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Observationes analyticae variae de combinationibus

Leonhard Euler

OBSERVATIONES ANALYTICAE VARIAE DE COMBINATIONIBVS

Auctore
L. Euler.

§. 1.

Proposita nobis sit series quantitatum quarumcunque siue finita siue in infinitum excurrens haec:

$$a, b, c, d, e, f, g, h, \text{etc.}$$

quae litterae denotent quantitates quascunque siue inter se aequales siue inaequales. Interim tamen quantitates, quae diuersis litteris indicantur, inter se inaequales vocabo, etiamsi in exemplis earum loco numeros aequales substituere liceat.

§. 2. Nunc primo ex his quantitatibus formentur potestatibus sumendis nouae series, quarum summae designentur litteris maiusculis A, B, C, D etc. vt sequitur: sit scilicet

$$A = a + b + c + d + e + \text{etc.}$$
$$B = a^2 + b^2 + c^2 + d^2 + e^2 + \text{etc.}$$
$$C = a^3 + b^3 + c^3 + d^3 + e^3 + \text{etc.}$$
$$D = a^4 + b^4 + c^4 + d^4 + e^4 + \text{etc.}$$
$$E = a^5 + b^5 + c^5 + d^5 + e^5 + \text{etc.}$$
$$\text{etc.}$$

quae series singulae erunt infinitae, si numerus quantitatum a, b, c, d, etc. assumtarum fuerit infinitus; sin autem numerus harum quantitatum sit finitus ac determinatus

natus puta $= n$, tum ſingulae iſtae ſeries totidem terminos complectentur.

§. 3. Deinde ſequenti modo ex quantitatibus aſſumtis $a, b, c, d,$ etc. productis inaequalium ſumendis formentur ſeries. Primo ſcilicet colligantur quantitates ſingulae, tum facta ex binis inaequalibus; tertio ex ternis inaequalibus; quarto ex quaternis inaequalibus et ita porro; atque hae ſeries litteris graecis α, β, γ, δ, etc. indicentur vt ſequitur.

$$\alpha = a + b + c + d + \text{etc.}$$
$$\beta = ab + ac + ad + ac + bd + \text{etc.}$$
$$\gamma = abc + abd + abe + bcd + \text{etc.}$$
$$\delta = abcd + abce + bcde + \text{etc.}$$
$$\varepsilon = abcde + \text{etc.}$$
$$\text{etc.}$$

quae ſeries ſi quantitatum aſſumtarum a, b, c, d, etc. numerus fuerit infinitus, non ſolum omnes in infinitum excurrent, ſed etiam ipſarum ſerierum hoc modo formandarum numerus erit infinitus. Quodſi autem numerus quantitatum a, b, c, d, etc. fuerit finitus puta $= n$, tum ſeries α continebit n terminos, ſecunda ſeries β conſtabit ex $\frac{n(n-1)}{1 \cdot 2}$ terminis, tertia γ ex $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$ terminis, quarta δ ex $\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$ terminis, et ita porro, donec tandem ad ſeriem perueniatur ex vnico termino conſtantem, quam ſequentes omnes euaneſcent terminis omnino carentes. Perſpicuum autem eſt ſerierum, quae hoc modo generantur, numerum fore $= n$, earumque vltimam vnico conſtare termino, qui ſit pro-

 ductum

ductum ex omnibus quantitatibus assumtis a, b, c, d, e, etc.

§. 4. Quemadmodum autem hic producta ex quantitatibus inaequalibus tantum assumsimus, ex iisque series expositas formauimus; ita iisdem quantitatibus in productis quoties fieri poterit repetendis, nanciscemur nouas productorum ex singulis, binis, ternis, quaternis, etc. series, in quibus factores aequales non, vt ante excludantur; hae ergo series ita se habebunt.

$$\mathfrak{A} = a + b + c + d + e + \text{etc.}$$
$$\mathfrak{B} = a^2 + ab + b^2 + ac + bc + c^2 + \text{etc.}$$
$$\mathfrak{C} = a^3 + a^2b + ab^2 + b^3 + a^2c + abc + \text{etc.}$$
$$\mathfrak{D} = a^4 + a^3b + a^2b^2 + a^2bc + abcd + \text{etc.}$$
$$\mathfrak{E} = a^5 + a^4b + a^3b^2 + a^3bc + a^2bcd + \text{etc.}$$
etc.

in his nempe seriebus omnes continentur quantitates, quae per multiplicationem ex quantitatibus assumtis a, b, c, d, etc. produci possunt. Ceterum notandum est si numerus quantitatum a, b, c, d, etc. fuerit finitus $= n$, tum seriem primam $\mathfrak{A}$ esse habituram n terminos, secuuda autem $\mathfrak{B}$ habebit $\frac{n(n+1)}{1\cdot 2}$ terminos, tertia $\mathfrak{C}$ habebit $\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}$ terminos, quarta $\mathfrak{D}$ vero $\frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}$ terminos, et ita porro.

§. 5. Tres hi serierum, quas ex quantitatibus assumtis a, b, c, d, etc. triplici modo composuimus, ordines multifariam inter se connectuntur, ita vt vno serierum ordine cognito, bini reliqui ordines inde possint determinari. Atque in hoc negotio ad connexionis legem et rationem inuestigandam obseruatio atque inductio plu plurimum adhiberi ſolet; hocque pacto primum quidem certiſſime conſtat eſſe $A - \alpha = \mathfrak{A}$; ac de reliquis compertum eſt eſſe:

$$\alpha = A$$
$$\beta = \frac{\alpha A - B}{2}$$
$$\gamma = \frac{\beta A - \alpha B + C}{3}$$
$$\delta = \frac{\gamma A - \beta B + \alpha C - D}{4}$$
$$\varepsilon = \frac{\delta A - \gamma B + \beta C - \alpha D + E}{5}$$

etc. item

$$\mathfrak{A} = A$$
$$\mathfrak{B} = \frac{\mathfrak{A} A + B}{2}$$
$$\mathfrak{C} = \frac{\mathfrak{B} A + \mathfrak{A} B + C}{2}$$
$$\mathfrak{D} = \frac{\mathfrak{C} A + \mathfrak{B} B + \mathfrak{A} C + D}{4}$$
$$\mathfrak{E} = \frac{\mathfrak{D} A + \mathfrak{C} B + \mathfrak{B} C + \mathfrak{A} D + E}{5}$$

etc.

praetereaque

$$\mathfrak{A} = \alpha$$
$$\mathfrak{B} = \alpha \mathfrak{A} - \beta$$
$$\mathfrak{C} = \alpha \mathfrak{B} - \beta \mathfrak{A} + \gamma$$
$$\mathfrak{D} = \alpha \mathfrak{C} - \beta \mathfrak{B} + \gamma \mathfrak{A} - \delta$$
$$\mathfrak{E} = \alpha \mathfrak{D} - \beta \mathfrak{C} + \gamma \mathfrak{B} - \delta \mathfrak{A} + \varepsilon$$

etc.

Harumque relationum ope ex datis ſummis ſerierum cuiuscunque claſſis definiri poterunt ſummae ſerierum, quae in duabus reliquis claſſibus continentur.

§. 6. Ad naturam atque indolem harum ſerierum diligentius attendenti facile quidem per obſeruationem et inductionem veritas iſtius mutuae relationis patebit. Verum tamen quo magis de veritate huius nexus conuincamur, expediet ſequenti modo totum hoc negotium conſiderare; quo ſimul aliae inſuper proprietates nobis offerentur, ad quas ſola inductio non tam facile viam aperit. Aſſumtis ſcilicet pro libitu quantitatibus

$$a, b, c, d, e, \text{ etc.}$$

ex iisque formatis trium claſſium ſeriebus ſupra memoratis, contemplemur hanc expreſſionem:

$$P = \frac{az}{1-az} + \frac{bz}{1-bz} + \frac{cz}{1-cz} + \frac{dz}{1-dz} + \frac{ez}{1-ez} + \text{etc.}$$

cuius ſinguli termini in progreſſiones geometricas reſoluti more ſolito, dabunt

$$\begin{aligned} P = &+ z\,(a + b + c + d + e + \text{etc.}) \\ &+ z^2\,(a^2 + b^2 + c^2 + d^2 + e^2 + \text{etc.}) \\ &+ z^3\,(a^3 + b^3 + c^3 + d^3 + e^3 + \text{etc.}) \\ &+ z^4\,(a^4 + b^4 + c^4 + d^4 + e^4 + \text{etc.}) \end{aligned}$$

quae ſeries omnes in prima claſſe continentur. Quare ſi earum loco ſummae ſupra (2) poſitae ſcribantur fiet:

$$P = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$$

cuius idcirco ſeriei ſumma erit, vti ſumſimus

$$P = \frac{az}{1-az} + \frac{bz}{1-bz} + \frac{cz}{1-cz} + \frac{dz}{1-dz} + \text{etc.}$$

Simili autem modo ſi fuerit:

$$Q = \frac{az}{1+az} + \frac{bz}{1+bz} + \frac{cz}{1+cz} + \frac{dz}{1+dz} + \text{etc.}$$

erit per ſeries primae claſſis:

$$Q = Az - Bz^2 + Cz^3 - Dz^4 + Ez^5 - \text{etc.}$$

§. 7.

§. 7. Consideremus porro hanc expressionem:
$R=(1+az)(1+bz)(1+cz)(1+dz)(1+ez)$ etc. cuius factores, si actu in se multiplicentur; ac termini secundum exponentes ipsius z disponantur fiet coëfficiens ipsius z aequalis summae quantitatum assumtarum a, b, c, d, etc. Coëfficiens ipsius z^2 erit aggregatum omnium productorum ex binis inaequalibus; coëfficiens ipsius z^3 erit aggregatum omnium productorum ex ternis inaequalibus et ita porro: ex quibus sequitur fore

$$R=1+\alpha z+\beta z^2+\gamma z^3+\delta z^4+\varepsilon z^5+\text{ etc.}$$

secundum definitiones supra (§. 3) datas.
Quodsi autem ponatur:
$S(1-az)(1-bz)(1-cz)(1-dz)(1-ez)$ etc. erit faciendo tantum z negatiuo

$$S=1-\alpha z+\beta z^2-\gamma z^3+\delta z^4-\varepsilon z^5+\text{ etc.}$$

§. 8. Vt series hae R et S cum praecedentibus P et Q comparentur, notandum est esse
$lR=l(1+az)+l(1+bz)+l(1+cz)+l(1+dz)+$ etc. vnde sumendis differentialibus erit:

$$\frac{dR}{Rdz}=\frac{a}{1+az}+\frac{b}{1+bz}+\frac{c}{1+cz}+\frac{d}{1+dz}+\text{ etc.}$$

quae per z multiplicata dat illam ipsam expressionem quam supra Q vocauimus, ita vt sit: $Q=\frac{zdR}{Rdz}$. Simili autem modo erit $\frac{dS}{Sdz}=\frac{-a}{1-az}-\frac{b}{1-bz}-\frac{c}{1-cz}-$ etc. vnde habebitur $P=\frac{-zdS}{Sdz}$.

§. 9. Cum nunc sit $R=1+\alpha z+\beta z^2+\gamma z^3+$ etc. erit $\frac{zdR}{dz}=\alpha z+2\beta z^2+3\gamma z^3+4\delta z^4+5\varepsilon z^5+$ etc. ideoque $Q=Az-Bz^2+Cz^3-Dz^4+Ez^5-$ etc.

$$= \frac{\alpha z + 2\beta z^2 + 3\gamma z^3 + 3\delta z^4 + 5\varepsilon z^5 + \text{etc.}}{1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}}$$

At ex aequalitate harum expreſſionum ſequuntur ſequentes relationes inter litteras A, B, C, D, E, etc. et α, β, γ, δ, ε, etc.

$$A = \alpha$$
$$\alpha A - B = 2\beta$$
$$\beta A - \alpha B + C = 3\gamma$$
$$\gamma A - \beta B + \alpha C - D = 4\delta$$
$$\delta A - \gamma B + \beta C - \alpha D + E = 5\varepsilon$$
etc.

Simili vero modo ex altera aequatione $P = \frac{-z\,dS}{S\,dz}$ ſequitur $P = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$

$$= \frac{\alpha z - 2\beta z^2 + 3\gamma z^3 - 4\delta z^4 + 5\varepsilon z^5 - \text{etc.}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \varepsilon z^5 + \text{etc.}}$$

quae pariter easdem praebet determinationes, quas ſupra §. 5. tradidimus.

§. 10. Praeterea autem ex aequatione $Q = \frac{z\,dR}{R\,dz}$ conſequimur integrando $\int \frac{Q\,dz}{z} = l\,R$. Quoniam vero eſt $Q = Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}$ erit $\int \frac{Q\,dz}{z} = Az - \frac{Bz^2}{2} + \frac{Cz^3}{3} - \frac{Dz^4}{4} + \text{etc.}$ cuius ſeriei valor itaque exprimet logarithmum huius ſeriei $R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$

Quemadmodum igitur eſt:

$$l(1 + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.}) = Az - \tfrac{1}{2}Bz^2 + \tfrac{1}{3}Cz^3 - \tfrac{1}{4}Dz^4 + \text{etc.}$$

ita etiam ex aequatione $\int \frac{P\,dz}{z} = -\,l\,S$ erit

$$l(1 - \alpha z + \beta z^2 - \gamma z^3 + \text{etc.}) = -Az - \tfrac{1}{2}Bz^2 - \tfrac{1}{3}Cz^3 - \tfrac{1}{4}Dz^4 - \text{etc.}$$

Quare ſi k ſcribatur pro numero, cuius logarithmus hyperbolicus eſt $= 1$, habebitur:

$$1 + \alpha z + \beta z^2 + \gamma^3 + \delta z^4 + \text{etc.} = k^{Az - \frac{1}{2}Bz^2 + \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 + \text{etc.}}$$

et

et

$$1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.} = k^{-Az - \frac{1}{2}Bz^2 - \frac{1}{3}Cz^3 - \frac{1}{4}Dz^4 - \text{etc.}}$$

§. 11. Notatu praeterea dignae sant expressiones harum R et S reciprocae nempe $\frac{1}{R}$ et $\frac{1}{S}$. Est vero $\frac{1}{S} = \frac{1}{(1-az)(1-bz)(1-cz)(1-dz)\,\text{etc.}}$ ad cuius fractionis valorem per seriem, cuius termini secundum potestates ipsius z progrediantur, exprimendum, perspicuum est in se inuicem multiplicari oportere cunctas has progressiones geometricas

$$\frac{1}{1-az} = 1 + az + a^2z^2 + a^3z^3 + a^4z^4 + \text{etc.}$$

$$\frac{1}{1-bz} = 1 + bz + b^2z^2 + b^3z^3 + b^4z^4 + \text{etc.}$$

$$\frac{1}{1-cz} = 1 + cz + c^2z^2 + c^3z^3 + c^4z^4 + \text{etc.}$$

$$\frac{1}{1-dz} = 1 + dz + d^2z^2 + d^3z^3 + d^4z^4 + \text{etc.}$$

In Producto autem post primum terminum 1 coëfficiens ipsius z erit summa quantitatum $a + b + c + d +$ etc. coëfficiens ipsius z erit summa factorum ex binis non excipiendo factores aequales in eodem facto; coëfficiens ipsius z^3 etit summa factorum ex ternis, et ita porro quas productorum summas supra (§. 4) litteris alphabethi geometrici $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, etc. designauimus.

His itaque litteris introductis habebimus:

$$\frac{1}{S} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \mathfrak{E}z^5 + \text{etc.}$$

atque simili modo valorem ipsius R tractando erit:

$$\frac{1}{R} = 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \mathfrak{E}z^5 + \text{etc.}$$

§. 12. Hae ergo series reciprocae sunt earum, quas supra sub litteris R et S (§. 7) protulimus. Atque hanc ob causam erit:

1 =

$1 = (1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.})\,(1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.})$ pariterque

$1 = (1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.})\,(1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.})$

Ex vtraque autem ſequitur vna eademque relatio inter valores litterum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. et $\alpha, \beta, \gamma, \delta$, erit ſcilicet:

$$\mathfrak{A} - \alpha = 0$$
$$\mathfrak{B} - \alpha\mathfrak{A} + \beta = 0$$
$$\mathfrak{C} - \alpha\mathfrak{B} + \beta\mathfrak{A} - \gamma = 0$$
$$\mathfrak{D} - \alpha\mathfrak{C} + \beta\mathfrak{B} - \gamma\mathfrak{A} + \delta = 0$$

etc.

quam eandem relationem iam ſupra (§. 5) tradidimus.

§. 13. Quodſi ponamus $\frac{1}{R} = T$ et $\frac{1}{S} = V$, vt ſit

$$T = 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.}$$

et $V = 1 - \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$

erit $\frac{dR}{R} = -\frac{dT}{T}$ et $\frac{dS}{S} = -\frac{dV}{V}$; hincque

fiet $P = \frac{z\,dV}{V\,dz}$ et $Q = -\frac{z\,dT}{T\,dz}$. Quare cum

ſit $\frac{z\,dV}{dz} = \mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.}$

et $-\frac{z\,dT}{dz} = \mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}$

habebimus loco P et Q valores debitos ex (§. 6.) ſcribendo has aequationes

$$Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.} = \frac{\mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc}}{1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}}$$

et

$$Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.} = \frac{\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}}{1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.}}$$

ex quibus eadem ſequitur relatio inter litteras A, B, C, D, etc. et $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. quam ſupra (§. 5.) dedimus. Erit ſcilicet

$\mathfrak{A}$

$$\mathfrak{A} = A$$
$$2\,\mathfrak{B} = \mathfrak{A}A + \mathfrak{B}$$
$$3\,\mathfrak{C} = \mathfrak{B}A + \mathfrak{A}B + C$$
$$4\,\mathfrak{D} = \mathfrak{C}A + \mathfrak{B}B + \mathfrak{A}C + D$$
$$5\,\mathfrak{E} = \mathfrak{D}A + \mathfrak{C}B + \mathfrak{B}C + AD + E$$
etc.

§. 14 Ex aequationibus (§. 12.) datis ſequitur fore

$$l(1+\alpha z+\beta z^2+\gamma z^3+\text{etc.}) = -l(1-\mathfrak{A}z+\mathfrak{B}z^2-\mathfrak{C}z^3+\text{etc.})$$
et
$$l(1-\alpha z+\beta z^2-\gamma z^3+\text{etc.}) = -l(1+\mathfrak{A}z+\mathfrak{B}z^2+\mathfrak{C}z^3+\text{etc.})$$

His igitur ad §. 10. accommodatis erit:

$$l(1-\mathfrak{A}z+\mathfrak{B}z^2-\mathfrak{C}z^3+\text{etc.}) = -Az+\tfrac{1}{2}Bz^2-Cz^3+\tfrac{1}{4}Dz^4-\text{etc.}$$
et
$$l(1+\mathfrak{A}z+\mathfrak{B}z^2+\mathfrak{C}z^3+\text{etc.}) = Az+\tfrac{1}{2}Bz^2+\tfrac{1}{3}Cz^3+\tfrac{1}{4}Dz^4+\text{etc.}$$

Hincque ſumto k pro numero, cuius logarithmus $=1$ erit

$$1-\mathfrak{A}z+\mathfrak{B}z^2-\mathfrak{C}z^4+\text{etc.} = k^{-Az+\frac{1}{2}Bz^2-\frac{1}{3}Cz^3+\frac{1}{4}Dz^4-\text{etc.}}$$
atque
$$1+\mathfrak{A}z+\mathfrak{B}z^2+\mathfrak{C}z^4+\text{etc.} = k^{Az+\frac{1}{2}Bz^2+\frac{1}{3}Cz^3+\frac{1}{4}Dz^4+\text{etc.}}$$

§. 15. Si iam litterae R et S retineant valores ſupra aſſumtos (§. 7.) erit

$$1+\alpha z+\beta z^2+\gamma z^3+\delta z^4+\text{etc.} = R$$
$$1-\mathfrak{A}z+\mathfrak{B}z^2-\mathfrak{C}z^3+\mathfrak{D}z^4-\text{etc.} = \frac{1}{R}$$
et
$$1-\alpha z+\beta z^2-\gamma z^3+\delta z^4-\text{etc.} = S$$
$$1+\mathfrak{A}z+\mathfrak{B}z^2+\mathfrak{C}z^3+\mathfrak{D}z^4+\text{etc.} = \frac{1}{S}$$

Ex quibus deducuntur ſequentia conſectaria.

$$1+\beta z^2+\delta z^4+\zeta z^6+\theta z^8+\text{etc.} = \frac{R+S}{2}$$
$$\alpha z+\gamma z^3+\varepsilon z^5+\eta z^7+\iota z^9+\text{etc.} = \frac{R-S}{2}$$

$$1 + \mathfrak{B}z^2 + \mathfrak{D}z^4 + \mathfrak{F}z^6 + \mathfrak{H}z^8 + \text{etc.} = \frac{R+S}{2RS}$$

$$\mathfrak{A}z + \mathfrak{C}z^3 + \mathfrak{E}z^5 + \mathfrak{G}z^7 + \mathfrak{J}z^9 + \text{etc.} = \frac{R-S}{2RS}$$

hincque colligitur ista proportio:

$$1 + \beta z^2 + \delta z^4 + \zeta z^6 + \text{etc.} : \alpha z + \gamma z^3 + \varepsilon z^5 + \eta z^7 + \text{etc.} =$$

$$1 + \mathfrak{B}z^2 + \mathfrak{D}z^4 + \mathfrak{F}z^6 + \text{etc.} : \mathfrak{A}z + \mathfrak{C}z^3 + \mathfrak{E}z^5 + \mathfrak{G}z^7 + \text{etc.} =$$

Cum praeterea sit:

$$R - 1 = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

$$1 - \frac{1}{R} = \mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$$

erit

$$R = \frac{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}}{\mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}}$$

similique modo propter

$$1 - S = \alpha z - \beta z^2 + \gamma z^3 - \delta z^4 + \text{etc.}$$

$$\frac{1}{S} - 1 = \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$$

erit

$$S = \frac{\alpha z - \beta z^2 + \gamma z^3 - \delta z^4 + \text{etc.}}{\mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}}$$

§. 16. Deinde vero si vt supra (§. 6.) ponamus

$$P = Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}$$

$$Q = Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}$$

erit ex paragr. 9.

$$\alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.} = QR$$

$$\alpha z - 2\beta z^2 + 3\gamma z^3 - 4\delta z^4 + \text{etc.} = PS$$

similique modo ex paragr. 13 habebitur

$$\mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.} = \frac{P}{S}$$

$$\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.} = \frac{Q}{R}$$

Exquibus sequentia corollaria facile deriuantur

$$\frac{\alpha z - 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.}}{Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}} = S = \frac{Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.}}{\mathfrak{A}z + 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 + 4\mathfrak{D}z^4 + \text{etc.}}$$

$$\frac{\alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.}}{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}} = R = \frac{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}}{\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}}$$

Pro

Pro litteris igitur R et S habemus quintuplices valores hos:

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

$$R = \frac{1}{1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \text{etc.}}$$

$$R = \frac{\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc}}{\mathfrak{A}z - \mathfrak{B}z^2 + \mathfrak{C}z^3 - \mathfrak{D}z^4 + \text{etc.}}$$

$$R = \frac{\alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + \text{etc.}}{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}}$$

$$R = \frac{Az - Bz^2 + Cz^3 - Dz^4 + \text{etc.}}{\mathfrak{A}z - 2\mathfrak{B}z^2 + 3\mathfrak{C}z^3 - 4\mathfrak{D}z^4 + \text{etc.}}$$

qui posito $-z$ loco z totidem praebent valores pro S. Atque ex horum quinque valorum multiplici combinatione quam plurimae proprietates elici possunt, quas terni litterarum nostrarum ordines, scillcet: A, B, C, D etc. α, β, γ, δ, etc. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. inter se tenent, quibus autem euoluendis hic supersedemus.

§. 17. His, quae latissime ptent, paraemissis atque expositis ad magis particularia descendamus, ac primo quidem pro serie litterarum $a, b, c, d,$ etc. accipiatur progressio geometrica infinita haec: $n, n^2, n^3, n^4, n^5, n^6,$ etc. qua in formulas superiores successiue introducta habebimus:

$$A = n + n^2 + n^3 + n^4 + n^5 + \text{etc.} = \frac{n}{1-n}$$

$$B = n^2 + n^4 + n^6 + n^8 + n^{10} + \text{etc.} = \frac{nn}{1-nn}$$

$$C = n^3 + n^6 + n^9 + n^{12} + n^{15} + \text{etc.} = \frac{n^3}{1-n^3}$$

$$D = n^4 + n^8 + n^{12} + n^{16} + n^{20} + \text{etc.} = \frac{n^4}{1-n^4}$$

etc.

Iam ex §. 6 duplices pro litteris P et Q nanciscimur vares, qui erunt:

$$P = \frac{nz}{1-nz} + \frac{n^2z}{1-n^2z} + \frac{n^3z}{1-n^3z} + \frac{n^4z}{1-n^4z} + \text{etc.}$$

 Q =

$Q = \frac{nz}{1-nz} + \frac{n^2z}{1+n^2z} + \frac{n^3z}{1+n^3z} + \frac{n^4z}{1+n^4z} + \text{etc.}$
hincque ex inuentis litterarum A, B, C, D, etc. valoribus nafcentur hi alteri:

$P = \frac{nz}{1-n} + \frac{nnz^2}{1-nn} + \frac{n^3z^3}{1-n^3} + \frac{n^4z^4}{1-n^4} + \text{etc.}$

$Q = \frac{nz}{1-n} - \frac{n^2z^2}{1-nn} + \frac{n^3z^3}{1-n^3} - \frac{n^4z^4}{1-n^4} + \text{etc.}$

§. 18. Ex paragr. porro 7 habebimus pro R et S fequentes expreffiones:

$R = (1+nz)(1+n^2z)(1+n^3z)(1+n^4z)$ etc.

$S = (1-nz)(1-n^2z)(1-n^3z)(1-n^4z)$ etc.

qui factores actu in fe multiplicati, et producta fecundum dimenfiones ipfius z ordinata praebebunt pro R et S has feries:

$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$

$S = 1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.}$

vbi litterae α, β, γ, δ, etc ex ferie affumta

$n, n^2, n^3, n^4, n^5, n^6, n^7,$ etc.

ita determinabuntur, vt fit:

I. α = fummae fingulorum terminorum; vnde erit:

$\alpha = n + n^2 + n^3 + n^4 + n^5 + n^6 + n^7 + \text{etc.}$

quae eft ipfa progreffio geometrica affumta in qua quaevis poteftas ipfius n occurrit, atque coëfficientem habet $+1$.

II. β = fummae factorum ex binis terminis: vnde erit:

$\beta = n^3 + n^4 + 2n^5 + 2n^6 + 3n^7 + 3n^8 + 4n^9 + 4n^{10} + \text{etc.}$

in qua ferie poft poteftatem tertiam omnes fequentes ipfius n poteftates occurrunt: quaelibet autem poteftas toties occurrit, quoties ex multiplicatione binorum terminorum feriei α oriri poteft. Cum autem multiplicatio poteftatum confiftat in exponentium additione, coëfficiens cuiusque poteftatis ipfius n in ferie β oftendet, quot variis

riis modis exponens ipsius n possit in duas partes inaequales distribui, seu quoties iste exponens ex additione duorum numerorum integrorum inaequalium produci queat. Sic potestatis decimae n^{10} coefficiens est 4, quia 10 quatuor modis in duas partes inaequales distribui potest nempe $10 = 1 + 9$; $10 = 3 + 7$.
$10 = 2 + 8$; $10 = 4 + 6$.

III. $\gamma =$ summae factorum ex ternis terminis seriei α inaequalibus; vnde erit:

$$\gamma = n^6 + n^7 + 2n^8 + 3n^9 + 4n^{10} + 5n^{11} + 7n^{12} + 8n^{13} + \text{etc.}$$

in qua post potestatem sextam omnes sequentes ipsius n potestates occurrunt. Cuiuslibet autem potestatis coëfficiens indicat, quot variis modis exponens distribui possit in tres partes inaequales, seu quoties idem exponens produci queat ex additione trium numerorum integrorum inter se inaequalium. Sic potestas n^{12} coëfficientem habet 7, quia exponens 12 septem modis in tres partes inaequales partiri potest: vti

$12 = 1 + 2 + 9$; $12 = 2 + 3 + 7$
$12 = 1 + 3 + 8$; $12 = 2 + 4 + 6$
$12 = 1 + 4 + 7$; $12 = 3 + 4 + 5$
$12 = 1 + 5 + 6$;

IV. $\delta =$ summae factorum ex quatuor terminis seriei α inaequalibus inter se, vnde erit

$$\delta = n^{10} + n^{11} + 2n^{12} + 3n^{13} + 5n^{14} + 6n^{15} + 9n^{16} + \text{etc.}$$

cuius prima potestas est n^{10}, quippe cuius exponens est $1+2+3+4$ seu numerus trigonalis quartus. Sequentium potestatum quaelibet toties adest, quoties eius exponens oriri potest ex additione quatuor numerorum integrorum inter se inaequalium. Sic potestas sexta decima n^{16} coëfficientem habet 9, quia 16 nouem modis in quatuor partes inter ſe inaequales diſpertiri poteſt, quae nouem partitiones ſunt:

$16 = 1 + 2 + 3 + 10$; $16 = 1 + 3 + 4 + 8$
$16 = 1 + 2 + 4 + 9$; $16 = 1 + 3 + 5 + 7$
$16 = 1 + 2 + 5 + 8$; $16 = 1 + 4 + 5 + 6$
$16 = 1 + 2 + 6 + 7$; $16 = 2 + 3 + 4 + 7$
$16 = 2 + 3 + 5 + 6$

Simili modo res ſe habet in ſequentium litterarum ε, ζ, η, etc. valoribus qui erunt

$$\varepsilon = n^{15} + n^{16} + 2n^{17} + 3n^{18} + 5n^{19} + 7n^{20} + 10n^{21} + \text{etc.}$$
$$\zeta = n^{21} + n^{22} + 2n^{23} + 3n^{24} + 5n^{25} + 7n^{26} + 11n^{27} + \text{etc.}$$
$$\eta = n^{28} + n^{29} + 2n^{30} + 3n^{31} + 5n^{32} + 7n^{33} + 11n^{34} + \text{etc.}$$
etc.

in quibus ſeriebus omnibus cuiusuis ipſius n poteſtatis coëfficiens indicat, quot variis modis exponens ipſius n poſſit reſolui in tot partes inaequales, quota ſeries eſt a principio numerata. Seu coëfficiens cuiusque termini declarat, quoties exponens ipſius n oriri queat ex additione tot numerorum integrorum inter ſe inaequalium quota ipſa ſeries, ex qua terminus deſumitur, eſt, numerando a prima α. Sic in ſerie ſeptima coëfficiens poteſtatis n^{34} eſt 11, quia numerus 34 vndecim modis diſtribui poteſt in ſeptem partes inaequales, quae diſtributiones ſunt:

$34 = 1 + 2 + 3 + 4 + 5 + 6 + 13$
$34 = 1 + 2 + 3 + 4 + 5 + 7 + 12$
$34 = 1 + 2 + 3 + 4 + 5 + 8 + 11$
$34 = 1 + 2 + 3 + 4 + 5 + 9 + 10$
$34 = 1 + 2 + 3 + 4 + 6 + 7 + 11$
$34 = 1 + 2 + 3 + 4 + 6 + 8 + 10$

$34 = 1 + 2 + 3 + 4 + 7 + 8 + 9$
$34 = 1 + 2 + 3 + 5 + 6 + 7 + 10$
$34 = 1 + 2 + 3 + 5 + 6 + 8 + 9$
$34 = 1 + 2 + 4 + 5 + 6 + 7 + 9$
$34 = 1 + 3 + 4 + 5 + 6 + 7 + 8$

Atque ex his natura ſerierum, quae hoc pacto pro litteris α, β, γ, δ, etc. prodeunt, facile perſpicitur.

§. 19. Inueſtigando igitur, quot variis modis quisque numerus in partes inaequales numero datas diſtribui poſſit, ſeries iſtae litteris α, β, γ, δ, etc. ſignatae formari poterunt: quod autem opus foret ſummopere moleſtum. Viciſſim autem ex his ſeriebus aliunde cognitis et formatis reſolui poterit problema hoc non inelegans, quod mihi a Viro Clar. Naudaeo propoſitum ita ſehabet: „Definire, quot variis modis datus numerus produci queat ex additione aliquot numerorum integrorum inter ſe „inaequalium; quorum numerus detur.
Sic Clariſſ Propoſitor quaerit, quot variis modis numerus 50 oriri poſſit ex additione ſeptem numerorm integrorum inaequalium. Ad quam quaeſtionem reſoluendam manifeſtum eſt in ſubſidium vocari debere ſeriem η in qua coëfficiens cuiusque termini indicat, quot variis modis exponens ipſius n reſolui poſſit in 7 partes inaepuales. Quare ſeries illa

$\eta = n^{28} + n^{29} + 2n^{30} + 3n^{31} + 5n^{32} + 7n^{33} + 11n^{34} +$ etc.

cotninuari debebit vsque ad terminum, in quo poteſtas quinquageſima ipſius n continetur, cuius coëfficiens qui erit 522 oſtendet numerum 50 omnino 522 modis diverſis ex additione ſeptem numerorum integrorum interſe inaequalium produci poſſe. Ex quo perſpicuum eſt,

ſi

si modus habeatur commodus et facilis formandi illas series α, β, γ, δ, etc. eo ipso problema istud Naudaeanum perfectissime solutum iri.

§. 20. Cum igitur supra §. §. 5 et 9. modus traditus sit inueniendi valores litterarum α, β, γ, δ, etc. ex cognitis valoribus litterarum A, B, C, D, etc. in praesenti negotio resolutionem facile expedire poterimus propterea quod ex §. 17 cognitos habemus valores A, B, C, D, etc. atque praeterea est, vt sequitur

$$\alpha = A$$
$$\beta = \frac{\alpha A - B}{2}$$
$$\gamma = \frac{\beta A - \alpha B + C}{3}$$
$$\delta = \frac{\gamma A - \beta B + \alpha C - D}{4}$$
$$\varepsilon = \frac{\delta A - \gamma B + \beta C - \alpha D + E}{5}$$

etc.

Ex his igitur obtinebimus:

$$\alpha = \frac{n}{1-n}$$
$$2\beta = \frac{\alpha n}{1-n} - \frac{nn}{1-nn}$$
$$3\gamma = \frac{\beta n}{1-n} - \frac{\alpha n^2}{1-n^2} + \frac{n^3}{1-n^3}$$
$$4\delta = \frac{\gamma n}{1-n} - \frac{\beta n^2}{1-n^2} + \frac{\alpha n^3}{1-n^3} - \frac{n^4}{1-n^4}$$

etc.

Quod si autem loco α, β, γ, etc. successiue substituantur valores ante reperti, prodibunt:

$$\alpha = \frac{n}{1-n}$$
$$\beta = \frac{n^3}{(1-n)(1-nn)}$$
$$\gamma = \frac{n^6}{(1-n)(1-nn)(1-n^3)}$$
$$\delta = \frac{n^{10}}{(1-n)(1-n^2)(1-n^3)(1-n^4)}$$
$$\varepsilon = \frac{n^{15}}{(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)}$$

etc.

Ex

Ex his itaque intelligitur esse in hoc casu

$\alpha = A$

$\beta = AB$

$\gamma = ABC$

$\delta = ABCD$

$\varepsilon = ABCDE$

etc.

§. 21. Lex haec, qua valores litterarum $\alpha, \beta, \gamma, \delta$ etc. progredi sunt inuenti, compluribus formulis euolutis obseruatur, eiusque veritas nisi per inductionem adhuc non constat. Quo igitur haec veritas firmius confirmetur, conueniet eandem progressionis legem alio modo planissimo, in quo inductioni nullus locus relinquatur, elicere. Cum itaque nobis propositum sit valores literarum $\alpha, \beta, \gamma, \delta$, etc. indagare, quas sortiuntur in serie

$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$

Si fuerit vti initio assumsimus.

$R = (1 + nz)(1 + n^2 z)(1 + n^3 z)(1 + n^4 z) \ldots$

notandum est si loco z scribatur nz, expressionem, cui modo R erat aequalis, mutari in hanc formam

$(1 + n^2 z)(1 + n^3 z)(1 + n^4 z)(1 + n^5 z) \ldots$

quae multiplicata per $1 + nz$ ipsam priorem expressionem producit. Quamobrem recte concludimus, si in serie

$1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$

loco z scribamus nz vt habeamus

$1 + \alpha nz + \beta n^2 z^2 + \gamma n^3 z^3 + \delta n^4 z^4 + \varepsilon n^5 z^5 + \text{etc.}$

hancque expressionem per $1 + nz$ multiplicemus, tum productum, quod erit

$$\begin{aligned} 1 &+ \alpha nz + \beta n^2 z^2 + \gamma n^3 z^3 + \delta n^4 z^4 + \varepsilon n^5 z^5 + \text{etc.} \\ &+ nz + \alpha n^2 z^2 + \beta n^3 z^3 + \gamma n^4 z^4 + \delta n^5 z^5 + \text{etc.} \end{aligned}$$

aequale esse debere illi ipsi priori seriei

$1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 +$ etc.

Quodsi ergo actu coëfficientes ttrminorum homologorum coëquemus, nanciscemur sequentes pro α, β, γ, etc determinationes.

$$\alpha = \frac{n}{1-n} = \frac{n}{1-n}$$

$$\beta = \frac{\alpha n^2}{1-n^2} = \frac{n^3}{(1-n)(1-n^2)}$$

$$\gamma = \frac{\beta n^3}{1-n^3} = \frac{n^6}{(1-n)(1-n^2)(1-n^3)}$$

$$\delta = \frac{\gamma n^4}{1-n^4} = \frac{n^{10}}{(1-n)(1-n^2)(1-n^3)(1-n^4)}$$

etc.

§. 22. Hoc igitur modo inuenimus summas serierum illarum α, β, γ, δ etc. satis commode expressas ex quibus vicissim ipsae illae series formari poterunt. Nam cum illae series secundum potestates ipsius n progrediantur eae prodire debebunt, si istae expressiones summarum per diuisionem more consueto euoluantur atque in series infinitas secundum potestates ipsius n procedentes conuertantur. Quae operatio cum diuisione absoluatur manifestum est omnes illas series α, β, γ, δ, etc. ad id genus pertinere, quod nomine serierum recurrentium indicari solet; atque adeo quilibet terminus ex aliquot praecedentibus determinabitur. Vt autem pateat, quomodo in singulis his seriebus quisque terminus ex praecedentibus sit formandus, denominatores illarum expressionum pro litteris α, β, γ, δ, etc. inuentarum per multiplicationem actu euolui debent, quo facto habebitur:

$$\alpha = \frac{n}{1-n}$$

$$\beta = \frac{n^3}{1-n-n^2+n^3}$$

$$\gamma = \frac{n^6}{1-n-n^2+n^4+n^5-n^6}$$

$$\delta = \frac{n^{10}}{1-n-n^2+2n^5-n^8-n^9+n^{10}}$$

$\varepsilon =$

$$\varepsilon = \frac{n^{15}}{1-n-n^2+n^5+n^6+n^7-n^8-n^9-n^{10}+n^{13}+n^{14}-n^{15}}$$

$$\zeta = \frac{n^{21}}{1-n-n^2+n^5+2n^7-n^9-n^{10}-n^{11}-n^{12}+2n^{14}+n^{16}-n^{19}-n^{20}+n^{21}}$$

etc.

Atque ex his denominatoribus intelligitur, quomodo in singulis seriebus quisque terminus ex praecedentibus componi debeat, si praecepta, quae de formatione serierum recurrentium habentur, in subsidium vocentur.

§. 23. At ex forma expressionum pro litteris α, β, γ, δ, etc. inuentarum, qua quaelibet est productum ex praecedente in nouum quempiam factorem, alius deducitur modus satis idoneus ex quauis sirie iam inuenta seriem sequentem inueniendi. Sic, cum series $\alpha = \frac{n}{1-n}$ sit progressio geometrica

$$\alpha = n + n^2 + n^3 + n^4 + n^5 + n^6 + n^7 + \text{etc.}$$

ex hac reperietur series β, si ea multiplicetur per $\frac{n^2}{1-n^2}$, vel si multiplicetur per hanc progressionem geometricam:

$$n^2 + n^4 + n^6 + n^8 + n^{10} + n^{12} + n^{14} + \text{etc.}$$

Ex serie porro β hoc pacto inuenta, si ea multiplicetur per $\frac{n^3}{1-n^3} = n^3 + n^6 + n^9 + n^{12} + n^{15} + n^{18} + \text{etc.}$ producetur series γ. Haecque multiplicata per

$$\frac{n^4}{1-n^4} = n^4 + n^8 + n^{12} + n^{16} + n^{20} + n^{24} + \text{etc.}$$

producet seriem δ. Atque ita porro seriem cuiusque ordinis multiplicando per certam quamdam progressionem geometricam reperietur series sequens. Hocque pacto non difficulter has series quousque libuerit, continuare licebit: atque sic problema supra memoratum a Clar. Naudaeo propositum resoluetur.

§. 24. Facilius autem quaelibet series ex se ipsa ope praecedentis poterit continuari, si ad modum respeciamus quo valor cuiusque litterrarum α, β, γ, δ, etc. ex praecedente determinatur. Sic cum ſit $\beta = \frac{\alpha n^2}{1 - n^2}$ eri $\beta = \beta n n + \alpha n n$; quare ſi ad ſeriem β per $n n$ mutiplicatam addatur ſeries α per $n n$ multiplicata, ipſa ſeries β oriri debebit. Cum igitur conſtet ſeriei β primum terminum eſſe n^3 ponamus:

$$\beta = \mathfrak{a} n^3 + \mathfrak{b} n^4 + \mathfrak{c} n^5 + \mathfrak{d} n^6 + \mathfrak{e} n^7 + \mathfrak{f} n^8 + \mathfrak{g} n^9 + \text{etc.}$$

eritque

$$\beta n^2 = \qquad + \mathfrak{a} n^5 + \mathfrak{b} n^6 + \mathfrak{c} n^7 + \mathfrak{d} n^8 + \mathfrak{e} n^9 + \text{etc.}$$
$$\alpha n^2 = n^3 + n^4 + n^5 + n^6 + n^7 + n^8 + n^9 + \text{etc.}$$

Aequatis iam terminis propter $\beta = \beta n n + \alpha n n$ habebimus:

$\mathfrak{a} = 1$	$\mathfrak{e} = \mathfrak{c} + 1 = 3$
$\mathfrak{b} = 1$	$\mathfrak{f} = \mathfrak{d} + 1 = 3$
$\mathfrak{c} = \mathfrak{a} + 1 = 2$	$\mathfrak{g} = \mathfrak{e} + 1 = 4$
$\mathfrak{d} = \mathfrak{b} + 1 = 2$	$\mathfrak{h} = \mathfrak{f} + 1 = 4$

etc.

Simili modo cum ſit $\gamma = \frac{\beta n^3}{1 - n^3}$ ſeu $\gamma = \gamma n^3 + \beta n^3$; ex ſerie β formabitur ſeries γ, atque porro ex ſerie γ ope aequationis $\delta = \delta n^4 + \gamma n^4$ producetur ſeries δ; pariterque ſequentes omnes conficientur.

§. 25. Quoniam in expreſſione

$$R = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$$

valores litterarum α, β, γ, δ, etc. inuenimus, ſitque

$$R = (1 + nz)(1 + n^2 z)(1 + n^3 z)(1 + n^4 z) \ldots$$

conuertetur productum hoc ex infinitis factoribus conſtans:

$$(1 + nz)(1 + n^2 z)(1 + n^3 z)(1 + n^4 z) \ldots$$

in ſeriem hanc ſecundum poteſtates ipſius z procedentem

$$1 + \frac{nz}{1-n} + \frac{n^3 z^2}{(1-n)(1-n^2)} + \frac{n^6 z^3}{(1-n)(1-n^2)(1-n^3)} + \frac{n^{10} z^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)} + \text{etc}$$

Atque ſummae huius ſeriei logarithmus hyperbolicus ex §. 10 erit $= \frac{nz}{1-n} - \frac{nnz^2}{2(1-n^2)} + \frac{n^3 z^3}{3(1-n^3)} - \frac{n^4 z^4}{4(1-n^4)} + \text{etc.}$

Vel ſi k ſcribatur pro numero, cuius logar. $= 1$. erit

k

$$k^{\frac{nz}{1-n} - \frac{n^2z^2}{2(1-n^2)} + \frac{n^3z^3}{3(1-n^3)} - \frac{n^4z^4}{4(1-n^4)} + \text{etc.}} = R$$

ſeu iſta expreſſio exponentialis eſt aequalis ſummae illius ſeriei, in quam valorem ipſius R transmutanimus.

§. 26. Verum vt ad propoſitum problema reuertamur quo definiendum ſit, quot variis modis datus numerus m, partiri queat in μ partes inaequales inter ſe et integras; indicemus hunc modorum numerum, quem quaerimus, huiusmodi ſcriptione $m^{(\mu)i}$ qua nobis perpetuo numerus modorum indicetur, quibus numerus m per additionem produci queat ex μ numeris integris inter ſe inaequalibus; atque ad hanc partium inaequalitatem denotandam ſupra litteram i adiunximus; quae omittetur ſi quaeſtio formabitur, de numero modorum inveniendo, quibus datus numerus m omnino in μ partes tam aequales quam inaequales diſtribui queat. Quod problema poſtea pari facilitate ſolutum exhibebitur.

§. 27. Iſte ergo modorum numerus $m^{(\mu)i}$ erit coëfficiens poteſtatis n^m in illa ſerierum α, β, γ, δ, ε, etc. quae a prima α numerata in ordine eſt *tota* quot μ continet vnitates. Huius autem ſeriei ſumma eſt $=$

$$\frac{n^{\frac{\mu(\mu+1)}{1.2}}}{(1-n)(1-n^2)(1-n^3)(1-n^4)\ldots\ldots\ldots(1-n^\mu)}$$

ideoque ſeriei, quae ex hac forma naſcitur terminus generalis eſt $= m^{(\mu)i}n^m$. Seriei autem quae naſcitur ex hac forma

$$\frac{n^{\frac{\mu(\mu-1)}{1.2}}}{(1-n)(1-n^2)(1-n^3)(1-n^4)\ldots\ldots(1-n^\mu)}$$

terminus generalis erit $= m^{(\mu)i}n^{m-\mu}$, ſeu pro eadem ipſius n poteſtate erit terminus generalis $= (m+\mu)^{(\mu)i}n^m$. Subtrahatur prior expreſſio a poſteriore, atque reſiduae expreſſionis

$$\frac{n^{\frac{\mu(\mu-1)}{1\cdot 2}}}{(1-n)(1-n^2)(1-n^3)(1-n^4)\ldots\ldots(1-n^{\mu-1})}$$

terminus generalis erit $= n^m((m+\mu)^{(\mu)i} - m^{(\mu)i})$ huius autem eiusdem seriei terminus generalis est $m^{(\mu-1)i}\,n^m$, quocirca habebimus:

$$m^{(\mu-1)i} = (m+\mu)^{(\mu)i} - m^{(\mu)i}$$

vnde hanc adipiscimur regulam, vt sit

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-1)i}$$

cuius ope, si constiterit, quod varies modis numerus m distribui possit cum in μ partes, tum in $\mu-1$ partes inaequales, hos binos modorum numeros addendo reperietur, quot variis modis numerus maior $m+\mu$ distribui possit in μ partes inaequales. Atque ita resolutio casuum difficiliorum ad simpliciores reducitur, atque tandem ad simplicissimos per se notos; quippe constat, si fuerit $m < \frac{\mu\mu+\mu}{2}$ tum fore $m^{(\mu)i} = 0$, et si fuerit $m = \frac{\mu\mu+\mu}{2}$, tum erit $m^{(\mu)i} = 1$.

§. 28. Cum formula $m^{(\mu)i}$ sit terminus generalis huius expressionis

$$\frac{n^{\frac{\mu(\mu+1)}{2}}}{(1-n)(1-n^2)(1-n^3)\ldots\ldots(1-n^{\mu})}$$

Videamus qualem seriem praebeat ista expressio

$$\frac{1}{(1-n)(1-n^2)(1-^3)\ldots\ldots(1-n^{\mu})}$$

si euoluatur, atque secundum dimensiones ipsius n disponatur. Ponamus autem prodire hanc seriem

$$1 + an + \mathfrak{C}n^2 + 2n^3 + n^4 + \;\; n^5 + \text{etc.}$$

ex cuius generatione perspicitur, coëfficientem cuiusque potesta-

poteſtatis ipſius n monſtare quot variis modis exponens ipſius n per additionem produci queat ex his datis numeris

$$1, 2, 3, 4, 5, 6, \ldots \mu$$

hicque nec certus partium numerus praeſcribitur, ex quibus componatur, nec iſta conditio ponitur, vt partes ſint inter ſe inaequales. Hanc itaque ob cauſam expreſſio $m^{(\mu)i}$ ſimul indicabit, quot variis omnino modis numerus $m - \frac{\mu(\mu+1)}{2}$ per additionem prodici queat ex numeris $1, 2, 3, 4, 5, \ldots \mu$. Sic ſi quaeratur quot variis modis numerus 50 diſtribui poſſit in 7 partes inaequales, propter $m = 50$ et $\mu = 7$, quaeſtio eo reducitur, vt inueſtigetur quod varis modis nummerus $50 - 28$ ſeu 22 oriri queat per additionem ex his ſeptem numeris: $1, 2, 3, 4, 5, 6, 7$. Hoc ergo pacto duplicis generis quaeſtiones vna eademque opera reſolvuntur.

§. 29. Definitis hoc pacto littiris $\alpha, \beta, \gamma, \delta$, etc. pro caſu, quo loco litterarum a, b, c, d, etc. progreſſionem geometricam n, n^2, n^3, n^4, n^5, etc. infinitam aſſumſimus, ordo poſtulat, vt etiam in valores tertii ordinis $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$, etc. inquiramus. Adhibuimus autem has litteras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. in ſeriebus, his valoribus $\frac{1}{R}$, et $\frac{1}{S}$ aequalibus, ſumſimus enim ſupra §. 11. eſſe

$$\frac{1}{S} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \mathfrak{E}z^5 + \text{etc.}$$

et

$$\frac{1}{R} = 1 - \mathfrak{A}z + \mathfrak{B}z^2 - \mathfrak{C}z^3 + \mathfrak{D}z^4 - \mathfrak{E}z^5 + \text{etc.}$$

obtinentibus R et S valores primum aſſumtos quibus erat:

$R = (1 + nz)(1 + n^2 z)(1 + n^3 z)(1 + n^4 z)$ etc.
$S = (1 - nz)(1 - n^2 z)(1 - n^3 z)(1 - n^4 z)$ etc.
Intelligitur autem hinc feriem

$$\frac{1}{S} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.}$$

oriri, fi innumerabiles iftae progreffiones geometricae in fe inuicem multiplicentur.

$$\frac{1}{1-nz} = 1 + nz + n^2 z^2 + n^3 z^3 + n^4 z^4 + \text{etc.}$$
$$\frac{1}{1-n^2 z} = 1 + n^2 z + n^4 z^2 + n^6 z^3 + n^8 z^4 + \text{etc.}$$
$$\frac{1}{1-n^3 z} = 1 + n^3 z + n^6 z^2 + n^9 z^3 + n^{12} z^4 + \text{etc.}$$
$$\frac{1}{1-n^4 z} = 1 + n^4 z + n^8 z^2 + n^{12} z^3 + n^{16} z^4 + \text{etc.}$$
etc.

Pofito autem $-z$ loco z prodit fimili modo feries $\frac{1}{R}$.

§. 30. Ex ifta harum ferierum generatione manifeftum eft effe:

I. $\mathfrak{A} = n + n^2 + n^3 + n^4 + n^5 +$ etc.
quae eft progreffio geometrica omnes ipfius n poteftates complectens fingulas per coëfficientem $+ 1$ multiplicatas.

II. $\mathfrak{B} = n^3 + n^4 + 2n^5 + 2n^6 + 3n^7 + 3n^8 + 4n^9 + 4n^{10} +$ etc.
in qua coëfficiens cuiusque ipfius n poteftatis tot continet vnitates, quot variis modis exponens ipfius n in duas partes fiue aequales fiue inaequales partiri poteft. Sic poteftatis n^8 coëfficiens eft 4 quia 8 quatuor modis in 2 partes partitur

$8 = 1 + 7$; $8 = 3 + 5$
$8 = 2 + 6$; $8 = 4 + 4$

III. $\mathfrak{C} = n^6 + n^7 + 2n^8 + 3n^9 + 4n^{10} + 5n^{11} + 7n^{12}$ etc.
in qua cuiusque poteftatis ipfius n coëfficiens tot continet vnitates, quot variis modis exponens ipfius n in tres

tres partes siue aequales siue inaequales distribui potest, sic n^9 coëfficientem habet 7, quia 7 modis 9 in tres partes dispertiri se patitur.

$9 = 1 + 1 + 7$; $9 = 2 + 2 + 5$
$9 = 1 + 2 + 6$; $9 = 2 + 3 + 4$
$9 = 1 + 3 + 5$; $9 = 3 + 3 + 3$
$9 = 1 + 4 + 4$;

IV. $\mathfrak{D} = n^4 + n^5 + 2n^6 + 3n^7 + 5n^8 + 6n^9 + 9n^{10} +$ etc. vbi cuiusque potestatis ipsius n coëfficiens tot continet vnitates, quot variis modis exponens ipsius n in quatuor partes siue aequales siue inaequales resolui potest. Atque similis est ratio sequentium serierum; quae pro litteris $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, etc. reperiuntur.

§. 31. Harum ergo serierum ope alterum problema, quod simul cum praecedente Vir Cl. Naudaeus mihi proposuit, resolui potest, quot ita se habet.

„Inuenire quot variis modis datus numers m partiri „possit in μ partes tam aequales quam inaequales:
„Siue inueniri quot variis modis datus numerus m „per additionem μ numerorum integrorum siue aequa„lium siue inaequalium produci queat

Quod problema a praecedente eo tantum discrepat, quod in praecedente partitio ad partes tantum inter se inaequales sit restricta, haec autem partes quoque aequales admittat. Ad numerum autem omnium modorum in hoc problemate quaesitum signo exprimendum vtamur hac forma:

$$m^{(\mu)}$$

quae scilicet declaret, quot variis modis numerus m partiri queat in μ partes integras, partium aliquot aequalitate non exclusa: quamobrem in signo supra affixo (μ) ante adnexa littera i, qua inaequalitas partium indicabatur, hic est praetermissa.

§. 32. Solutio ergo huius problematis ad formationem serierum 𝔄, 𝔅, ℭ, 𝔇, 𝔈, etc. reducitur, at supra iam ostendimus (§. 5) quomodo harum litterarum valores ex valoribus litterarum α, β, γ, δ, etc. iam cognitis definiantur. Quanquam autem iste modus est generalis et ex rei natura petitus, tamen non satis dilucide legem, qua hi valores progrediuntur, ob oculos ponit. Quamobrem valores harum litterarum 𝔄, 𝔅, ℭ, 𝔇, 𝔈, etc. via huic casui propria inuestigabo, simili ei, qua supra (§. 21) vsus sum.

Quoniam est

$$\frac{1}{s} = \frac{1}{(1-nz)(1-n^2z)(1-n^3z)(1-n^4z)\,etc.}$$

perspicuum est, si in hac forma loco z scribatur nz, tum prodituram esse hanc formam

$$\frac{1}{(1-n^2z)(1-n^3z)(1-n^4z)(1-n^5z)\,etc.}$$

Ad ipsam autem hanc formam prior $\frac{1}{s}$ perducitur, si ea multiplicetur per $1-nz$. Hancobrem cum assumserimus esse

$$\frac{1}{s} = 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \mathfrak{E}z^5 + \text{etc.}$$

ponamus in hac nz loco z, habebimusque

$$1 + \mathfrak{A}nz + \mathfrak{B}n^2z^2 + \mathfrak{C}n^3z^3 + \mathfrak{D}n^4z^4 + \text{etc.}$$

Iam priorem seriem $\frac{1}{s}$ multiplicemus per $1-nz$

$$\begin{array}{l} 1 + \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \text{etc.} \\ \quad - nz - \mathfrak{A}nz^2 - \mathfrak{B}nz^3 + \mathfrak{C}nz^4 + \text{etc.} \end{array}$$

Quae forma cum illi esse debeat aequalis, erit

𝔄 =

$$\mathfrak{A} = \frac{n}{1-n} = \frac{n}{1-n}$$
$$\mathfrak{B} = \frac{\mathfrak{A}\,n}{1-n^2} = \frac{n^2}{(1-n)(1-n^2)}$$
$$\mathfrak{C} = \frac{\mathfrak{B}\,n}{1-n^3} = \frac{n^3}{(1-n)(1-n^2)(1-n^3)}$$
$$\mathfrak{D} = \frac{\mathfrak{C}\,n}{1-n^4} = \frac{n^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)}$$
etc.

§. 33. Hinc igitur noua percipitur relatio inter valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. et litterarum $\alpha, \beta, \gamma, \delta$, etc. quae eo magis eſt notatu digna quo minus hi valores a ſe inuicem diſcrepant Collato enim (§. 21) intelligitur eſſe:

$$\alpha = \mathfrak{A}$$
$$\beta = n\,\mathfrak{B}$$
$$\gamma = n^3\,\mathfrak{C}$$
$$\delta = n^6\,\mathfrak{D}$$
$$\varepsilon = n^{1}\,\mathfrak{E}$$
etc.

Manifeſtum ergo eſt ratione coëfficientium ſeries $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. omnino cum ſeriebus $\alpha, \beta, \gamma, \delta$, etc. congruere, totumque diſcrimen in exponentibus ipſius n ſitum eſſe. In ſerie quidem $\mathfrak{A}$, exponentes quoque aequales ſunt exponentibus in ſerie α; at in ſerie $\mathfrak{B}$ exponentes vnitate deficiunt ab exponentibus ſeriei β: in ſerie $\mathfrak{C}$ exponentes ternario deficiunt ab exponentibus ſeriei γ: et ita porro defectus ſecundum numeros trigonales progrediuntur.

§. 34. Ex ſeriebus ergo $\alpha, \beta, \gamma, \delta$, etc. quas ſupra formare docuimus, et quibus prius problema Naudaeanum reſoluitur, ſimul hoc poſterius problema a Naudaeo propoſitum ita reſolui poteſt, vt eius ſolutio reducatur ad ſolu-

ſolutionem prioris. Erit nempe

$$m^{(1)} = m^{(1)i}$$
$$m^{(2)} = (m+1)^{(2)i}$$
$$m^{(3)} = (m+3)^{(3)i}$$
$$m^{(4)} = (m+6)^{(4)i}$$

et generaliter

$$m^{(\mu)} = \left(m+\frac{\mu(\mu-1)}{2}\right)^{(\mu)i}$$

et viciſſim

$$m^{(\mu)i} = \left(m-\frac{\mu(\mu-1)}{2}\right)^{(\mu)}$$

Quoniam autem porro inuenimus eſſe:

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-1)i}$$

erit reductione ad caſum praeſentem facta:

$$\left(m-\frac{\mu(\mu-3)}{2}\right)^{(\mu)} = \left(m-\frac{\mu(\mu-1)}{2}\right)^{(\mu)} + \left(m-\frac{(\mu-1)(\mu-2)}{2}\right)^{(\mu-1)}$$

ſeu commodius

$$m^{(\mu)} = (m-\mu)^{(\mu)} + (m-1)^{(\mu-1)}$$

ex qua proprietate etiam facile ſeries litterarum 𝔄, 𝔅, ℭ, etc. formabuntur, ſicque hoc alterum problema reſoluetur.

§. 35. Ad exemplum huius problematis quaeſtionem Vir Clar. affert, vt determinetur, quot variis modis numerus 50 in ſeptem omnino partes ſiue aequales ſiue inaequales diſpertiri queat. Haec ergo quaeſtio ad prius problema reducetur, ob $m=50$ et $\mu=7$, ſi quaeratur quot variis modis numerus $50+21$, ſeu numerus 71, in ſeptem partes inaequales partiri queat. Vtrumque autem fieri poſſe 8946 modis diuerſis. Praeterea vero hic idem numerus 8946 indicat (§. 28), quot variis modis $71-28=43$ per additionem produci queat ex his numeris 1, 2, 3, 4, 5, 6, 7. Atque generaliter numerus modorum $m^{(\mu)}$, quibus numerus m in μ partes ſiue aequales ſiue in-

inaequales resoluitur, simul ostendit, quot variis modis numerus $m-\mu$ produci queat per additionem ex his numeris definitis

$$1, 2, 3, 4, 5, \ldots \mu.$$

§. 36. Finem huic dissertationi faciat obseruatio notatu digna, quam quidem rigore geometrico demostrare mihi nondum licuit. Obseruaui scilicet hoc infinitorum factorum productum

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}$$

si per multiplicationem actu euoluatur, praebere hanc seriem:

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+n^{51}+\text{etc.}$$

vbi eae tantum ipsius n potestates occurrunt, quarum exponentes continentur hac forma: $\frac{3xx\pm x}{2}$. Ac si x sit numerus impar, potestates ipsius n, quae sunt $n^{\frac{3xx\pm x}{2}}$ coëfficientem habent -1; si autem x sit numerus par, tum potestates $n^{\frac{3xx\pm x}{2}}$ coëfficientem habent $+1$.

§. 37. Praeterea notari meretur series huius reciproca, quae oritur ex euolutione huius fractionis

$$\frac{1}{(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}}$$

prodibit scilicet ista series recurrens:

$$1+1n+2n^2+3n^3+5n^4+7n^5+11n^6+15n^7+22n^8+\text{etc.}$$

quippe quae per seriem superioem

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-\text{etc.}$$

multiplicata producit vnitatem. In illa autem serie coëfficiens cuiusque potestatis ipsius n tot continet vnitates, quot variis modis exponens ipsius n in partes dispertiri potest; sic 5 septem modis in partes resolui potest, vti

$5 = 5$ | $5 = 3+2$ | $5 = 2+2+1$
$5 = 4+1,$ | $5 = 3+1+1$ | $5 = 2+1+1+1$
$5 = 1+1+1+1+1$

nec numerus scilicet partium hic praescribitur nec inaequalitas.