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James E. Hetrick
University of the Pacific, jhetrick@pacific.edu

Y. Hosotani
University of Minnesota

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QED on a circle

James E. Hetrick and Yutaka Hosotani
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
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QED with massless fermions on a circle ($0 \le x < L$) is exactly solved. The nonintegrable phase $\exp[ie \int_0^L dx\, A_1(t,x)]$, which is a dynamical degree of freedom on a circle, couples through the anomaly to the zero mode of the fermion-antifermion bound states, leading to the θ vacuum.

Almost three decades ago Aharonov and Bohm[1] showed that the electromagnetic field strength in quantum mechanics underdescribes electromagnetism. The vector potential which gives a vanishing field strength in the region where the electrons propagate still affects the motion of the electrons through the nonintegrable phase $\exp(ie \int A_\mu dx^\mu)$, an integral along a noncontractable loop. In the Aharonov-Bohm experiment it is related to the total magnetic flux inside the solenoid, and is an external parameter of the system. Years later, it was recognized by one of the authors that if one of the spatial dimensions is given by a circle, the nonintegrable phase is promoted to a dynamical degree of freedom, leading to dynamical gauge symmetry breaking in non-Abelian gauge theory.[2,3] It was shown that the value of the nonintegrable phase is determined by quantum effects. In this paper we attempt to clarify the dynamics behind this phenomenon, analyzing the simplest example: QED with massless fermions on a circle. This model is exactly solvable[4–8] and was previously analyzed by Manton.[9] It turns out that the nonintegrable phase couples through the anomaly to the zero mode of the fermion-antifermion bound state, leading to the well-known θ vacuum. The infrared divergence, which sometimes plagues analysis of two-dimensional gauge theory, is absent on a circle. We confirm most of Manton's results. The main difference between Manton's analysis and ours lies in the way of bosonizing the fermions [Eq. (12) below]. We believe that our derivation is much simpler and more deductive. Moreover, it clarifies the connection between the nonintegrable phase and the θ vacuum.

The Lagrangian density of the model is

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi \, . \tag{1}$$

On a circle boundary conditions must be specified to define the theory. We require that $\mathcal{L}$, in addition to $F_{\mu\nu}$ and $j^\mu = \bar{\psi}\gamma^\mu\psi$, be single valued:

$$A_\mu(t,x+L) = A_\mu(t,x) \, , \tag{2a}$$

$$\psi(t,x+L) = -e^{2\pi i\alpha}\psi(t,x) \, . \tag{2b}$$

Parity invariance is maintained.

Within (2) one can always find a gauge in which

$$A_1(t,x) = b(t) \, . \tag{3}$$

There still remains the residual gauge symmetry which shifts b by $2\pi/eL$:

$$A_\mu \to A_\mu + \frac{2\pi}{eL}\delta_{\mu 1}, \quad \psi \to e^{-2\pi i x/L}\psi \, . \tag{4}$$

The zero mode $b(t)$, now being a dynamical degree of freedom, is related to the nonintegrable phase by $\exp(ie \int_0^L dx\, A_1) = \exp(iebL)$, and cannot be gauged away.[2,9] The electric field $E = F_{01} = \dot{b}(t) - A_0'(t,x)$ satisfies

$$E' = ej^0, \quad \dot{E} = -ej^1 \, . \tag{5}$$

The first equation implies

$$Q = \int_0^L dx\, j^0 = 0 \, . \tag{6}$$

Since $A_0'' = -ej^0$ and $A_0(t,x+L) = A_0(t,x)$,

$$A_0(t,x) = -e \int_0^L dy\, D(x,y;\lambda) j^0(t,y) \, , \tag{7}$$

$$D(x,y;\lambda) = \tfrac{1}{2}|x-y| + \frac{(x-\lambda)(y-\lambda)}{L} \, . \tag{8}$$

λ is arbitrary due to (6). A_0 is completely determined by j^0 up to an irrelevant constant. The second equation in (5), with $\partial_\mu j^\mu = 0$, then leads to

$$\ddot{b}(t) = -\frac{e}{L}\int_0^L dx\, j^1(t,x) = -eQ_5/L \, . \tag{9}$$

The time evolution of the nonintegrable phase is controlled by the total current or the axial charge.

The Hamiltonian reads

$$H = \frac{F^2}{2L} + \int_0^L dx\, \bar{\psi}\gamma^1(-i\partial_1 + eb)\psi - \frac{e^2}{2}\int_0^L dx \int_0^L dy\, j^0(t,x) D(x,y;\lambda) j^0(t,y) \, , \tag{10}$$

where $F(t) \equiv L\dot{b}(t)$ is canonically conjugate to $b(t)$. The proper antisymmetrization of fermion operators has been understood.

It is instructive to quote some of the results in Ref. 2. The effective potential for a constant b due to one-loop diagrams of fermions satisfying (2b) is

$$V_{\text{eff}}[b] = i\,\mathrm{Tr}\ln(p\!\!\!/ + eA\!\!\!/) = \frac{2}{\pi L^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n(ebL - 2\pi\alpha) + \text{const} \, . \tag{11}$$

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It has minima at $ebL = 2\pi\alpha$ (mod2π) (Ref. 10). The value of b is dynamically adjusted such that the physics is independent of α in two-dimensional QED. This is a consequence of the invariance of the action under a boundary-condition-changing gauge transformation, $\Lambda = 2\pi\alpha x/eL$. The periodicity of $V_{\text{eff}}[b]$ is due to the residual gauge symmetry (4). Does this imply that all the minima are just gauge copies of one physical configuration? We will show that in massless QED ground states are still infinitely degenerate, but a small fermion mass term lifts the degeneracy. To understand this and the dynamics involved in this phenomenon we need to keep track of both fermion degrees of freedom and the nonintegrable phase.

To this end we bosonize the fermions[8,9,11] in the interaction picture:

$$\psi_\pm = C_\pm \frac{1}{\sqrt{L}} e^{\pm i[q_\pm + 2\pi p_\pm (t\pm x)/L]} :e^{\pm i\phi_\pm(t,x)}: ,$$
$$C_+ = 1, \quad C_- = e^{i\pi(p_+ - p_-)} , \tag{12}$$
$$\phi_\pm(t,x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (a_{\pm,n} e^{-2\pi i n(t\pm x)/L} + \text{H.c.}) ,$$

where $[q_\pm, p_\pm] = i$, $[a_{\pm,n}, a^\dagger_{\pm,m}] = \delta_{nm}$, and all other commutators vanish. Operators in the interaction picture are given by $p^I_\pm(t) = p_\pm$, $q^I_\pm(t) = q_\pm + 2\pi p_\pm t/L$, and

$$a^I_{\pm,n}(t) = a_{\pm,n} \exp(-2\pi i n t/L) .$$

$\psi^t = (\psi_+, \psi_-)$ satisfies the free massless Dirac equation with $\gamma^0 = \sigma_1$ and $\gamma^1 = i\sigma_2$. Since

$$\psi_\pm(t, x+L) = -e^{2\pi i p_\pm} \psi_\pm(t,x) = -\psi_\pm(t,x) e^{2\pi i p_\pm} , \tag{13}$$

$p_\pm$ must take discrete values α (mod1) in physical states, which conforms to the angular nature of $q_\pm$. Our choice of the Klein factor C_- simplifies the discussion of the residual gauge symmetry, which now reads

$$ebL \to ebL + 2\pi, \quad p_\pm \to p_\pm - 1 . \tag{14}$$

In rewriting the Hamiltonian in terms of bosonic operators, due caution must be taken to respect gauge invariance. For instance,

$$j^\mu(x) = \lim_{y\to x} \exp[-ieb(y-x)] \tfrac{1}{2} [\bar\psi(y), \gamma^\mu \psi(x)] ,$$
$$j^0 = -\frac{1}{\sqrt{\pi}} \frac{\partial\phi}{\partial x}, \quad j^1 = +\frac{1}{\sqrt{\pi}} \frac{\partial\phi}{\partial t} + \frac{e}{\pi} b , \tag{15}$$
$$\phi = \frac{1}{\sqrt{4\pi}} \left[q_+ + q_- + 2\pi(p_+ + p_-) \frac{t}{L} + 2\pi(p_+ - p_-) \frac{x}{L} + \phi_+ + \phi_- \right] .$$

It follows that

$$Q = \int_0^L dx\, j^0 = -p_+ + p_- ,$$
$$Q_5 = \int_0^L dx\, \psi^\dagger \gamma_5 \psi = \int_0^L dx\, j^1 = p_+ + p_- + \frac{ebL}{\pi} . \tag{16}$$

ebL/π in Q_5 represents a contribution from the anomaly.

Similar manipulations lead to the expression for H in the interaction picture. For instance, the kinetic term is

$$\frac{i}{4}[\psi^\dagger_\pm, (\partial_1 + ieb)\psi_\pm] - \frac{i}{4}[(\partial_1 - ieb)\psi^\dagger_\pm, \psi_\pm]$$
$$= \mp \frac{\pi}{12L^2} \pm \frac{1}{4\pi} : \left[\frac{2\pi p_\pm}{L} + eb \pm \frac{\partial\phi_\pm}{\partial x} \right]^2 : .$$

Going back to the Schrödinger picture, one obtains[12]

$$H = \frac{F^2}{2L} + \frac{\pi}{2L}(Q^2 + Q_5^2) + \int_0^L dx \tfrac{1}{2} N_{e/\sqrt{\pi}} \left[\bar\Pi^2 + \bar\phi'^2 + \frac{e^2}{\pi} \bar\phi^2 \right] , \tag{17}$$

where $\bar\phi(x) = (4\pi)^{-1/2}(\phi_+ + \phi_-)$, $\int_0^L dx\, \bar\phi(x) = 0$, and

$$[\bar\phi(x), \bar\Pi(y)] = i \left[\delta(x-y) - \frac{1}{L} \right] .$$

N_μ in (17) indicates normal ordering in the Schrödinger picture[13] with respect to a mass parameter μ. Notice that the Coulomb interaction induces a mass term only for the oscillatory part of ϕ. The zero mode of ϕ, on the other hand, couples to the nonintegrable phase b through the anomaly. The residual gauge symmetry (14) is represented by

$$UHU^{-1} = H, \quad U = \exp\left[i \left[q_+ + q_- + \frac{2\pi F}{eL} \right] \right] . \tag{18}$$

The Hamiltonian (17) can be exactly solved. The oscillatory part of ϕ is just a free scalar field corresponding to fermion-antifermion bound states with mass $e/\sqrt{\pi}$, hence we focus on q_+, q_-, and b. As remarked before, $Q\,|\,\text{phys}\rangle = 0$, and therefore $p_+\,|\,\text{phys}\rangle = p_-\,|\,\text{phys}\rangle$. Since $[H, p_\pm] = 0$ and

$$e^{2\pi i p_\pm}\,|\,\text{phys}\rangle = e^{2\pi i\alpha}\,|\,\text{phys}\rangle ,$$

eigenfunctions of $p_\pm$ and H can be written as

$$\Phi_n = \frac{1}{2\pi} u(ebL - 2\pi n - 2\pi\alpha) e^{-i(q_+ + q_-)(n+\alpha)} , \tag{19}$$

where n is an integer. It is easy to see that $u(x)$ obeys the harmonic-oscillator equation with frequency $e/\sqrt{\pi}$. For the ground-state wave function

$$u(x) = (eL\pi^{3/2})^{-1/4} \exp[-x^2/(2eL\pi^{1/2})] ,$$

the Φ_n's satisfy

$$\langle \Phi_n \,|\, \Phi_m \rangle = \int_{-\infty}^{\infty} d(ebL) \int_0^{2\pi} dq_+ \int_0^{2\pi} dq_-\, \Phi_n^* \Phi_m = \delta_{nm} . \tag{20}$$

The ground states are infinitely degenerate. For each Φ_n, $\langle \Phi_n \,|\, ebL \,|\, \Phi_n \rangle = 2\pi(\alpha + n)$, exactly as was found in the computation of the effective potential (11). Each Φ_n is related to the others by the residual gauge symmetry: $U\Phi_n = \Phi_{n-1}$.

The degeneracy is lifted by a small fermion mass $m\bar\psi\psi$ (Refs. 7 and 8). The bosonization (12) immediately leads to

$$\bar{\psi}\frac{1-\gamma_5}{2}\psi=\exp[-i\pi(p_+-p_-)+2\pi i(p_+-p_-)x/L]$$
$$\times e^{i(q_++q_-)}\frac{1}{L}N_0(e^{i\sqrt{4\pi}\bar{\phi}}) \tag{21}$$

in the Schrödinger picture. Making use of the identity[13]

$$\frac{1}{L}N_0(e^{i\sqrt{4\pi}\bar{\phi}})=B(\mu,L)N_\mu(e^{i\sqrt{4\pi}\bar{\phi}}) ,$$
$$B(\mu,L)=\frac{\mu}{4\pi}\exp\left[\gamma+\frac{\pi}{\mu L}+2\int_0^\infty dx\,(1-e^{\mu L\cosh x})^{-1}\right] , \tag{22}$$

one finds

$$\left\langle\Phi_n\left|\bar{\psi}\frac{1-\gamma_5}{2}\psi\right|\Phi_l\right\rangle=\delta_{n,l-1}e^{-\pi/\mu L}B(\mu,L) , \tag{23}$$

where $\mu=e/\sqrt{\pi}$. True eigenstates are θ states:

$$|\Phi_\theta\rangle=\frac{1}{\sqrt{2\pi}}\sum_n(-1)^n e^{-in\theta}|\Phi_n\rangle . \tag{24}$$

Φ_θ is gauge invariant: $U\Phi_\theta=-e^{-i\theta}\Phi_\theta$ [U defined in (18)].

We define $\langle A\rangle_\theta=\langle\Phi_\theta|A|\Phi_\theta\rangle/\langle\Phi_\theta|\Phi_\theta\rangle$. Then,

$$\left\langle\bar{\psi}\frac{1-\gamma_5}{2}\psi\right\rangle_\theta=-e^{-i\theta}e^{-\pi/\mu L}B(\mu,L) . \tag{25}$$

In the presence of the mass term $m\bar{\psi}\psi$ the $\theta=0$ state has the lowest energy, i.e., $|\text{vac}\rangle=|\Phi_{\theta=0}\rangle$. When the spatial volume is finite, $\theta\neq0$ states are unstable.

The result (25) shows that the appearance of the θ vacuum is closely related to chiral-symmetry breaking.[5,7,8] Indeed, p_++p_-, the generator of chiral transformations ($q_\pm\to q_\pm+\beta$), does not annihilate the vacuum. It is also easy to see, in the Heisenberg picture,

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi)=\frac{1}{\sqrt{\pi}}(\partial_0^2-\partial_1^2)\phi+\frac{e}{\pi}\dot{b}$$
$$=-\frac{1}{\sqrt{\pi}}\frac{e^2}{\pi}\bar{\phi}+\frac{e}{\pi}\dot{b}=\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} , \tag{26}$$

which, integrated over x, implies that Q_5-ebL/π is constant, in accordance with (16) and (17).

It is known that in Minkowski spacetime the θ parameter is related to the background electric field.[7,8] To make this connection more transparent we introduce a static source

$$\rho_{\text{ext}}(x)=-\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial x}\phi_{\text{ext}} ,$$

where $\int_0^L dx\,\rho_{\text{ext}}(x)=0$. In the Coulomb interaction term in (10) j^0 is replaced by $j^0+\rho_{\text{ext}}$, and the mass term in (17) now becomes $(e^2/2\pi)(\bar{\phi}+\bar{\phi}_{\text{ext}})^2$, where

$$\bar{\phi}_{\text{ext}}(x)=\phi_{\text{ext}}(x)-L^{-1}\int_0^L dy\,\phi_{\text{ext}}(y) .$$

The background electric field is given by $E_{\text{ext}}=-(e/\sqrt{\pi})\bar{\phi}_{\text{ext}}$. Let us introduce two point charges:

$$\rho_{\text{ext}}(x)=q[\delta(x)-\delta(x-d)] .$$

They yield

$$E_{\text{ext}}=eq\left[-\frac{d}{L}+\theta(d-x)\right]\quad(0<x<L) .$$

Suppose that $eL\gg1$ and we are far away from the sources so that we are examining the physics in a uniform background electric field. Then the only change brought about in the Hamiltonian density in (17) is the shift $\bar{\phi}\to\bar{\phi}+\bar{\phi}_{\text{ext}}\equiv\bar{\phi}_{\text{new}}$. We rewrite everything in terms of $\bar{\phi}_{\text{new}}$. Since the difference between $\bar{\phi}$ and $\bar{\phi}_{\text{new}}$ is a c number, one has, in (22),

$$N_\mu(e^{-i\sqrt{4\pi}\bar{\phi}})=e^{-i\sqrt{4\pi}\bar{\phi}_{\text{ext}}}N_\mu^{\text{new}}(e^{i\sqrt{4\pi}\bar{\phi}_{\text{new}}}) ,$$

where $\mu=e/\sqrt{\pi}$. It follows that

$$\langle\bar{\psi}\psi\rangle_\theta=-2\cos\left[\theta-\frac{2\pi E_{\text{ext}}}{e}\right]e^{-\pi/\mu L}B(\mu,L) , \tag{27}$$

which establishes the fact that in the presence of a small mass perturbation, $\theta=2\pi E_{\text{ext}}/e$.

With the exact vacuum wave function $\Phi_{\theta=0}$ in hand, one can evaluate various correlation functions. We quote some of the results:

$$\langle E\rangle_0=\langle F\rangle_0=0,\quad\langle e^{i\pi Q_5}\rangle_0=e^{-\pi\mu L/4} , \tag{28}$$

where $\mu=e/\sqrt{\pi}$. For the nonintegrable phase $A=\exp(iebL)$,

$$\langle A\rangle_0=e^{2\pi i\alpha-\pi\mu L/4} ,$$
$$\langle A^\dagger(t)A(0)\rangle_0=\exp\left[-\frac{\pi\mu L}{2}(1-e^{-i\mu t})\right] , \tag{29}$$
$$\langle A(t)A(0)\rangle_0=\exp\left[4\pi i\alpha-\frac{\pi\mu L}{2}(1+e^{-i\mu t})\right] .$$

Notice that $\langle A\rangle_0\to0$ as $L\to\infty$. For the order parameter $M=\bar{\psi}\frac{1}{2}(1-\gamma_5)\psi$ of the chiral-symmetry breaking, one finds

$$\langle M^\dagger(x)M(0)\rangle_0=B(\mu,L)^2\exp\left[\sum_{n\neq0}\frac{1}{[n^2+(\mu/2\pi)^2]^{1/2}}e^{-2\pi inx/L}\right]$$
$$\sim\left[\frac{\mu}{4\pi}e^\gamma\right]^2\exp\left[\frac{1}{\sqrt{\mu|x|}}e^{-\mu|x|}\right] , \tag{30}$$

for $1 \ll \mu |x| \ll \mu L$.

Finally, we stress that the nonintegrable phase plays a crucial dynamical role on a circle. It links the residual gauge invariance, chiral-symmetry breaking, and the θ vacuum. The $L \to \infty$ limit is well defined only for matrix elements of physical operators in Minkowski spacetime, as the above results indicate.

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