u)^n \end{cases}$$

$$\lambda = 7; \begin{cases} x = (nn-49)u^7 - 42u^5 \\ y = 7u^5(nn\sqrt{(1+uu)}-6-7uu)(\sqrt{(1+uu)}+u)^n \end{cases}$$

&c.

30. Dividantur formulae primi casus per $nn-1$, atque prodibunt isti valores:

$$x = u \;\&\; (nn-1)y = (n\sqrt{(1+uu)}-u)(\sqrt{(1+uu)}+u)^n$$

unde ob $u = x$ variabilis u facillime eliminatur; orieturque sequens aequatio inter x & y

$$(nn-1)y = (n\sqrt{(1+xx)}-x)(\sqrt{(1+xx)}+x)^n$$

quae est eadem aequatio, quam jam supra bis elicuimus, & quam maxime foecundam esse linearum algebraicarum simpliciorum, fusius ostendi. Haec enim aequatio ad rationalitatem reducta, si fuerit n numerus integer, ascendet ad $n+2$ dimensiones, sin autem n sit numerus fractus, puta $= \frac{n}{m}$ numerus dimensionum erit $= n + 2m$. Cum autem posi-

positio $n=1$ fit inanis, curva simplicissima hinc orienda erit linea quarti ordinis.

31. Jam conjunctim sumantur formulae casuum: $\lambda=1$ & $\lambda=3$, ex iisque reperietur:

$$x=(nn-1)\alpha u+(nn-9)\beta u^3-6\beta u$$

$$y=\left\{\begin{array}{l}\alpha(n\sqrt{(1+uu)}-u\\+3\beta u(nu\sqrt{(1+uu)}-2-3uu)\end{array}\right\}(\sqrt{(1+uu)}+u)^n$$

Quo hinc facilius variabilis u eliminari possit, ponatur $\beta=\frac{1}{nn-9}$ & $(nn-1)\alpha=6\beta=\frac{6}{nn-9}$; ut sit $\alpha=\frac{6}{(nn-1)(nn-9)}$, & $x=u^3$, ideoque $u=\sqrt[3]{x}$.

Quare ob $(\sqrt{(1+uu)}+u)^n=(\sqrt{(1+uu)}-u)^{-n}$ erit:

$$y(\sqrt{(1+uu)}-u)^n=\frac{6n\sqrt{(1+uu)}-6u}{(nn-1)(nn-9)}+$$

$$\frac{3nuu\sqrt{(1+uu)}-6u-9u^3}{nn-9}=\frac{3n(2+(nn-1)uu)\sqrt{(1+uu)}}{(nn-1)(nn-9)}$$

$$-\frac{3u(2nn+3(nn-1)uu)}{(nn-1)(nn-9)}.$$ Hancobrem habebimus:

$$\frac{(nn-1)(nn-9)}{3}y(\sqrt{(1+uu)}-u)^n=\begin{array}{l}+n(2+(nn-1)uu)\sqrt{(1+uu)}\\-u(2nn+3(nn-1)uu)\end{array}$$

existente $u=\sqrt[3]{x}$.

32. Conjungamus cum his formulis casus $\lambda = 5$ eritque

$$x = \alpha u^3 + (nn-25)\beta u^5 - 10\beta u^3$$

$$\frac{(nn-1)(nn-9)}{3}\, y\,(\sqrt{(1+uu)}-u)^n =$$

$$+n\alpha(2+(nn-1)uu)\sqrt{(1+uu)}$$
$$-\alpha u(2nn+3(nn-1)uu)$$
$$+\tfrac{5}{3}\beta(nn-1)(nn-9)u^3(nu\sqrt{(1+uu)}-4-5uu)$$

Sit $\beta = \frac{1}{nn-25}$ & $\alpha = 20\beta = \frac{20}{nn-25}$, erit $x = u^5$ &

$$\frac{(nn-1)(nn-9)(nn-25)}{5}\, y(\sqrt{(1+uu)}-u)^n =$$

$$+n(24+12(nn-1)uu+(nn-1)(nn-9)u^4)\sqrt{(1+uu)}$$
$$-u(24nn+4nn(nn-1)uu+5(nn-1)(nn-9)u^4)$$

Simili autem modo ulterius pergere licet, his formulis cum casu $\lambda = 7$ conjungendis: prodibit autem

$x = u^7$ atque

$$\frac{(nn-1)(nn-9)(nn-25)(nn-49)}{7}\, y(\sqrt{(1+uu)}-u)^n =$$

$$n(720+360(nn-1)u^2+30(nn-1)(nn-9)u^4+(nn-1)(nn-9)(nn-25)u^6)\sqrt{(1+uu)}$$

$$-u$$

$$-u\left(720nn+120nn(nn-1)u^{2}+6nn(nn-1)(nn-9)u^{4}+7(nn-1)(nn-9)(nn-25)u^{6}\right)$$

unde non difficulter lex sequentium formularum colligi potest.

33. Praeter hos vero casus, quibus variabilis u facile eliminatur, ex formulis ante §. 29 inventis, alii eadem praerogativa gaudentes derivari possunt. Cum enim sit:

$$x=-(nn-\lambda\lambda)u^{\lambda}+\lambda(\lambda-1)u^{\lambda-2}$$

$$y=\lambda u^{\lambda-2}(\lambda-1-nu\surd(1+uu)+\lambda uu)(\surd(1+uu)+u)^{n}$$

quoties exponens n est numerus impar sive integer sive fractus, statuere licebit $\lambda=n$. Sit igitur n numerus impar, ac fiat $\lambda=n$ erit:

$$x=n(n-1)u^{n-2} \quad \&$$

$$y=nu^{n-2}(n-1-nu\surd(1+uu)+nuu)(\surd(1+uu)+u)^{n}$$

seu per $n(n-1)$, dividendo habebitur:

$$x=u^{n-2} \quad \&$$

$$(n-1),y=u^{n-2}(n-1-nu\surd(1+uu)+nuu)(\surd(1+uu)+u)^{n}$$

34. Si, $n=1$ ex his formulis nihil oritur; fiat ergo, ut exemplum afferamus, $n=3$, eritque $x=u$ & $y=3x(2-3x\surd(1+xx)+3xx)(\surd(1+xx)+x)^{3}$.

 Cum

Cum igitur sit:

$$(\sqrt{(1+xx)}+x)^3 = 3x+4x^3+(1+4xx)\sqrt{(1+xx)}, \text{ erit}$$

$$y = 3x(3x+2x^3+2(1+xx)\sqrt{(1+xx)}) \quad \text{ideoque}$$

$$y - 9xx - 6x^4 = 6x(1+xx)\sqrt{(1+xx)}$$

Quae aequatio ad rationalitatem perducta fit:

$$yy - 18xxy - 12x^4y = 36xx + 27x^4$$

quae est nova linea quinti ordinis in numerum trajectoriarum reciprocarum referenda. Innumerabiles autem aliae aequationes rationales inter x & y hinc erui possunt; ponendis pro n aliis numeris imparibus, quae autem ad multo altiores dimensiones assurgent.

35. Simpliciores provenient, si in prima aequatione ponatur $\lambda = n-2$, unde fit:

$$x = (n-2)(n-3)u^{n-4} - 4(n-1)u^{n-2} \quad \&$$

$$y = (n-2)u^{n-4}(n-3+(n-2)uu-(n-2)u\sqrt{(1+uu)})(\sqrt{(1+uu)}+u)^n$$

Si enim hae formulae cum prius inventis conjungantur, fiatque

$$x = \alpha u^{n-2} + (n-2)(n-3)\beta u^{n-4} - 4(n-1)\beta u^{n-2}$$

$$\left.\begin{array}{l}(n-1)y = \alpha u^{n-2}(n-1+nuu-nu\sqrt{(1+uu)}) \\ +(n-1)(n-2)\beta u^{n-4}(n-3+(n-2)uu- \\ \quad (n-2)u\sqrt{(1+uu)})\end{array}\right\}(\sqrt{(1+uu)}+u)^n$$

po-

ponamus β:
$x = u^{n-4}$
$(n-2)(n-3)$
$((n-2)^2$
Quod si jam
$6y = x(6+$
$(\sqrt{}$
Simili autem
ut posito $n =$
buerit, progre
35. Ex
tam x quam y
impar, n vero
algebraicae con
mulis $dx = vd$
& $n = 1$, ut fi
formulae non
nerales. Cu
$x = t -$
perspicuum est
fuudt fuerit int

ponamus $\beta = \frac{1}{(n-2)(n-3)}$ & $\alpha = \frac{4(n-1)}{(n-2)(n-3)}$, ut sit $x = u^{n-4}$ erit:

$$(n-2)(n-3)y = u^{n-2}((n-2)(n-3) + nnuu + 4nu^4 - u((n-2)^2 + 4nuu)\sqrt{(1+uu)})(\sqrt{(1+uu)} + u)^n$$

Quod si jam ponatur $n = 5$, fiet $x = u$ ideoque

$$6y = x(6 + 25xx + 20x^4 - (9x + 20x^3)\sqrt{(1+xx)})(\sqrt{(1+xx)} + x)^5.$$

Simili autem modo hic ita adjungi potest casus $\lambda = n - 4$, ut posito $n = 7$ fiat $x = u$, atque pari modo, quousque libuerit, progredi licet.

35. Ex reliquis formulis §. 9 exhibitis, in quibus tam x quam y per tertiam variabilem t, cujus u est functio impar, v vero functio par, exprimitur, in genere formulae algebraicae commode erui non possunt: sin autem in his formulis $dx = vdt$ & $dy = \left(\frac{1+u}{1-u}\right)^n vdt$ ponatur $v = 1 - uu$, & $n = 1$, ut fiat $dx = (1 - uu)dt$ & $dy(1 + u)^2 dt$, quae formulae non obstante hac determinatione sunt maxime generales. Cum igitur hinc sit:

$$x = t - \int uudt \;\&\; y = t + 2\int udt + \int uudt$$

perspicuum est curvas algebraicas prodire, si tam $\int udt$ quam $\int uudt$ fuerit integrabile. Ad hoc utrumque praestandum

in

In formula: $\int u\,dt = tu - \int t\,du$ | ponatur $\int t\,du = p$; ut sit $t = \frac{dp}{du}$

ubi, quia t est functio impar ipsius u, necesse est ut sit p functio par. Jam hic valor ipsius t in altera formula substituatur, eritque

$$\int uu\,dt = tuu - 2\int tu\,du = \frac{uu\,dp}{du} - 2\int u\,dp = \frac{uu\,dp}{du} - 2pu + 2\int p\,du$$

ponatur $\int p\,du = q$, ut sit $p = \frac{dq}{du}$, eritque q functio impar ipsius u; & sumto du constante fiet $dp = \frac{ddq}{du}$ atque substitutionibus retro factis erit $t = \frac{ddq}{du^2}$ &

$$\int uu\,dt = \frac{uu\,ddq}{du^2} - \frac{2u\,dq}{du} + 2q \quad \& \quad \int u\,dt = \frac{u\,ddq}{du^2} - \frac{dq}{du}$$

37. His valoribus substitutis habebimus

Quartam Regulam pro inveniendis trajectoriis reciprocis algebraicis:

Sumatur q functio quaecunque impar ipsius u, positoque elemento du constante erit curvae quaesitae

Abscissa AP $= x = \frac{ddq(1-uu)}{du^2} + \frac{2u\,dq}{du} - 2q$

Applicata PM $= y = \frac{ddq(1+u)^2}{du^2} - \frac{2dq(1+u)}{du} + 2q$

Ponamus $q = u^\lambda$, existente λ numero impari, erit

$x =$

$$x = \lambda(\lambda-1)u^{\lambda-2}(1-uu) + 2\lambda u^{\lambda} - 2u^{\lambda}$$

$$y = \lambda(\lambda-1)u^{\lambda-2}(1+u)^{2} - 2\lambda u^{\lambda-1}(1+u) + 2u^{\lambda}$$

five

$$x = \lambda(\lambda-1)u^{\lambda-2} - (\lambda-1)(\lambda-2)u^{\lambda}$$

$$y = \lambda(\lambda-1)u^{\lambda-2} + 2\lambda(\lambda-2)u^{\lambda-1} + (\lambda-1)(\lambda-2)u^{\lambda}$$

38. Ex his formulis variabilis u non difficulter eliminatur, fi enim altera per alteram dividatur, erit

$$\lambda(\lambda-1)(x-y) + 2\lambda(\lambda-2)xu + (\lambda-1)(\lambda-2)uu(x+y) = 0$$

hincque $uu = \dfrac{-2\lambda(\lambda-2)xu - \lambda(\lambda-1)(x-y)}{(\lambda-1)(\lambda-2)(x+y)}$;

inveniturque

$$u = \frac{-\lambda(\lambda-2)x \pm \sqrt{\lambda(\lambda-2)((\lambda-1)^{2}yy - xx)}}{(\lambda-1)(\lambda-2)(x+y)}$$

qui valor fi in altera æquatione, vel in fumma amborum $x + y = 2\lambda u^{\lambda-2}(\lambda-1+(\lambda-2)u)$ fubftituatur, orietur æquatio inter x & y, quæ femel tantum quadratis fumendis ad rationalitatem reducetur. Quatuor autem hifce regulis pro inventione trajectoriarum reciprocarum algebraicarum traditis, quicquid in folutione hujus problematis adhuc defiderari poterat, hic abunde præftitiffe mihi videor.

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