1750

Demonstratio gemina theorematis Neutoniani, quo traditur relatio inter coefficientes cuiusvis aequationis algebraicae et summas potestatum radicum eiusdem

Leonhard Euler

Demonstratio Gemina
THEOREMATIS NEUTONIANI
quo traditur relatio inter coëfficientes cujusvis
æquationis algebraicæ & summas potestatum
radicum ejusdem.

§. I.

Postquam æquatio algebraica tam a fractionibus quam ab irrationalitate fuerit liberata, atque ad hujusmodi formam reducta:

$$x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} - E x^{n-5} + \cdots + N = 0$$

demonstrari solet in analysi, hujusmodi æquationem tot semper habere radices, sive sint reales sive imaginariæ, quot unitates contineantur in potestatis summæ exponente n. Tum vero non minus certum est, si hujus æquationis omnes radices fuerint $\alpha, \varsigma, \gamma, \delta, \epsilon, \cdots \nu$, coëfficientes terminorum æquationis A, B, C, D, E, &c. ex his radicibus ita conflari, ut sit:

A = summæ omnium radicum = $\alpha + \varsigma + \gamma + \delta + \cdots + \nu$
B = summæ productorum ex binis = $\alpha\varsigma + \alpha\gamma + \alpha\delta + \varsigma\gamma + $&c.
C = summæ productorum ex ternis = $\alpha\varsigma\gamma + $ &c.
D = summæ productorum ex quaternis = $\alpha\varsigma\gamma\delta + $ &c.
E = summæ productorum ex quinis = $\alpha\varsigma\gamma\delta\epsilon + $ &c.

&c.

Ultimumque tandem terminum absolutum $\pm$ N esse productum ex omnibus radicibus $\alpha\varsigma\gamma\delta \ldots \nu$.

§. II

§. II. Quo jam theorema, cujus demonstrationem hic tradere conftitui, facilius ac brevius enunciare queam; defignet $f\alpha$ fummam omnium radicum; $f\alpha^2$ fummam quadratorum earumdem radicum; $f\alpha^3$ fummam cuborum radicum; $f\alpha^4$ fummam biquadratorum iftarum radicum, & ita porro: ita ut fit:

$$f\alpha = \alpha + \varepsilon + \gamma + \delta + \epsilon + \cdots + \nu$$

$$f\alpha^2 = \alpha^2 + \varepsilon^2 + \gamma^2 + \delta^2 + \epsilon^2 + \cdots + \nu^2$$

$$f\alpha^3 = \alpha^3 + \varepsilon^3 + \gamma^3 + \delta^3 + \epsilon^3 + \cdots + \nu^3$$

$$f\alpha^4 = \alpha^4 + \varepsilon^4 + \gamma^4 + \delta^4 + \epsilon^4 + \cdots + \nu^4$$

$$f\alpha^5 = \alpha^5 + \varepsilon^5 + \gamma^5 + \delta^5 + \epsilon^5 + \cdots + \nu^5$$

&c.

§. II. Hac fignandi ratione expofita Neutonus affirmat Istas poteftatum, quæ ex fingulis radicibus formantur, fummas per coefficientes æquationis A, B, C, D, E, &c. ita definiri, ut fit.

$$f\alpha = A$$

$$f\alpha^2 = A f\alpha - 2B$$

$$f\alpha^3 = A f\alpha^2 - B f\alpha + 3C$$

$$f\alpha^4 = A f\alpha^3 - B f\alpha^2 + C f\alpha - 4D$$

$$f\alpha^5 = A f\alpha^4 - B f\alpha^3 + C f\alpha^2 - D f\alpha + 5E$$

$$f\alpha^6 = A f\alpha^5 - B f\alpha^4 + C f\alpha^3 - D f\alpha^2 + E f\alpha - 6F$$

&c.

O 3

Cujus

Cujus theorematis demonstrationem Newtonus non folum nul-
lam tradit, fed etiam ipfe videtur ejus veritatem ex continua
illatione conclufiffe.   Primum enim demonftratione non eget
effe $f_a = A$: & cum fit

$$A = a + \overset{\bullet}{c} + \overset{\bullet}{r} + \overset{\bullet}{\delta} + \overset{\bullet}{e} + \&c. + 2ac + 2ar + 2a\delta + 2cr +$$

$2c\delta + \&c.$ erit $A = f_a + 2B$, ideoque $f_a = A - 2B = A f_a$
$- 2B$: fimilique modo veritas fequentium formularum evinci
poteft; fed continuo majore opus erit labore.

§. IV. Cum plures jam hujus theorematis utiliffimi verita-
tem oftenderint, eorum demonftrationes autem regulis combina-
tionum plerumque innitantur, quae etiamfi verae fint, tamen ab
inductione plurimum pendeant, duplicem hic afferam demonftratio-
nem, in quarum utraque inductioni nihil tribuitur.   Altera qui-
dem ex analyfi infinitorum eft petita, quae etfi nimis longe remo-
ta videatur, tamen totum negotium perfecte conficit: verum ta-
men cum contra eam jure objici queat, hujus theorematis verita-
tem evictam effe oportere, antequam ad analyfin infinitorum per-
veniatur; alteram demonftrationem adjungam, in qua nihil affu-
mitur, nifi quod ftatim ab initio in explicatione naturae aequati-
onum tradi folet.

Demonftratio I.

§. V. Ponatur $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + \ldots$
$+ N = Z$ & cum aequationis $Z = 0$ radices feu valores ipfius
x fint, $a, c, v, \delta, \ldots v$, quorum numerus eft $= n$, erit ex
natura aequationum:

$$Z = (x - a)(x - c)(x - v)(x - \delta) \ldots (x - v)$$

et

et logarithmis sumendis habebitur:

$$lZ = l(x-\alpha)+l(x-\varepsilon)+l(x-\gamma)+l(x-\delta)+\cdots$$
$$+l(x-\nu)$$

Quod si jam harum formularum differentialia capiantur erit:

$$\frac{dZ}{Z} = \frac{dx}{x-\alpha}+\frac{dx}{x-\varepsilon}+\frac{dx}{x-\gamma}+\frac{dx}{x-\delta}+\&c.$$
$$+\frac{dx}{x-\nu}$$

Ideoque per dx dividendo fiet:

$$\frac{dZ}{Zdx} = \frac{1}{x-\alpha}+\frac{1}{x-\varepsilon}+\frac{1}{x-\gamma}+\frac{1}{x-\delta}+\cdots+\frac{1}{x-\nu}$$

— Convertantur tunc singulae hæ fractiones more solito in series geometricas in finitas: ob

$$\frac{1}{x-\alpha}=\frac{1}{x}+\frac{\alpha}{x^2}+\frac{\alpha^2}{x^3}+\frac{\alpha^3}{x^4}+\frac{\alpha^4}{x^5}+\frac{\alpha^5}{x^6}+\&c.$$

$$\frac{1}{x-\varepsilon}=\frac{1}{x}+\frac{\varepsilon}{x^2}+\frac{\varepsilon^2}{x^3}+\frac{\varepsilon^3}{x^4}+\frac{\varepsilon^4}{x^5}+\frac{\varepsilon^5}{x^6}+\&c.$$

$$\frac{1}{x-\gamma}=\frac{1}{x}+\frac{\gamma}{x^2}+\frac{\gamma^2}{x^3}+\frac{\gamma^3}{x^4}+\frac{\gamma^4}{x^5}+\frac{\gamma^5}{x^6}+\&c.$$

$$\&c.$$

$$\frac{1}{y-\alpha}=\frac{1}{x}+\frac{\nu}{x^2}+\frac{\nu^2}{x^3}+\frac{\nu^3}{x^4}+\frac{\nu^4}{x^5}+\frac{\nu^5}{x^6}+\&c.$$

His

His igitur feriebus colligendis, fignisque ante expofitis f_a, $\overset{2}{f_a}$, $\overset{3}{f_a}$ &c. introducendis invenietur, quia numerus harum ferierum eft $= n$:

$$\frac{dZ}{Zdx} = \frac{n}{x} + \frac{1}{x^2}f_a + \frac{1}{x^3}\overset{2}{f_a} + \frac{1}{x^4}\overset{3}{f_a} + \frac{1}{x^5}\overset{4}{f_a} + \&c.$$

§. VI. Cum autem ftatuerimus:

$$Z = x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} - \cdots \pm N$$

erit fimiliter differentialibus fumendis:

$$\frac{dZ}{dx} = n x^{n-1} - (n-1) A x^{n-2} + (n-2) B x^{n-3} - (n-3) .$$
$$C x^{n-4} + (n-4) D x^{n-5} - \&c.$$

hincque colligetur fuperior formula $\dfrac{dZ}{Zdx}$ ita expreffa ut fit:

$$\frac{dZ}{Zdx} =$$

$$\frac{n x^{n-1} - (n-1) A x^{n-2} + (n-2) B x^{n-3} - (n-3) C x^{n-4} + (n-4) D x^{n-5} - \&c.}{x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} - \&c.}$$

quæ igitur fractio æqualis effe debet feriei fupra inventæ:

$$\frac{n}{x} + \frac{1}{x^2}f_a + \frac{1}{x^3}\overset{2}{f_a} + \frac{1}{x^4}\overset{3}{f_a} + \frac{1}{x^5}\overset{4}{f_a} + \&c.$$

Quare

Quare fi utraque expreffio pro $\dfrac{dZ}{Zdx}$ inventa per alterius denominatorem $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + \&c.$ multiplicetur refultabit hæc æquatio:

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c.$$

$$= nx^{n-1} + x^{n-2}f_a + x^{n-3}f_a{}^2 + x^{n-4}f_a{}^3 + x^{n-5}f_a{}^4 + \&c.$$

$$- nAx^{n-2} - Ax^{n-3}f_a - Ax^{n-4}f_a{}^2 - Ax^{n-5}f_a{}^3 - \&c.$$

$$+ nBx^{n-3} + Bx^{n-4}f_a + Bx^{n-5}f_a{}^2 + \&c.$$

$$- nCx^{n-4} - Cx^{n-5}f_a - \&c.$$

$$+ nDx^{n-5} + \&c.$$

§. VII. Quemadmodum jam utrinque termini primi nx^{n-1} funt æquales, neceffe eft ut & fecundi, tertii, quarti &c. inter fe feorfim æquentur; unde fequentes nafcentur æquationes:

$$-(n-1)A = f_a - nA$$
$$+(n-2)B = f_a{}^2 - Af_a + nB$$
$$-(n-3)C = f_a{}^3 - Af_a{}^2 + Bf_a - nC$$
$$+(n-4)D = f_a{}^4 - Af_a{}^3 + Bf_a{}^2 - Cf_a + nD$$
$$\&c.$$

harumque æquationum lex, qua progrediuntur, fponte eft

Euleri Opuscula Tom. II.   P   mani-

manifesta. Ex iis autem obtinentur formulæ illæ ipfæ, quibus theorema Neutonianum conftat; fcilicet:

$$\int a = A$$
$$\int a^2 = A \int a - 2B$$
$$\int a^3 = A \int a^2 - B \int a + 3C$$
$$\int a^4 = A \int a^3 - B \int a^2 + C \int a - 4D$$
$$\int a^5 = A \int a^4 - B \int a^3 + C \int a^2 - D \int a + 5E$$

Quæ eft altera theorematis propofiti demonftratio.

Demonftratio II.

§. VIII. Quo hujus demonftrationis vis clarius perfpiciatur, eam ad æquationem determinati gradus accommodabo, ita tamen ut ea intelligatur ad quosvis gradus æque patere. Sit ergo propofita æquatio quinti gradus:

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

cujus quinque radices fint a, c, γ, δ, e. Quia igitur quælibet radix loco x fubftituta æquationi fatisfacit, erit:

$$a^5 - A a^4 + B a^3 - C a^2 + D a - E = 0$$
$$c^5 - A c^4 + B c^3 - C c^2 + D c - E = 0$$
$$\gamma^5 - A \gamma^4 + B \gamma^3 - C \gamma^2 + D \gamma - E = 0$$
$$\delta^5 - B \delta^4 + B \delta^3 - C \delta^2 + D \delta - E = 0$$
$$e^5 - A e^4 + B e^3 - C e^2 + D e - E = 0$$

Colligentur hæ æquationes in unam fummam, & ob figna fupra recepta (§. 2.) habebitur:

$$\int a^5 - A \int a^4 + B \int a^3 - C \int a^2 + D \int a - 5E = 0$$
$$\text{feu } \int a^5 = A \int a^4 - B \int a^3 + C \int a^2 - D \int a + 5E.$$

§. IX.

§. IX. Hinc dilucidè patet, si æquatio proposita fuerit gradus cujuscunque

$$x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} - \ldots \pm M x \mp N = 0$$

ubi in ultimis terminis signorum ambiguorum superiora valent, si exponens summus n fuerit numerus impar, inferiora si par; fore pariter:

$$f^n = A f^{n-1} - B f^{n-2} + C f^{n-3} - \ldots \mp M f^a \pm n N$$

si quidem per a indicetur radix quælibet istius æquationis sicque veritas Theorematis Neutoniani jam pro uno casu est ostensa. Superest igitur, ut ejusdem veritatem tam pro altioribus quam pro inferioribus radicum potestatibus demonstremus.

§. X. Pro altioribus quidem potestatibus res pari modo patet, si enim valores $a, \xi, \gamma, \delta, \varepsilon$ satisfaciant æquationi

$$x^5 - A x^4 + B x^3 - C x^2 + D x - E = 0$$

satisfacient quoque sequentibus æquationibus:

$$x^6 - A x^5 + B x^4 - C x^3 + D x^2 - E x = 0$$
$$x^7 - A x^6 + B x^5 - C x^4 + D x^3 - E x^2 = 0$$
$$x^8 - A x^7 + B x^6 - C x^5 + D x^4 - E x^3 = 0$$

&c.

Ac propterea si in unaquaque æquatione pro x singuli valores $a, \xi, \gamma, \delta, \varepsilon$ substituantur, & aggregata colligantur, erit

$f a$

$$f\alpha^6 = A f\alpha^5 - B f\alpha^4 + C f\alpha^3 - D f\alpha^2 + E f\alpha$$

$$f\alpha^7 = A f\alpha^6 - B f\alpha^5 + C f\alpha^4 - D f\alpha^3 + E f\alpha^2$$

$$f\alpha^8 = A f\alpha^7 - B f\alpha^6 + C f\alpha^5 - D f\alpha^4 + E f\alpha^3$$

&c.

§. XI. Si ergo α denotet radicem quamcunque hujus æqua-tionis:

$$x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} - \ldots + M x \mp N = 0$$

erit non folum, uti jam invenimus:

$$f\alpha^n = A f\alpha^{n-1} - B f\alpha^{n-2} + C f\alpha^{n-3} - D f\alpha^{n-4} \ldots \mp M f\alpha \pm nN$$

fed etiam ad altiores quoque poteftates progrediendo erit:

$$f\alpha^{n+1} = A f\alpha^n - B f\alpha^{n-2} + C f\alpha^{n-3} - D f\alpha^{n-4} \ldots \mp M f\alpha^2 \pm N f\alpha$$

$$f\alpha^{n+2} = A f\alpha^{n+1} - B f\alpha^n + C f\alpha^{n-1} - D f\alpha^{n-2} + \ldots \mp M f\alpha^3 \pm N f\alpha^2$$

$$f\alpha^{n+3} = A f\alpha^{n+2} - B f\alpha^{n+1} + C f\alpha^n - D f\alpha^{n-1} + \ldots \mp M f\alpha^4 \pm N f\alpha^3$$

&c.

& in genere quidem, fi ad n addatur numerus quicunque m, erit

$$f\alpha^{n+m} = A f\alpha^{n+m-1} - B f\alpha^{n+m-2} + C f\alpha^{n+m-3} - \ldots \mp M f\alpha^{m+1} \pm N f\alpha^m.$$

Ubi quidem notandum eft, fi fit $m = 0$, ob fingulas poteftates $\alpha^0 = 1$, $c^0 = 1$, $\gamma^0 = 1$; &c. numerumque harum litterarum $= n$

fore

fore $f\alpha = n$, quo casu formula primo inventa in hac expressione continetur.

§. XII. Quamquam autem haec expressio aeque veritati est consentanea, si pro *m* accipiatur numerus negativus: hincque pro aequatione quinti gradus assumta

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

sequentes formulae pariter locum habent:

$$f\alpha^4 = Af\alpha^3 - Bf\alpha^2 + Cf\alpha^1 - Df\alpha^0 + Ef\alpha^{-1}$$

$$f\alpha^3 = Af\alpha^2 - Bf\alpha^1 + Cf\alpha^0 - Df\alpha^{-1} + Ef\alpha^{-2}$$

$$f\alpha^2 = Af\alpha^1 - Bf\alpha^0 + Cf\alpha^{-1} - Df\alpha^{-2} + Ef\alpha^{-3}$$

$$\&c.$$

tamen hae formulae sunt diversae ab iis, quas theorema continet. Demonstrandum enim est esse:

$$f\alpha^4 = Af\alpha^3 - Bf\alpha^2 + Cf\alpha - 4D$$

$$f\alpha^3 = Af\alpha^2 - Bf\alpha + 3C$$

$$f\alpha^2 = Af\alpha - 2B$$

$$f\alpha = A$$

Harum igitur formularum veritatem sequenti modo ostendo.

§. XIII. Proposita scilicet aequatione quinti gradus:

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0.$$

For-

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Formentur retinendis iisdem coëfficientibus fequentes æquatio-
nes inferiorum graduum:

I. $x - A = 0.$ Radix fit p

II. $x^2 - Ax + B = 0.$ Radix quælibet fit q

III. $x^3 - Ax^2 + Bx - C = 0.$ Sit radix quælibet r

IV. $x^4 - Ax^3 + Bx^2 - Cx + D = 0.$ Radix quælibet s

Quarum æquationum radices, etiamfi inter fe maxime difcre-
pent, tamen in his fingulis æquationibus eandem conftituent fum-
mam $= A$.   Deinde remota prima fumma productorum ex bi-
nis radicibus ubique erit eadem $= B$: Tum fumma productorum
ex ternis radicibus ubique erit $= C$, præter æquationes fcilicet I
& II, ubi C non occurrit.   Similiter in IV & propofita fum-
ma productorum ex quaternis radicibus erit eadem $= D$.

§. XIV.   In quibus autem æquationibus non felum
fumma radicum eft eadem, fed etiam fumma productorum ex bi-
nis radicibus, ibi quoque fumma quadratorum radicum eft eadem.
Sin autem præterea fumma productorum ex ternis radicibus fue-
rit eadem, tum fumma quoque cuborum omnium radicum erit
eadem. Atque fi infuper fumma productorum ex quaternis ra-
dicibus fuerit eadem, tum quoque fumma biquadratorum omnium
radicum erit eadem atque ita porro.   Hic fcilicet affumo, quod
facile concedetur, fummam quadratorum per fummam radicum
& fummam productorum ex binis determinari; fummam cubo-
rum autem præterea requirere fummam factorum ex ternis radici-
bus; ac fummam biquadratorum præterea fummam factorum ex qua-
ternis

ternis radicibus, & ita porro; quod quidem demonſtratu non eſ-
ſet difficile.

§. XV. In æquationibus ergo inferiorum graduum, qua-
rum radices denotantur reſpective per litteras p, q, r, s, dum ipſius
propoſitæ quinti gradus quælibet radix littera * indicatur, erit:

$$\int a = \int s = \int r = \int q = \int p$$

$$\int a^2 = \int s^2 = \int r^2 = \int q^2$$

$$\int a^3 = \int s^3 = \int r^3$$

$$\int a^4 = \int s^4$$

At per ea quæ ante §. 9 demonſtravimus eſt

$$\int p = A$$

$$\int q^2 = A\int q - 2B$$

$$\int r^3 = A\int r^2 - B\int r + 3C$$

$$\int s^4 = A\int s^3 - B\int s^2 + C\int s - 4D$$

Hinc ergo nanciſcimur pro æquatione quinti gradus propo-
ſita: $x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$ has formulas

$$\int a$$

$$f^{\underline{a}} = A$$

$$f^{\underline{a}} = Af^{\underline{a}} - 2B$$

$$f^{\underline{a}} = Af^{\underline{a}} - Bf^{\underline{a}} + 3C$$

$$f^{\underline{a}} = Af^{\underline{a}} - Bf^{\underline{a}} + Cf^{\underline{a}} - 4D$$

§. XVI. In æquatione ergo cujuscunque gradus propoſita:

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. \cdots \pm N = 0$$

ſi quælibet radix littera a indicetur erit:

$$f^{\underline{a}} = A$$

$$f^{\underline{a}} = Af^{\underline{a}} = 2B$$

$$f^{\underline{a}} = Af^{\underline{a}} - Bf^{\underline{a}} + 3C$$

$$f^{\underline{a}} = Af^{\underline{a}} - Bf^{\underline{a}} + Cf^{\underline{a}} - 4D$$

$$f^{\underline{a}} = Af^{\underline{a}} - Bf^{\underline{a}} + Cf^{\underline{a}} - Df^{\underline{a}} + 5E$$

&c.

Hocque modo veritas Theorematis Neutoniani pariter habetur demonſtrata.

E|llip
ut
neis rect
expectar
jus integ
ab irratic
regration
quidem
tenendu
me ſequi
trum elli
cularibus
curvæ al
ellipſis,
ri queant

§.
tam pori
lipſis per
ſum etia
nes finit
hil admo
præter
Euleri

Ani-