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Solutio problematis difficillimi a Fermatio propositi

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SOLVTIO PROBLEMATIS DIFFICILLIMI A FERMATIO PROPOSITI.

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§. 1.

Quamquam problemata, quae olim ſolutu difficilia ſunt habita, hodie plerumque ob fines analyſeos tantopere promotos nihil vel parum difficultatis habere ſolent; tamen hoc in eo problematum genere, quae ad methodum Diophanti pertinent, non vſu venit. In hac enim analyſeos parte poſt Fermatii tempora, qui plurimum ſtudii et operae in ea feliciſſimo cum ſucceſſu conſum ſit, non ſolum nihil vltra praeſtitum eſſe videtur, ſed etiam hoc ſtudium a geometris, qui eum ſunt ſecuti, fere penitus eſt neglectum. Et ſi autem ea analyſeos pars, in qua mathematici hodie potiſſimum verſantur, ob ſummam vtilitatem, quam ad reliquas ſcientias atque artes copioſiſſime affert, omni laude maxime digna eſt habenda: tamen altera quoque pars, quae in numeris eſt occupata, et ad problemata indeterminata ſoluenda adhiberi ſolet, idcirco minime eſt contemnenda, cum in ea plerumque ſumma ingenii vis cernatur atque ab analyſta non mediocris ſagacitas requiratur.

§. 2. Quae cum ita ſint comparata, ea huius generis problemata, quae a Fermatio ſummopere difficilia ſunt indicata, eadem et hodie non magis facta ſunt facilia; hincque ſtudium, quod in eorum ſolutione ponitur, non male collocatur. Proponit autem Fermatius in annotationibus ſuis ad Diophantum Bacheti ſequens problema tanquam ſolutu difficillimum

Inuenire triangulum rectangulum in numeris rationalibus expreſſum, cuius vterque cathetus area ipſius trianguli minutus producat numerum quadratum.

Huius ergo problematis ſequentes, quas mihi quidem efficere contigit, ſolutiones in medium afferre viſum eſt,

Praeparatio ad ſolutionem.

§. 3. Notum eſt triangulum rectangulum in numeris rationalibus exprimi, ſi ponatur cathetorum alter $= 2ab$, et alter $= aa-bb$, tum enim prodibit hypotenuſa $= aa+bb$. Generalius catheti ambo poni poſſunt $\frac{2ab}{z}$ et $\frac{aa-bb}{z}$, prodeunte hypotenuſa $= \frac{aa+bb}{z}$. Ponam autem, quoniam naturam trianguli rectanguli vltimo loco in computum vocare expedit,

vnum cathetum $= \frac{2x}{z}$

alterum cathetum $= \frac{y}{z}$

eritque area $= \frac{xy}{zz}$

Ac primo per conditionem problematis hae quantitates

$$\left.\begin{array}{l} \text{I. } \frac{2x}{z} - \frac{xy}{zz} \text{ ſeu } 2xz - xy \\ \text{II. } \frac{y}{z} - \frac{xy}{zz} \text{ ſeu } yz - xy \end{array}\right\} \text{ quadrata effici debent.}$$

Tum vero, quia hypotenuſa fit $= \frac{\sqrt{(4xx+yy)}}{z}$, haec quantitas

III. $4xx+yy$ reddi debet quadratum.

§. 4. Quoniam hae ambae quantitas $2xz-xy$ et $yz-xy$ eſſe debent quadrata, earum productum pariter erit quadratum. Ordior ergo a producto,

$2xy$

$$2xyzz - 2xxyz - xyyz + xxyy$$

quod quadratum reddi debet, ponoque eius radicem $= xy - \frac{p}{q}yz$, vt ex euolutione valor ipsius z commode definiri queat; fiet autem

$$2xyzz - 2xxyz - xyyz + xxyy = xxyy - \frac{2p}{q}xyyz + \frac{pp}{qq}yyzz$$

Ac deleto vtrinque termino communi $xxyy$ et reliqua aequatione per yz diuisa obtinebitur

$$2xz - 2xx - xy = \frac{-2p}{q}xy + \frac{pp}{qq}yz$$

vnde fit $z = \frac{2qqxx + qqxy - 2pqxy}{2qqx - ppy}$

§. 5. Inuento iam valore ipsius z, fiet

$2z - y = \frac{4qqxx - 4pqxy + ppyy}{2qqx - ppy} = \frac{(2qx - py)^2}{2qqx - ppy}$

$z - x = \frac{ppxy + qqxy - 2pqxy}{2qqx - ppy} = \frac{xy(p - q)^2}{2qqx - ppy}$

hincque porro habebitur:

$2xz - xy = \frac{x(2qx - py)^2}{2qqx - ppy} = \frac{xx(2qx - py)^2}{2qqxx - ppxy}$

$yz - xy = \frac{xyy(p - q)^2}{2qqx - ppy} = \frac{xxyy(p - q)^2}{2qqxx - ppxy}$

Quarum quantitatum cum vtraque esse debeat quadratum; hoc efficietur, dummodo communis denominator: $2qqxx - ppxy$ fiat quadratum. Ponatur in hunc finem $2qqxx - ppxy = rrxx$, ac diuisione facta per x erit $(2qq - rr)x = ppy$, et $\frac{x}{y} = \frac{pp}{2qq - rr}$

§. 6. Sufficiet autem ad nostram solutionem nosse relationem inter x et y, quia in calculum iam introductus est communis denominator z, quare ponere licebit:

$$x = pp \text{ et } y = 2qq - rr$$

vnde fiet $z - x = \frac{pp(2qq - rr)(p - q)^2}{pprr}$ ideoque

$z =$

$$z = pp + \frac{(2qq-rr)(p-q)^2}{rr}$$

ideoque superest tantum, vt $4xx + yy$ reddatur quadratum, vnde sequens expressio debet esse quadratum

$$4p^4 + 4q^4 - 4qqrr + r^4$$

vnde sequentes solutiones particulares adornabuntur.

Solutio prima.

§. 7. Quoniam igitur quaestio huc est reducta, vt pro litteris p, q, r eiusmodi valores assignentur, qui hanc expressionem

$$4p^4 + 4q^4 - 4qqrr + r^4$$

reddant quadratum, solutio generalis, quae omnes omnino valores idoneos harum litterarum complectatur, tradi nequit. Cum igitur solutionibus specialibus acquiescere debeamus, ponam primo radicem huius expressionis esse $= 2pp \mp rr$, vt termini $4p^4$ et r^4 vtrinque se destruant, ac prodibit haec aequatio

$$4q^4 - 4qqrr = \mp 4pprr$$

vnde fit $pp = \mp \frac{qq}{rr}(qq-rr)$, et habebimus

vel $p = \frac{q}{r}\sqrt{(qq-rr)}$ vel $p = \frac{q}{r}\sqrt{(rr-qq)}$

§. 8. Priori formulae $p = \frac{q}{r}\sqrt{(qq-rr)}$ satis fit ponendo $q = cc + dd$, et $r = 2cd$, vnde fit $p = \frac{(cc+dd)(cc-dd)}{2cd}$

Ex his ergo valoribus:

$p = (cc+dd)(cc-dd)$

seu $q = 2cd(cc+dd)$

$r = 4ccdd$

$q =$

$$\left.\begin{array}{l} q = cc + dd \\ r = 2cd \\ p = \frac{(cc+dd)(cc-dd)}{2cd} \end{array}\right| \text{ erit } \begin{array}{l} x = pp \\ y = 2qq - rr \\ \sqrt{(4xx+yy)} = 2pp + rr \\ z = x + \frac{y(-q)^2}{rr} \end{array}$$

quibus inuentis erit pro triangulo rectangulo quaesito:

I. cathetus $= \frac{2x}{z}$, II. cathetus $= \frac{y}{z}$

Exemplum. I.

§. 9. Sit $c = 2$, et $d = 1$, ac prodibunt hi valores:

$p = 5 \cdot 3 = 15$; $x = 225$

$q = 4 \cdot 5 = 20$; $y = 544$

$r = 4 \cdot 4 = 16$; $z = 225 + \frac{544 \cdot 25}{256} = \frac{25 \cdot 89}{8} = \frac{2225}{8}$

atque $\sqrt{(4xx+yy)} = 2pp + rr = 706$

ex quibus conficitur hoc triangulum rectangulum in numeris:

I. cath. $\frac{2x}{z} = \frac{144}{89}$; II. cath. $\frac{y}{z} = \frac{43,52}{25 \cdot 89}$; III. hypot. $= \frac{5648}{25 \cdot 89}$

area ergo erit $= \frac{72 \cdot 4352}{25 \cdot 89^2}$, et problemati ita satisfit.

I. cath. - area $= \frac{144}{25 \cdot 89^2}(25.89 - 2176) = \frac{144 \cdot 49}{25 \cdot 89^2} = \left(\frac{12 \cdot 7}{5 \cdot 89}\right)^2$

II. cath. - area $= \frac{4352}{25 \cdot 89^2}(89 - 72) = \frac{17 \cdot 17 \cdot 256}{25 \cdot 89^2} = \left(\frac{16 \cdot 17}{5 \cdot 89}\right)^2$.

Exemplum. 2.

§. 10. Sit $c = 3$, et $d = 1$, ac sequentes prodibunt valores

$p = 10 \cdot 8$ | | $p = 20$
$q = 6 \cdot 10$ | qui per 4 diuisi ad minores | $q = 15$
$r = 6 \cdot 6$ | terminos hos reducuntur | $r = 9$

 ex

ex his fit $x = 400$; $y = 369$; et $z = \frac{4625}{9}$; $\sqrt{(4xx + yy)} = 881$; vnde triangulum rectangulum erit

I. cath. $\frac{2x}{z} = \frac{32\cdot 9}{185}$; II. cath. $\frac{y}{z} = \frac{81\cdot 41}{25\cdot 185}$; III. hyp. $= \frac{9\cdot 881}{25\cdot 185}$

atque area $= \frac{16\cdot 9\cdot 81\cdot 41}{25\cdot 185^2}$; quare problemati ita satisfit :

I. cath. $-$ area $= \frac{2\cdot 16\cdot 9\cdot 25\cdot 185 - 16\cdot 9\cdot 81\cdot 41}{25\cdot 185^2} = \frac{16\cdot 9\cdot 5929}{25\cdot 185^2} = \left(\frac{4\cdot 3\cdot 77}{5\cdot 185}\right)^2$

II. cath. $-$ area $= \frac{81\cdot 41\cdot 185 - 16\cdot 9\cdot 81\cdot 41}{25\cdot 185^2} = \frac{81\cdot 41\cdot 41}{25\cdot 185^2} = \left(\frac{9\cdot 41}{5\cdot 185}\right)^2$

Solutio secunda.

§. 11. Sumatur ex solutione praecedente casus posterior $p = \frac{q}{r}\sqrt{(rr - qq)}$, qui requirit hos valores :

$$\left.\begin{array}{l} r = cc + dd \\ q = 2cd \\ p = \frac{2cd(cc-dd)}{cc+dd} \end{array}\right\} \text{ seu } \begin{array}{ll} r = (cc+dd)^2 ; & x = pp \\ q = 2cd(cc+dd) ; & y = 2qq - rr \\ p = 2cd(cc-dd) ; & \sqrt{(4xx+yy)} = 2pp - rr \end{array}$$

et vt ante $z = x + \frac{y(p-q)^2}{rr}$

Quia autem esse debet $2qq > rr$ erit $8ccdd > (cc+dd)^2$ et $2cd\sqrt{2} > cc + dd$, seu $0 > cc - 2cd\sqrt{2} + dd$, quod huc redit, vt fit $dd > (c - d\sqrt{2})^2$; ergo vel $d > c - d\sqrt{2}$ seu $\frac{d}{c} > \frac{1}{1+\sqrt{2}}$

vel $d > d\sqrt{2} - c$ seu $\frac{d}{c} < \frac{1}{\sqrt{2}-1}$

Ergo si $d = 1$ necesse est vt sit vel $c < \sqrt{2} + 1$ vel $c > \sqrt{2} - 1$. At est $c > 1$, vnde semper erit $c > \sqrt{2} - 1$, et $2qq - rr$ fiet quantitas positiua. Erit itaque

I. cath. $= \frac{2x}{z}$; II. cath. $= \frac{y}{z}$ et III. hypot : $= \frac{\sqrt{(4xx+yy)}}{z}$.

Exemplum. 1.

§. 12. Sit $c = 2$, et $d = 1$, ac prouenient hi valores ;

$r =$

$$\left.\begin{array}{l} r = 5.5 = 25 \\ q = 4.5 = 20 \\ p = 4.3 = 12 \end{array}\right\} \text{hincque} \left\{\begin{array}{l} x = 144 \\ y = 175 \\ \surd(4xx + yy = 337 \end{array}\right.$$

atque $z = 144 + \frac{175 \cdot 64}{625} = \frac{4048}{25}$

Vnde trianguli quaesiti erit

I. cath. $= \frac{2x}{z} = \frac{288 \cdot 25}{4048} = \frac{18 \cdot 25}{253} = \frac{450}{253}$

II. cath. $= \frac{y}{z} = \frac{25 \cdot 175}{4048} = \frac{4377}{4048}$

III. hypot. $= \frac{\surd(4xx + yy)}{z} = \frac{25 \cdot 337}{4048} = \frac{8425}{4048}$

Area itaque erit $= \frac{225 \cdot 4375}{253 \cdot 4048} = \frac{225 \cdot 4375}{16 \cdot 253^2}$

Vnde problemati hoc modo satisfit, vt sit:

I. cath. $-$ area $= \frac{225(32 \cdot 253 - 4375)}{16 \cdot 253^2} = \frac{225 \cdot 61^2}{16 \cdot 253^2} = \left(\frac{15 \cdot 61}{4 \cdot 253}\right)^2$

II. cath. $-$ area $= \frac{25(175 \cdot 253 - 9 \cdot 4375)}{253 \cdot 4048} = \frac{25 \cdot 25 \cdot 7 \cdot 28}{16 \cdot 253^2} = \left(\frac{25 \cdot 14}{4 \cdot 253}\right)^2$

Exemplum. 2.

§. 13. Sit $c = 3$ et $d = 1$, ac prodibunt hi valores:

$$\begin{array}{l|l} r = 10.10 & r = 25 \\ q = 6.10 & q = 15 \\ p = 6.8 & p = 8 \end{array}\Bigg\} \text{hincque} \left\{\begin{array}{l} x = 144 \\ y = 175 \\ \surd(4xx + yy) = 337 \end{array}\right.$$

qui valores cum sint iidem, qui in exemplo praecedente, hinc nulla noua oritur solutio. Maiores autem numeros pro c et d non substituo, quod inde nimis complicati valores pro x, y et z prodeunt; praecipua enim cura in hoc debet poni, vt triangula in minimis, quantum fieri potest, numeris expressa reperiantur.

Solutio

Solutio tertia.

§. 14. Cum $4xx+yy=4p^4+4q^4-4qqrr+r^4$ esse debeat quadratum, eius radicem ponamus hic $=2pp\pm 2qq$, vt sit $\sqrt{(4xx+yy)}=2pp\pm 2qq$; atque prodibit haec aequatio $r^4-4qqrr=\pm 8ppqq$; vnde fit $pp=\pm\frac{2rr(rr-4qq)}{16qq}$ et vel $p=\frac{r}{4q}\sqrt{(2rr-8qq)}$ vel $p=\frac{r}{4q}\sqrt{(8qq-2rr)}$ Quia vero ob $y=2qq-rr$ esse oportet $2qq>rr$, prior valor erit inutilis, habebimusque

$$p=\frac{r}{4q}\sqrt{(8qq-2rr)};\ x=pp;\ y=2qq-rr;$$
$$\text{et } \sqrt{(4xx+yy)}=2pp-2qq$$

atque vt ante $z=x+\frac{y(p-q)^2}{rr}$. Erit ergo

I cathetus $=\frac{2x}{z}$; II cath. $=\frac{y}{z}$; hypot. $=\frac{\sqrt{(4xx+yy)}}{z}$

Nunc ergo huc deuenimus, vt $8qq-2rr$ reddatur quadratum: sit eius radix $=\frac{c}{d}(2q+r)$ eritque $4q-2r=\frac{cc}{dd}(2q+r)$ seu $4ddq-2ddr=2ccq+ccr$, hincque $q=cc+2dd$ et $r=4dd-2cc$; $2q+r=8dd$ atque $\sqrt{(8qq-2rr)}=8cd$, hincque $p=\frac{4cd(2dd-cc)}{2dd+cc}$ Quare in integris multiplicando per $2dd+cc$ fiet

$$p=4cd(2dd-cc) \qquad x=pp$$
$$q=(2dd+cc)^2 \qquad y=2qq-rr$$
$$r=2(2dd-cc)(2dd+cc) \qquad \sqrt{(4xx+yy)}=2pp-2qq$$
$$z=x+\frac{y(p-q)^2}{rr}$$

Exemplum. I.

§. 15. Sit $c=1$; $d=1$, erit:

$p=$

$p = 4$; $x = 16$
$q = 9$; $y = 126$ et $z = 16 + \frac{126 \cdot 25}{36} = \frac{207}{2} = \frac{9 \cdot 23}{2}$
$r = 6$; $\sqrt{(4xx + yy)} = 130$
I cath. $\frac{64}{207}$; II cath. $= \frac{252}{207}$; III hypot. $= \frac{260}{207}$
Area vero erit $= \frac{64 \cdot 126}{207 \cdot 207} = \frac{64 \cdot 14}{9 \cdot 23^2}$: ficque fiet
I. cath. $-$ area $= \frac{64}{9 \cdot 23^2}(23 - 14) = \frac{64}{23^2} = (\frac{8}{23})^2$
II. cath. $-$ area $= \frac{252 \cdot 23 - 64 \cdot 14}{9 \cdot 23^2} = \frac{28 \cdot 175}{9 \cdot 23^2} = \frac{4 \cdot 7^2 5^2}{9 \cdot 23^2} = (\frac{2 \cdot 5 \cdot 7}{9 \cdot 23})^2$
Hocque exemplum fine dubio in numeris minimis exiftit, vti deinceps oftendam.

Exemplum. 2.

§. 16. Quia debet effe $2qq > rr$, oportet vt fit $\frac{c}{d} > 2 - \sqrt{2}$; nihilque refert, fiue fit $2dd > cc$ fiue minus, quia nihil obftat, quo minus p, q, r, effe queant numeri negatiui.
Sit igitur $d = 2$; $c = 3$; erit $2dd - cc = -1$; $2dd + cc = 17$ atque

$p = -24 . 1 = -24$ | $x = 576$
$q = 17 . 17 = 289$ | $y = 2 . 7 . 41 . 17^2$
$r = -2 . 17 = -34$ | $\sqrt{(4xx + yy)} = 2 . 5 . 53 . 313$

$z = \frac{90983}{2}$

I. cath. $= \frac{2304}{90983}$; II. cath. $= \frac{28 \cdot 41 \cdot 17^2}{90983}$; III. hyp. $= \frac{4 \cdot 5 \cdot 53 \cdot 313}{90983}$

§. 17. In his omnibus exemplis notari meretur, perinde effe, fiue litterarum c et d valores capiantur affirmatiui, fiue negatiui, inde enim tantum valores p, vel q, vel r prodeunt negatiui; neque propterea valores x et

y alterantur. Verum valor ipsius z variationem subit ex quo pro z semper duplex valor assignari poterit, alter qui iam est exhibitus $z = x + \frac{y(p-q)^2}{rr}$ alter vero $z = x + \frac{y(p+q)^2}{rr}$: sicque ob duplicem valorem ipsius z singula exempla allata duplicabuntur.

§. 18. Huiusmodi solutiones particulares plures adhuc elicere licet, dum aliae idoneae quantitates pro radice quadrata huius formae $4p^4 + 4q^4 - 4qqrr + r^4$ assumuntur. Veluti si haec radix ponatur $rr + 2qq \pm 2pp$, obtinebitur haec aequatio $-4qqrr = 4qqrr \pm 4pp(2qq + rr)$, seu $pp(2qq + rr) = \mp 2qqrr$; vnde patet signum inferius valere, esseque $\surd(4xx + yy) = rr + 2qq - 2pp$, existente vel $p = \frac{2qr}{\surd 2(2qq+rr)}$ vel $q = \frac{pr}{\surd 2(rr-pp)}$ quae formulae iam facile rationales redduntur. Hic ergo si ponatur $r = 3$, $p = 1$, erit $q = \frac{3}{4}$, et in integris

$p = 4$	$x = 16$	qui casus ob y negatiuum
$q = 3$	$y = -126$	non conuenit quaestioni
$r = 12$	$\surd(4xx + yy) = 130$	

§. 19. Qoniam cardo quaestionis in hoc versatur, vt haec expressio reddatur quadratum, $4p^4 + (2qq - rr)^2$, potest hoc generaliter ita effici, vt eius radix ponatur $= 2qq - rr + \frac{2m}{n} pp$, vnde fiet $pp = \frac{m}{n}(2qq - rr) + \frac{mm}{nn} pp$ seu $(nn - mm)pp = mn(2qq - rr)$, et $p = \surd \frac{mm(2qq-rr)}{nn-mm}$ $= mn \surd \frac{2qq - rr}{mn(nn-mm)}$, cui conditioni satisfiet eiusmodi numeros pro m et n quaerendo, vt sit $mn(nn - mm)$ numerus huius formae $2ff - gg$. Verum haec solutio facilius obtinetur ex ipsa praeparatione ad solutionem tradita, quae, si recte tractetur, omnes solutiones non solum in-

in se complectitur; sed etiam solutiones in minoribus numeris omnes commode exhibet. Eam data opera euoluam.

Solutio generalis.

§. 20. Assumtis cathetis trianguli quaesiti $\frac{2x}{z}$ et $\frac{y}{z}$ ponatur statim, vt anguli recti ratio habeatur:
$x = ab$; $y = aa - bb$; eritque trianguli
I. cath. $= \frac{2ab}{z}$; II. cath. $= \frac{aa-bb}{z}$. hypot. $= \frac{aa+bb}{z}$
et area huius trianguli erit $= \frac{ab(aa-bb)}{zz}$
Inuenimus autem primo (§. 4.)

$$z = \frac{2qqxx + qqxy - 2pqxy}{2qqx - ppy}$$

seu $z = x + \frac{xy(p-q)^2}{2qqx - ppy}$.
Vel, quia q tam negatiue quam affirmatiue accipere licet, erit

$$z = x + \frac{xy(p \pm q)^2}{2qqx - ppy}$$

existente $x = ab$ et $y = aa - bb$.

§. 21. Tum vero (§. 5.) hanc quantitatum x et y indolem inuenimus, vt sit $2qqxx - ppxy = rrxx$, vnde fit $z = x + \frac{y(p \pm q)^2}{rr} = ab + \frac{(aa-bb)(p \pm q)^2}{rr}$.
Nihil aliud ergo efficiendum restat, nisi vt haec aequatio $2qqxx - ppxy = rrxx$, seu haec:

$$xy = \frac{xx}{pp}(2qq - rr) \text{ conficiatur.}$$

Vbi cum sit $xy = ab(aa-bb)$, eiusmodi numeros pro a et b inuestigari oportet, vt fiat $ab(aa-bb)$ numerus huius formae $2ff - gg$, seu $(2ff - gg)bb$.

§. 22. Ponamus igitur pro a et b iam huiusmodi valores esse erutos, vt sit

$$ab(aa-bb)=(2ff-gg)bb$$

Cum igitur ob $x=ab$ sit:

$$(2ff-gg)bb\frac{aabb}{pp}(2qq-rr)$$

hinc statim sponte se prodit

$$\frac{abq}{p}=fb \text{ et } \frac{abr}{p}=gb$$

sit ergo $p=ab$ erit $q=fb$ et $r=gb$

atque $z=ab+\frac{(aa-bb)(ab\pm fb)^2}{ggbb}$

Eruntque trianguli rectanguli quaesiti latera:

$$\text{I. cath.} = \frac{2ab}{z} = \frac{2abggbb}{2abggbb+(aa-bb)(ab\pm fb)^2}$$

$$\text{II. cath.} = \frac{aa-bb}{z} = \frac{(aa-bb)ggbb}{2abggbb+(aa-bb)(ab\pm fb)^2}$$

$$\text{III. hypot.} = \frac{aa+bb}{z} = \frac{(aa+bb)ggbb}{2abggbb+(aa-bb)(ab\pm fb)^2}$$

§. 23. Possunt etiam ex huiusmodi valoribus ipsarum a et b quibusuis innumerabilia triangula rectangula, quae quaesito satisfaciant, erui. Posito enim $p=ab$, si sit $ab(aa-bb)=(2ff-ggbb)$, erit

$$(ff-ggbb)=2q-rr$$

seu $2(ffbb-qq)=ggbb-rr$

Ponatur $2(fb+q)\frac{m}{n}(gb+r)$ eritque $fb-q=\frac{n}{m}(gb-r)$ et hinc reperietur:

(2ff

$$q = \frac{2mngb-(2nn+mm)fb}{2nn-mm}$$

$$r = \frac{(2nn+mm)gb-4mnfb}{2nn-mm}$$

Vel in numeris integris erit

$$p = (2nn-mm)ab,$$
$$q = 2mngb-(2nn+mm)fb$$
$$r = (2nn+mm)gb-4mnfb$$

§. 24. Inuentis sic valoribus his p, q, et r, erit

$$z = \frac{abrr+(aa-bb)(p+q)^2}{rr}$$

atque trianguli quaesiti latera erunt:

I. cath. $= \frac{2ab}{z}$, II. cath. $= \frac{aa-bb}{z}$ et III. hyp. $\frac{aa+bb}{z}$ vnde pro singulis idoneis valoribus ipsarum a et b, vt sit $ab(aa-bb) = (2ff-gg)bb$, ob m et r numeros pro arbitrio assumendos, innumerabilia triangula exhiberi poterunt.

§. 25. Quoniam igitur totum negotium huc redit, vt pro a et b eiusmodi numeri assumantur, vt productum $ab(aa-bb)$ siue $ab(a+b)(a-b)$ fiat numerus huius formae $(2ff-gg)bb$. Quo hoc facilius effici possit, indolem numerorum, qui in hac forma generali $(2ff-gg)bb$ seu hac $2tt-uu$ continentur, attentius considerari conueniet. Ac primo quidem perspicuum est, in forma $2tt-uu$

$-uu$ contineri omnes numeros quadratos, quippe qui prodeunt, si $u=t$; tum vero etiam in hac forma continentur omnes numeri quadrati duplicati, ponendo $u=0$. Praeterea vero infiniti alii occurrunt numeri, qui vsque ad 200 sunt sequentes:

1, 2, 4, 7, 8, 9, 14, 16, 17, 18, 23, 25, 28, 31, 32, 34, 36, 41, 46, 47, 49, 50, 56, 62, 63, 64, 68, 71, 72, 73, 79, 81, 82, 89, 92, 94, 97, 98, 100, 103, 112, 113, 119, 121, 124, 126, 127, 128, 136, 137, 142, 144, 146, 151, 153, 158, 161, 162, 164, 167, 169, 175, 178, 184, 188, 191, 193, 194, 196, 199, 200.

§. 26. Si numeri primi considerentur, qui occurrunt, ii non solum omnes in hac forma $8m \pm 1$ continentur, sed etiam viciſsim omnes numeri primi in hac gemina forma $8m \pm 1$ contenti ibi occurrunt, ideoque in forma $2tt-uu$ comprehenduntur. Praeterea vero horum numerorum primorum dupla adsunt, item eorum producta, tam per quosuis numeros quadratos, quam per se ipsos; nec non horum productorum dupla. Qua proprietate animaduersa non difficile erit hos numeros quousque libuerit continuare.

§. 27. Hinc porro colligitur numeros non primos in forma $2tt-uu$ contentos alios diuisores, qui quidem inter se sint primi, non admittere, nisi qui ipsi sint numeri in eadem forma $2tt-uu$ contenti. Quare cum productum $ab(a+b)(a-b)$ esse debeat numerus formae $2tt$

$-uu$,

$-uu$, hique factores a, b, $a+b$, $a-b$, sint vel primi inter se vel ad summum binarium pro communi diuisore habeant, qui ipsi in forma $2tt-uu$ continetur, necesse est, vt hi singuli factores a, b, $a+b$, $a-b$ sint numeri eiusdem formae $2tt-uu$. Quo cognito ex tabula tradita non erit difficile idoneos valores pro a et b excerpere, vt non solum a et b sed etiam $a+b$ et $a-b$ in eadem tabula existant.

§. 28. Quod si autem a, b, et $a+b$, $a-b$ singuli sint numeri formae $2tt-uu$, tum quoque eorum productum $ab(a+b)(a-b)$ in eadem forma continebitur, quod generatim ita ostendi potest: sint propositi duo numeri huius formae, velut $2\alpha\alpha-\beta\beta$ et $2\gamma\gamma-\delta\delta$ erit eorum productum $(2\alpha\alpha-\beta\beta)(2\gamma\gamma-\delta\delta)=(2\alpha\gamma+\beta\delta)^2-2(\beta\gamma+\alpha\delta)^2=2(2\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta)^2-(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta\delta)^2$. Est enim generaliter

$$xx-2yy=2(x+y)^2-(x+2y)^2$$

ita vt hae duae formae $2tt-uu$ et $tt-2uu$ inter se congruant. Cum igitur productum ex duobus numeris formae $2tt-uu$ facile ad eandem formam reuocetur, etiam si quotcunque numeri huius formae in se inuicem multiplicentur, eorum productum in eadem forma comprehendi reperietur

§. 29. Tribuatur ergo primo ipsi b valor quidam ex tabula numerorum allata (§. 25), et in eadem tabula facile dispicietur, vtrum insint tres numeri $a-b$, a, $a+b$, qui differant illo numero b. Verum hanc tabulam inspicienti mox patet pro b vel numeros impares, vel per 8

diuisi-

diuiſibiles tantum aſſumi poſſe, ſiquidem *a* et *b* numeri debent eſſe inter ſe primi. Huiusmodi igitur valoribus pro *b* ſubſtitutis, pro *a* ſequentes prodibunt valores

b	valores ipſius *a*
1	8, 17, 63, 72, 127,
7	9, 16, 25, 144,
8	9, 17, 71, 81, 89, 161,
9	16, 23, 25, 32, 41, 73, 103, 112, 128, 137, 184,
16	25, 47, 63, 97, 137, 153,
17	64, 81, 144, 161,
23	41. 121, 144,
25	56, 72, 119, 128, 137, 144, 153, 169,
31	32, 63, 72, 81, 113, 144,
32	41, 49, 81, 121,
41	72, 103, 112, 153,
47	56, 72, 79, 81, 97, 128, 144,
49	72, 113, 146,
56	81, 97, 137,
63	64, 79, 136,
71	73,
72	79, 89, 97, 103, 119, 121
73	89
79	—
81	97, 112, 113

Ex-

Exemplum. 1.

§. 30. Quo vſus huius tabulae ad ſolutionem problematis clarius appareat, ſit $b=1$; $a=8$, eritque: $ab=8$; $aa-bb=63$; $ab(aa-bb)=8.9.7=4.9.14$ Fiet ergo $4.9.14=bb(2ff-gg)$, ideoque $b=6$ et $2ff-gg=14$, vnde colligitur $f=3$, $g=2$, et ex §. 23 obtinebimus.

$p=8(2nn-mm)$; $q=24mn-18(2nn+mm)$; $r=12(2nn+mm)-72mn$
qui ſublato communi diuiſore 2, erit

$$p=8nn-4mm$$

$$q=12mn-18nn-9mm;\ r=12nn+6mm-36mn$$

$$p+q=12mn-10nn-13mm$$

$$-p+q=12mn-26nn-5mm$$

$$\text{et } z=8+\tfrac{63(p\pm q)^2}{rr}$$

Hinc ergo innumerabiles prodeunt valores ipſius z, ex quorum quouis conficitur triangulum rectangulum.

I. cath. $=\frac{16}{z}$; II. cath. $=\frac{63}{z}$; III. hypot. $=\frac{65}{z}$

Caſusque omnium ſimpliciſſimus oritur ponendo $n=0$ et $m=1$, vnde fit $r=6$, $p\pm q=\frac{5}{13}$, et $z=8+\frac{7}{4}.\frac{25}{169}$ ergo vel $z=\frac{207}{4}$ vel $z=\frac{1215}{4}$, quorum valorum prior eſt pro caſu ſimpliciſſimo iam §. 15. expoſito.

Exemplum. 2.

§. 31. Cum pro quibusque valoribus litterarum a et b infiniti exhiberi poſſint valores idonei ipſius z, quorum inuentio nulla difficultate laborat per ea, quae §. §. 23 & 24 ſunt tradita, hic tantum valorem §. 22 datum, $z=$

 ab

$ab + \frac{(aa-bb)(ab+fb)^2}{ggbb}$ adhibere ſufficiet, ob $ab(aa-bb) = (2ff-gg)bb$; vnde erunt trianguli catheti, I $= \frac{2ab}{z}$; II $= \frac{aa-bb}{z}$ et hypoth. $= \frac{aa+bb}{z}$. Sit igitur $b = 7$ et $a = 9$ erit $ab = 63$; $aa-bb = 32$, et $ab(aa-bb), = 63.32 = 16.9.14 = (2ff-gg)bb$, vnde fiet $b = 12$; $f = 3$ et $g = 2$; ergo $z = 63 + \frac{32(63 \pm 36)^2}{24.24}$ ſeu $z = 63 + \frac{9(27 \pm 4)^2}{2}$; ideoque vel $z = \frac{207}{2}$ vel $z = \frac{1215}{2}$ conſequenter triangulum quaeſitum erit vt ante:

I cath. $= \frac{126}{207}$; II cath. $= \frac{64}{207}$ III hypot. $= \frac{260}{207}$.

Exemplum. 3.

§. 32. Quo vſus tabulae §. 29. exhibitae clarius perſpiciatur, ſumamus pro a et b maiores numeros, ſitque $b = 41$ et $a = 112$, vt ſit $ab = 7.16.41$; $aa-bb = 71.9.17$ erit $ab(aa-bb) = 16.9.7.17.41.71 = (2ff-gg)bb$, et $b = 12$ atque $7.17.41.71 = 2ff-gg$. At eſt $7 = 3^2 - 2.1^2$; $17 = 2.3^2 - 1^2$; $41 = 7^2 - 2.2^2$; $71 = 2.6^2 - 1^2$, vnde fit $7.41 = (21 \pm 2.2)^2 - 2(6 \pm 7)^2 = 17^2 - 2.1^2 = 2.16^2 - 15^2$; $17.71 = (2.18 \pm 1)^2 - 2(6 \pm 3)^2 = 35^2 - 2.3^2 = 2.32^2 - 29^2$ Atque $7.17.41.71 = (17.35 - 2.3)^2 - 2(51 - 35)^2 = 589^2 - 2.16^2$; ergo $7.17.41.71 = 2.573^2 - 557^2$. Haec autem reductio ad formam $2tt - uu$ infinitis aliis modis fieri poteſt, quorum ſimpliciſſimus eſt hic:

$7.17.41.71 = 2.417^2 - 37^2$ vt ſit $f = 417$ et $g = 37$.

Ergo

Ergo ob $h = 12$ erit $fh = 12.3.139$ et $gh = 12.37.$

ideoque $z = 16.7.41 + \frac{5.17.71(16.7.41 \pm 4.5.139)^2}{16.5.37.37}$

seu $z = 16.7.41 + \frac{17.71(4.7.41 - 5.139)^2}{37.37}$

vel $z = 16.7.41 + \frac{17.71.103.103}{37.37} = \frac{1909\,1511}{1369}$

Ex quo obtinebitur triangulum rectangulum:

I. cath. $= \frac{9184.1369}{19091511}$

II. cath. $= \frac{10863.1369}{19091511}$

III. hyp. $= \frac{14225.1369}{1909.511}$