



1750

# De seriebus quibusdam considerationes

Leonhard Euler

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## DE SERIEBVS QVIBVSDAM CONSIDERATIONES.

AVCTORE  
*Leonb. Eulero.*

§. I.

**P**ostquam inuenissem serierum reciprocarum hac forma contentarum

$$1 \pm \frac{1}{3^n} \pm \frac{1}{5^n} \pm \frac{1}{7^n} \pm \frac{1}{9^n} \pm \frac{1}{11^n} \pm \text{etc.}$$

vbi ambiguum signorum superiora valent, si  $n$  est numerus par, inferiora vero si  $n$  est numerus impar, summas a quadratura circuli pendere, ac per tantam peripheriae circuli  $\pi$  potestatem determinari, cuius exponens sit  $= n$ ; nonnullae se mihi obtulerunt obseruationes, cum ad has ipsas series, tam ad earum vsum in summandis aliis seriebus spectantes. Quae cum non admodum sint obuiaae, ac fortasse ad alia negotia vtilitatem non spernendam asferre queant, eas hic exponere non abs re fore sum arbitratus.

§. 2. Posita constanter ratione diametri ad circuli peripheriam vt 1 ad  $\pi$ , considero circulum, cuius radius seu semidiameter sit  $= x$ , et denotabit  $\pi$  eius semicircumferentiam seu arcum 180 graduum. Quod si nunc accipiat in hoc circulo arcus  $= s$ , cuius sinus sit  $= y$ ; cosinus  $= x$ , et tangens  $= t$ ; erit

$$y = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

$$x = 1 - \frac{s^2}{1 \cdot 2} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

G 3

etque

atque hinc

$$0 = 1 - s - \frac{s^2}{1 \cdot 2} + \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

feu

$$0 = 1 - \frac{s}{1} + \frac{s^2}{1 \cdot 2} + \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

§. 3. Consideremus primo aequationem, qua relatio inter finem  $y$  et arcum  $s$  continetur, ac manifestum est, valorem  $s$  pro dato  $y$  non esse constantem, sed omnes eos arcus denotare, quorum idem est communis finis  $y$ . Sit arcuum horum minimus  $= \frac{m}{n} \pi$ , habebunt omnes sequentes arcus

$$\frac{m}{n} \pi, \frac{n-m}{n} \pi, \frac{2n-m}{n} \pi, \frac{3n-m}{n} \pi, \frac{4n-m}{n} \pi, \text{etc.}$$

$$-\frac{n-m}{n} \pi; -\frac{2n-m}{n} \pi; -\frac{3n-m}{n} \pi; -\frac{4n-m}{n} \pi; -\frac{5n-m}{n} \pi \text{ etc.}$$

eundem communem finem  $y$ , Quocirca huius aequationis:

$$0 = 1 - \frac{s}{1 \cdot y} + \frac{s^2}{1 \cdot 2 \cdot y^2} - \frac{s^3}{1 \cdot 2 \cdot 3 \cdot y^3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot y^4} - \text{etc.}$$

habebuntur sequentes innumerabiles factores:

$$\left(1 - \frac{ns}{m\pi}\right) \left(1 + \frac{ns}{(n-m)\pi}\right) \left(1 - \frac{ns}{(2n-m)\pi}\right) \left(1 + \frac{ns}{(3n-m)\pi}\right) \left(1 - \frac{ns}{(4n-m)\pi}\right) \text{etc.}$$

§. 4. Hinc itaque valores ipsius  $\frac{s}{y}$  constituent sequentem seriem:

$$\frac{n}{m\pi} + \frac{n}{(n-m)\pi} - \frac{n}{(2n-m)\pi} - \frac{n}{(3n-m)\pi} + \frac{n}{(4n-m)\pi} + \frac{n}{(5n-m)\pi} - \text{etc.}$$

Horum itaque summa aequalis erit coefficienti ipsius  $-s$  in aequatione, qui est  $= \frac{1}{y}$ : Summa factorum ex binis erit  $= 0$ , summa ex ternis  $= -\frac{1}{1 \cdot 2 \cdot 3 \cdot y}$ , etc. vti sequitur:

summ. terminorum	$= \frac{1}{y}$
summ. fact. ex binis	$= 0$
summ. fact. ex ternis	$= -\frac{1}{1 \cdot 2 \cdot 3 \cdot y}$
summ. fact. ex quaternis	$= 0$

sum.

summ. fact. ex quinis	=	$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$
summ. fact. ex senis	=	0
summ. fact. ex septenis	=	$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$
summ. fact. ex octonis	=	0

etc.

§. 5. Quod si autem generatim seriei cuiuscunque  $a + b + c + d + e +$  etc. fuerit

summa ipsorum terminorum	=	$\alpha$
summa factorum ex binis	=	$\xi$
summa factorum ex ternis	=	$\gamma$
summa factorum ex quaternis	=	$\delta$
summa factorum ex quinis	=	$\epsilon$
summa factorum ex senis	=	$\zeta$

etc.

poterint ex his summae quadratorum, cuborum, biquadratorum, et potestatum quarumvis terminorum huius seriei assignari. Quodsi enim fit

$a + b + c + d$	etc.	=	A
$a^2 + b^2 + c^2 + d^2$	+ etc.	=	B
$a^3 + b^3 + c^3 + d^3$	+ etc.	=	C
$a^4 + b^4 + c^4 + d^4$	+ etc.	=	D
$a^5 + b^5 + c^5 + d^5$	+ etc.	=	E
$a^6 + b^6 + c^6 + d^6$	+ etc.	=	F

etc.

sequenti modo istarum summarum valores determinabuntur.

$$\begin{aligned} A &= \alpha \\ B &= \alpha A - 2 \xi \\ C &= \alpha B - \xi A + 3 \gamma \\ D &= \alpha C - \xi B + \gamma A - 4 \delta \end{aligned}$$

$$E =$$

$$\begin{aligned} E &= \alpha D - \beta C + \gamma B - \delta A + 5 \varepsilon \\ F &= \alpha E - \beta D + \gamma C - \delta B + \varepsilon A - 6 \zeta \\ &\text{etc.} \end{aligned}$$

Quae progressio cum facili legem teneat, et ex terminis praecedentibus quivis terminus expedite definiri possit, poterimus seriei superioris valores ipsius  $z$  exhibentis summam potestatum quarumcunque terminorum definire.

§. 6. Antequam autem hanc generalem progressionem reliquamus, notari conveniet singularem proprietatem, quam valores litterarum A, B, C, D etc. inter se tenent. Oriuntur ii scilicet ex evolutione huius expressionis

$$\alpha - 2\beta z + 3\gamma z^2 - 4\delta z^3 + 5\varepsilon z^4 - 6\zeta z^5 + 7\eta z^6 - \text{etc.}$$

$$1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \varepsilon z^5 + \zeta z^6 - \text{etc.}$$

si quidem per divisionem actualem quotus secundum potestates ipsius  $z$  eruatur. Prodit namque divisione consueto more instituta sequens quotus  $A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$  ita ut ista series aequalis sit illi fractioni. Praeterea nosandum est, si seriei  $1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.}$  summa ponatur  $= Z$ , ita ut sit  $Z$  denominator illius fractionis, fore numeratorem  $= \frac{-dz}{bz}$ . Ex quo seriei  $A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$  summa erit  $= \frac{-dz}{Zbz}$ . Non solum itaque ex datis factis binorum, ternorum, quaternorum etc. summae potestatum seriei propositae  $a + b + c + d + \text{etc.}$  scilicet valores litterarum A, B, C, D, etc. poterunt inueniri, sed etiam summa seriei, quam hae ipsae potestates in nouam progressionem geometricam respecti-

speciè ducti, nimirum huius seriei  
 $A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$  summa poterit assignari. Hancque proprietatem probe notasse in sequentibus plurimum iuuabit, ubi in nouas series sumus inquisituri.

§. 7. Cum igitur huius seriei :

$\frac{x}{x} \left( \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} - \text{etc.} \right)$   
 dentur primo ipsorum terminorum summa, tum etiam summae factorum ex binis, ternis, quaternis et ita porro,

$$\begin{aligned} A &= \frac{1}{1y} \\ B &= \frac{A}{1y} \\ C &= \frac{B}{1y} - \frac{1}{1 \cdot 2y} \\ D &= \frac{C}{1y} - \frac{A}{1 \cdot 2 \cdot 3y} \\ E &= \frac{D}{1y} - \frac{B}{1 \cdot 2 \cdot 3y} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4y} \\ F &= \frac{E}{1y} - \frac{C}{1 \cdot 2 \cdot 3y} + \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y} \\ G &= \frac{F}{1y} - \frac{D}{1 \cdot 2 \cdot 3y} + \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6y} \quad \text{etc.} \end{aligned}$$

erit vt sequitur

$$\begin{aligned} \frac{x}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \text{etc.} &= \frac{A\pi}{n} \\ \frac{x}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.} &= \frac{B\pi^2}{n^2} \\ \frac{x}{m^3} + \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} + \text{etc.} &= \frac{C\pi^3}{n^3} \\ \frac{x}{m^4} + \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} - \frac{1}{(2n-m)^4} + \frac{1}{(2n+m)^4} + \text{etc.} &= \frac{D\pi^4}{n^4} \\ \frac{x}{m^5} + \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} - \frac{1}{(2n-m)^5} + \frac{1}{(2n+m)^5} + \text{etc.} &= \frac{E\pi^5}{n^5} \\ \frac{x}{m^6} + \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(2n-m)^6} + \frac{1}{(2n+m)^6} + \text{etc.} &= \frac{F\pi^6}{n^6} \end{aligned}$$

etc.

Tom. XII.

H

vbi

vbi pro potestatibus paribus omnes termini habent signum +, pro imparibus vero signa conueniunt cum signis ipsius seriei primae.

§. 8. Retineant litterae A, B, C, D, E, etc. valores, quos ipsis modo tribuimus, sitque nobis haec series proposita

$$A + Bz + Cz^2 + Dz^3 + Ez^4 \text{ etc.}$$

cuius summam ex regula §. 6 data inuestigemus. Huius

autem seriei summa inde est  $= \frac{dZ}{z dz}$  existente  $Z = 1 - \frac{z}{y}$   
 $+ \frac{z^2}{1+2z^2y} - \frac{z^4}{1+2z^4y} + \frac{z^6}{1+2z^6y} - \text{etc.} = 1 - \frac{z}{y} \sin. A \cdot z.$  Ex  
 quo ob  $y$  hoc loco constans ponendum erit  $dZ = \frac{-dz \cos. Az}{y}$   
 ac propterea summa seriei propositae

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc}$$

erit  $= \frac{\cos. A \cdot z}{y - \sin. Az}$ . Hinc erit istius seriei summa  $Az + Bz^2$

$$+ Cz^3 + Dz^4 + Ez^5 + \text{etc.} = \frac{z \cos. A \cdot z}{y - \sin. Az}$$

§. 9. Sit  $z = \frac{p\pi}{n}$ , exprimet haec series summam omnium harum seriei:

$$+ \frac{p}{m} + \frac{p}{n-m} - \frac{p}{n+m} - \frac{p}{2n-m} + \frac{p}{2n+m} + \text{etc.}$$

$$+ \frac{p^2}{m^2} + \frac{p^2}{(n-m)^2} + \frac{p^2}{(n+m)^2} + \frac{p^2}{(2n-m)^2} + \frac{p^2}{(2n+m)^2} + \text{etc.}$$

$$+ \frac{p^3}{m^3} + \frac{p^3}{(n-m)^3} - \frac{p^3}{(n+m)^3} - \frac{p^3}{(2n-m)^3} + \frac{p^3}{(2n+m)^3} + \text{etc.}$$

etc.

Haec autem series verticaliter additae dant

$$\frac{p}{m-p} + \frac{p}{n-m-p} - \frac{p}{n+m-p} - \frac{p}{2n-m-p} + \frac{p}{2n+m-p} + \text{etc.}$$

cuius seriei igitur summa est  $= \frac{p \pi \cos. A \cdot \frac{p\pi}{n}}{ny - n \sin. A \cdot \frac{p\pi}{n}}$  seu cum

$y$  sit

$y$  fit finus arcus  $\frac{m\pi}{n}$ , habebitur istius seriei summa

$$= \frac{p \pi \cos. A. \frac{p\pi}{n}}{n \sin. A. \frac{m\pi}{n} - n \sin. A. \frac{p\pi}{n}}$$

Quod si ponatur  $m - p = a$

et  $m + p = b$  ita vt fit  $m = \frac{a+b}{2}$  et  $p = \frac{b-a}{2}$  prodibit huius seriei

$$\frac{1}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.}$$

sive huius

$$\frac{a}{a} + \frac{2b}{n^2-b^2} - \frac{2a}{4n^2-a^2} + \frac{2b}{9n^2-b^2} - \frac{2a}{16n^2-a^2} + \frac{2b}{25n^2-b^2} - \text{etc.}$$

$$\text{summa} = \frac{\pi \cos. A. \frac{(b-a)\pi}{2n}}{n \sin. A. \frac{(b+a)\pi}{2n} - n \sin. A. \frac{(b-a)\pi}{2n}}$$

§. 10. Verum haec nimis sunt generalia, vt difficulter omnia, quae in iis comprehenduntur, perspici queant. Quamobrem ad specialiora descendamus, ac ponamus sinum  $y =$  sinui toti  $= 1$ : erit  $m = 1$  et  $n = 2$ . Hinc igitur sequentes nanciscimur series

$$\begin{aligned} \frac{1}{2} + \frac{1}{2} - \frac{1}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{5} - \frac{1}{7} - \text{etc.} &= \frac{A\pi}{2} \\ \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} &= \frac{B\pi^2}{2^2} \\ \frac{1}{3^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{7^3} - \frac{1}{9^3} - \text{etc.} &= \frac{C\pi^3}{2^3} \\ \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} &= \frac{D\pi^4}{2^4} \\ &\text{etc.} \end{aligned}$$

feu hae

$$\begin{aligned} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} &= \frac{A\pi}{2^2} \\ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} &= \frac{B\pi^2}{2^2} \\ 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} &= \frac{C\pi^3}{2^4} \\ 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} &= \frac{D\pi^4}{2^6} \end{aligned}$$

H 2

I



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$$\begin{aligned} I &= 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{E\pi^6}{26} \\ I &= 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = \frac{F\pi^6}{27} \\ I &= 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = \frac{G\pi^7}{28} \\ I &= 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = \frac{H\pi^8}{29} \end{aligned}$$

etc.

Valores autem litterarum A. B., C., D etc. ex sequenti lege inuenientur.

$$\begin{aligned} A &= I \\ B &= \frac{A}{I} \\ C &= \frac{B}{I} - \frac{1}{1 \cdot 2} \\ D &= \frac{C}{I} - \frac{A}{1 \cdot 2 \cdot 3} \\ E &= \frac{D}{I} - \frac{B}{1 \cdot 2 \cdot 3} + \frac{I}{1 \cdot 2 \cdot 3 \cdot 4} \\ F &= \frac{E}{I} - \frac{C}{1 \cdot 2 \cdot 3} + \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ G &= \frac{F}{I} - \frac{D}{1 \cdot 2 \cdot 3} + \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{I}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\ H &= \frac{G}{I} - \frac{E}{1 \cdot 2 \cdot 3} + \frac{C}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \end{aligned}$$

etc.

vnde reperiuntur sequentes valores litterarum.

$$\begin{aligned} A &= 1 \cdot \frac{\pi^2}{24} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} \\ B &= \frac{1}{1} \cdot \frac{\pi^2}{24} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} \\ C &= \frac{1}{1 \cdot 2} \cdot \frac{\pi^3}{24} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \text{etc.} \\ D &= \frac{2}{1 \cdot 2 \cdot 3} \cdot \frac{\pi^4}{25} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} \\ E &= \frac{5}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{\pi^5}{26} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.} \\ F &= \frac{16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{\pi^6}{27} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} \\ G &= \frac{61}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{\pi^7}{28} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.} \\ H &= \frac{272}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \cdot \frac{\pi^8}{29} = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.} \end{aligned}$$

I =

$$I = \frac{1385}{1 \cdot 2 \cdot 3 \dots 8} \cdot \frac{\pi^9}{2^{10}} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \text{etc.}$$

$$K = \frac{70 \cdot 6}{1 \cdot 2 \cdot 3 \dots 9} \cdot \frac{\pi^{10}}{2^{11}} = 1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.}$$

$$L = \frac{50 \cdot 21}{1 \cdot 2 \cdot 3 \dots 10} \cdot \frac{\pi^{11}}{2^{12}} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \text{etc.}$$

$$M = \frac{347 \cdot 292}{1 \cdot 2 \cdot 3 \dots 11} \cdot \frac{\pi^{12}}{2^{13}} = 1 + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \text{etc.}$$

$$N = \frac{2702 \cdot 765}{1 \cdot 2 \cdot 3 \dots 12} \cdot \frac{\pi^{13}}{2^{14}} = 1 - \frac{1}{3^{13}} + \frac{1}{5^{13}} - \frac{1}{7^{13}} + \text{etc.}$$

$$O = \frac{22364256}{1 \cdot 2 \cdot 3 \dots 13} \cdot \frac{\pi^{14}}{2^{15}} = 1 + \frac{1}{3^{14}} + \frac{1}{5^{14}} + \frac{1}{7^{14}} + \text{etc.}$$

§. 11. Denotant hic litterae A, B, C, etc. numerales tantum coefficientes potestatum  $\pi$  per potestates binarii diuisarum: quarum valores etsi satis commode ex lege data definiiri possunt, tamen alia lex potest exhiberi, quae magis ad calculum videtur expedita. Considero scilicet feriem  $A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$  cuius summa, quae tantisper designetur littera  $s$ , est per §. 8.  $= \frac{\cos A \cdot z}{1 - \sin A \cdot z}$ , ob  $y = 1$ . Quod si igitur ex hac aequatione  $s = \frac{\cos A \cdot z}{1 - \sin A \cdot z}$  valor ipsius  $s$  in serie exprimitur, quae secundum potestates ipsius  $z$  progrediatur, prodire debet ipsa series  $A + Bz + Cz^2 + Dz^3 + \text{etc.}$  Nulla enim alia series similis formae puta  $P + Qz + Rz^2 + Sz^3 + \text{etc.}$  assignari potest aequalis illi  $A + Bz + Cz^2 + Dz^3 + \text{etc.}$  quin simul coefficientes potestatum  $z$  congruant, sitque  $P = A$ ;  $Q = B$ ;  $R = C$ ;  $S = D$ , etc. At vero exprimit  $\frac{\cos A \cdot z}{1 - \sin A \cdot z}$  tangentem arcus  $\frac{\pi}{4} + \frac{z}{2}$  seu erit  $s = \text{tang. } A \left( \frac{\pi}{4} + \frac{z}{2} \right)$  et hancobrem conuertendo  $\frac{\pi}{4} + \frac{z}{2} = A \text{ tang. } s = \int \frac{ds}{1 + s^2}$  sumtisque differentialibus ob  $\frac{\pi}{4}$  constans seu arcum 45 graduum, habebitur  $\frac{dz}{2} = \frac{ds}{1 + s^2}$  siue  $dz + s^2 dz = 2 ds$ . Nunc ponatur  $s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$  erit

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$$\begin{aligned} \frac{z^2 ds}{dz} &= 2B + 4Cz + 6Dz^2 + 8Ez^3 + 10Fz^4 + \text{etc.} \\ ss &= A^2 + 2ABz + 2ACz^2 + 2ADz^3 + 2AEz^4 + \text{etc.} \\ r &= +1 + B^2z^2 + 2BCz^3 + 2BDz^4 + \text{etc.} \\ &+ C^2z^4 + \text{etc.} \end{aligned}$$

Comparatis nunc terminis homogeneis inter se valores litterarum ita definiantur, vt coefficientes singularum potestatum ipsius  $z$  euanescent; atque sequentes litterarum  $A, B, C, D, E$ , etc. obtinebuntur determinationes, existente vt iam inuenimus  $A = 1$ .

$$\begin{aligned} A &= 1 \\ B &= \frac{A^2 + 1}{2} \\ C &= \frac{2AB}{4} \\ D &= \frac{2AC + B^2}{6} \\ E &= \frac{2AD + 2BC}{8} \\ F &= \frac{2AE + 2BD + C^2}{10} \\ G &= \frac{2AF + 2BE + 2CD}{12} \\ H &= \frac{2AG + 2BF + 2CE + D^2}{14} \\ &\text{etc.} \end{aligned}$$

Atque hinc eadem prorsus determinationes litterarum  $A, B, C, D$  etc. prodibunt, quas altera lex supra data §. 10 suppeditat.

§. 12. Cum denominatores fractionum, quibus litterae  $A, B, C, D$ , etc. aequales sunt inuentae, satis regulariter progrediantur, potest hinc peculiaris regula ad inueniendos numeratores reperiri: Ponamus enim

$$A =$$

$A = \alpha$	$F = \frac{\alpha^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$	
$B = \frac{\alpha^2}{2}$	$G = \frac{\alpha^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$	
$C = \frac{\alpha^3}{1 \cdot 2}$	$H = \frac{\alpha^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$	
$D = \frac{\alpha^4}{1 \cdot 2 \cdot 3}$	$I = \frac{\alpha^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}$	
$E = \frac{\alpha^5}{1 \cdot 2 \cdot 3 \cdot 4}$	$K = \frac{\alpha^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}$	etc.

eritque factis substitutionibus haec lex :

$\alpha = 1$	$\varepsilon = \alpha \delta + 3 \beta \gamma$
$\beta = \frac{\alpha^2 + 1}{2}$	$\zeta = \alpha \varepsilon + 4 \beta \delta + 3 \gamma^2$
$\gamma = \alpha \beta$	$\eta = \alpha \zeta + 5 \beta \varepsilon + \frac{5 \cdot 4}{1 \cdot 2} \gamma \delta$
$\delta = \alpha \gamma + \beta^2$	$\theta = \alpha \eta + 6 \beta \zeta + \frac{6 \cdot 5}{1 \cdot 2} \gamma \varepsilon + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{\delta^2}{\beta}$
$\iota = \alpha \theta + 7 \beta \eta + \frac{7 \cdot 6}{1 \cdot 2} \gamma \zeta + \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \delta \varepsilon$	
$\kappa = \alpha \iota + 8 \beta \theta + \frac{8 \cdot 7}{1 \cdot 2} \gamma \eta + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \delta \zeta + \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{\varepsilon^2}{\beta}$	

Lex haec perspicua est, si hoc modo notetur, quoties postremus terminus sit quadratum, eum insuper per binarium diuidi debere.

§. 13. Consideremus nunc hanc seriem :

$$A z + B z^3 + G z^5 + D z^7 + E z^9 + \text{etc.}$$

cuius summam constat esse  $= \frac{z \cos A \cdot z}{1 - \sin A \cdot z}$ , ac ponatur

$$z = \frac{p\pi}{2}, \text{ erit } \frac{p\pi \cos A \cdot \frac{p\pi}{2}}{2 - 2 \sin A \cdot \frac{p\pi}{2}} =$$

$$\frac{A\pi}{2^2} \cdot 2p + \frac{B\pi^2}{2^3} \cdot 2p^3 + \frac{C\pi^3}{2^4} \cdot 2p^5 + \frac{D\pi^4}{2^5} \cdot 2p^7 + \text{etc.}$$

$$\text{seu } \frac{\pi \cos A \cdot \frac{p\pi}{2}}{4 - 4 \sin A \cdot \frac{p\pi}{2}} = \frac{A\pi}{2^2} + \frac{p^3 B\pi^2}{2^3} + \frac{p^5 C\pi^3}{2^4} + \frac{p^7 D\pi^4}{2^5} + \text{etc.}$$

Quodsi ergo loco singulorum terminorum substituantur series

ries ex §. 10 prodibit  $\frac{\pi \cos. A. \frac{p\pi}{2}}{4 - 4 \sin. A. \frac{p\pi}{2}} =$

$$\begin{aligned}
 &+ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} \\
 &+ p + \frac{p^2}{3} + \frac{p^2}{5} + \frac{p^2}{7} + \frac{p^2}{9} + \text{etc.} \\
 &+ p^2 - \frac{p^2}{3^3} + \frac{p^2}{5^3} + \frac{p^2}{7^3} + \frac{p^2}{9^3} - \text{etc.} \\
 &+ p^3 + \frac{p^3}{3^4} + \frac{p^3}{5^4} + \frac{p^3}{7^4} + \frac{p^3}{9^4} + \text{etc.} \\
 &\text{etc.}
 \end{aligned}$$

Omnes autem hae series deorsum additae abeunt in hanc:

$$\frac{x}{1-p} - \frac{x}{3+p} + \frac{x}{5-p} - \frac{x}{7+p} + \frac{x}{9-p} - \text{etc.}$$

cuius adeo summa est  $= \frac{\pi \cos. A. \frac{p\pi}{2}}{4 - 4 \sin. A. \frac{p\pi}{2}}$

§. 14. Plures huius generis series summabiles derivare licebit ex §. 9. serie sub finem exposita. Ponamus  $a = b = m$ : et habebimus hanc:

$$\frac{2m}{n^2 - m^2} - \frac{2m}{4n^2 - m^2} + \frac{2m}{9n^2 - m^2} - \frac{2m}{16n^2 - m^2} + \frac{2m}{25n^2 - m^2} - \text{etc.}$$

cuius summa erit  $= \frac{\pi}{n \sin. A. \frac{m\pi}{n}} \cdot \frac{1}{m}$  ob  $\cos. A. \circ \pi = 1$

et  $\sin. A. \circ \pi = 1$ . Quamobrem habebitur, si per  $2m$  diuidatur

$$\frac{1}{n^2 - m^2} - \frac{1}{4n^2 - m^2} + \frac{1}{9n^2 - m^2} - \frac{1}{16n^2 - m^2} + \frac{1}{25n^2 - m^2} - \text{etc.}$$

$$= \frac{\pi}{2mn \sin. A. \frac{m\pi}{n}} - \frac{1}{2m/m}. \text{ Ponamus porro } a = -m \text{ et}$$

$$b = +m, \text{ ac proveniet } - \frac{\pi \cos. A. \frac{m\pi}{n}}{n \sin. A. \frac{m\pi}{n}} + \frac{1}{m} =$$

$$\frac{2m}{n^2 - m^2} + \frac{2m}{4n^2 - m^2} + \frac{2m}{9n^2 - m^2} + \frac{2m}{16n^2 - m^2} + \frac{2m}{25n^2 - m^2} + \text{etc.}$$

feu

seu facta diuisione per  $2m$ , erit  $\frac{x}{2m^2} - \frac{\pi \cos A \cdot \frac{m\pi}{n}}{2mn \sin A \cdot \frac{m\pi}{n}}$

$$= \frac{x}{n^2 - m^2} + \frac{x}{4n^2 - m^2} + \frac{x}{9n^2 - m^2} + \frac{x}{16n^2 - m^2} + \frac{x}{25n^2 - m^2} + \text{etc.}$$

Quoties itaque euenit vt  $\cos A \cdot \frac{m\pi}{n}$  euanescat, toties seriei summa algebraice erit assignabilis quippe  $= \frac{x}{2m^2}$ .

Fit autem hoc, si fuerit  $\frac{m}{n} = \frac{2i+1}{2}$  seu  $m = 2i + 1$ , et  $n = 2$  vnde erit:

$$\frac{x}{9(2i+1)^2} + \frac{x}{4(2i+1)^2} + \frac{x}{16(2i+1)^2} + \frac{x}{36(2i+1)^2} + \frac{x}{64(2i+1)^2} + \text{etc.}$$

Ex quo sequens oritur propositio paradoxa: esse scilicet

$$\frac{x}{4-p} + \frac{x}{16-p} + \frac{x}{36-p} + \frac{x}{64-p} + \frac{x}{100-p} + \text{etc.} = \frac{x}{2p}$$

quoties fuerit  $p$  numerus quadratus integer et impar.

§. 15. Ponamus  $n = 1$ , atque  $m = p$ , erit

$$\frac{x}{1-p} + \frac{x}{4-p} + \frac{x}{9-p} + \frac{x}{16-p} + \frac{x}{25-p} + \text{etc.} = \frac{\pi\sqrt{p}}{2p \sin A \cdot \pi\sqrt{p}} - \frac{x}{2p}$$

$$\frac{x}{1-p} + \frac{x}{4-p} + \frac{x}{9-p} + \frac{x}{16-p} + \frac{x}{25-p} + \text{etc.} = \frac{x}{2p} - \frac{\pi\sqrt{p} \cos A \cdot \pi\sqrt{p}}{2p \sin A \cdot \pi\sqrt{p}}$$

quae series si addantur sequitur fore:

$$\frac{x}{1-p} + \frac{x}{9-p} + \frac{x}{25-p} + \text{etc.} = \frac{\pi\sqrt{p} \sin A \cdot \pi\sqrt{p}}{4p \sin A \cdot \pi\sqrt{p}}$$

at si eadem a se inuicem subtrahantur; erit

$$\frac{x}{4-p} + \frac{x}{16-p} + \frac{x}{36-p} + \text{etc.} = \frac{x}{2p} - \frac{\pi\sqrt{p}(1 + \cos A \cdot \pi\sqrt{p})}{4p \sin A \cdot \pi\sqrt{p}}$$

At est  $\frac{\sin A \cdot \pi\sqrt{p}}{\sin A \cdot \pi\sqrt{p}} = \text{tang. } A \cdot \frac{\pi\sqrt{p}}{2}$  et  $\frac{1 + \cos A \cdot \pi\sqrt{p}}{\sin A \cdot \pi\sqrt{p}} = \cos A \cdot \frac{\pi\sqrt{p}}{2}$

ex quo summae posteriores simpliciores reddentur.

§. 16. Possumus itaque hinc summare sequentes series

$$\frac{x}{1-p} + \frac{x}{4-p} + \frac{x}{9-p} + \frac{x}{16-p} + \text{etc.}$$

si quidem  $p$  significet numerum affirmatiuum quemcunque. At si loco  $p$  substituatur numerus negatiuus puta  $-q$ , tum sunt tam sinus et cosinus, quam ipsi arcus  $\pi\sqrt{p}$  seu  $\pi\sqrt{-q}$  quantitates imaginariae. Cum autem summae serierum nihilo

minus maneant reales et finitae, imaginaria sese destru-  
ent. Quamobrem inuestigari conueniet, cuiusmodi quan-  
titates reales in his formis  $\frac{\pi\sqrt{-q}}{\sin.A.\pi\sqrt{-q}}$  et  $\frac{\pi\sqrt{-q}}{\tanq.A.\pi\sqrt{-q}}$  contine-  
antur. Ad hoc ponamus  $u = \frac{\pi\sqrt{-q}}{\sin.A.\pi\sqrt{-q}}$  eritque  $\sin.A.\pi\sqrt{-q} = \frac{\pi\sqrt{-q}}{u}$  et  $\pi\sqrt{-q} = A \sin. \frac{\pi\sqrt{-q}}{u}$ ; sumantur differentia-  
lia positis  $\pi$  et  $u$  variabilibus, habebitur  $d\pi = \frac{u d\pi - \pi - \pi du}{u\sqrt{(uu+q\pi^2)}}$ . Po-  
natur  $u = \pi v$ , prodibit  $d\pi = \frac{-dv}{v\sqrt{(q+v^2)}}$  et  $\pi = \frac{1}{\sqrt{q}} \int \frac{1}{\sqrt{q+v^2}}$ . Hinc erit  $e^{\pi\sqrt{q}c} v = \sqrt{q+v^2}$  et  
 $v = \frac{2 e^{\pi\sqrt{q}c} \sqrt{q}}{e^{2\pi\sqrt{q}c} - 1}$  atque  $u = \frac{2 \pi e^{\pi\sqrt{q}c} \sqrt{q}}{e^{2\pi\sqrt{q}c} - 1}$ . Constans autem  $c$  ita  
debet esse comparata vt facto  $\pi = 0$  fiat  $u = 1$  ex quo fit  $c = 1$

Quamobrem erit  $\frac{\pi\sqrt{-q}}{\sin.A.\pi\sqrt{-q}} = \frac{2 e^{\pi\sqrt{q}} \pi \sqrt{q}}{e^{2\pi\sqrt{q}} - 1}$ . Simili modo  
ponatur  $\frac{\pi\sqrt{-q}}{\tanq.A.\pi\sqrt{-q}} = \frac{\pi}{v}$  erit  $v\sqrt{-q} = \tanq.A.\sqrt{-q}$  et  $\pi\sqrt{-q} = A \tanq. v\sqrt{-q}$  ac differentiando  $d\pi = \frac{dv}{1-qv^2}$ . Inte-  
gretur denuo, erit  $\pi = \frac{1}{2\sqrt{q}} \int \frac{1+v\sqrt{q}}{1-v\sqrt{q}}$  et  $e^{2\pi\sqrt{q}} - e^{2\pi\sqrt{q}v\sqrt{q}} =$   
 $1 + v\sqrt{q}$ , vnde fit  $v = \frac{1}{(e^{2\pi\sqrt{q}} + 1)\sqrt{q}}$  atque

$$\frac{\pi\sqrt{-q}}{\tanq.A.\pi\sqrt{-q}} = \frac{(e^{2\pi\sqrt{q}} + 1)\pi\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}$$

§. 17. Nacti igitur sumus octo sequentes series, qua-  
rum summae assignari possunt, quas cum summis conspe-  
ctui exponemus

$$\frac{1}{1-p} - \frac{1}{4-p} + \frac{1}{9-p} - \frac{1}{16-p} + \frac{1}{25-p} - \text{etc.} = \frac{\pi\sqrt{p}}{2p \sin.A.\pi\sqrt{p}} - \frac{1}{2p}$$

$$\frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{2p \tanq.A.\pi\sqrt{p}}$$

$$\frac{1}{1-p} + \frac{1}{9-p} + \frac{1}{25-p} + \frac{1}{49-p} + \text{etc.} = \frac{\pi\sqrt{p}}{4p \cos.A.\frac{\pi\sqrt{p}}{2}}$$

$$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \frac{1}{64-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{4p \operatorname{tang}. A. \frac{\pi\sqrt{p}}{2}}$$

$$\frac{1}{1+q} - \frac{1}{4+q} + \frac{1}{9+q} - \frac{1}{16+q} + \text{etc.} = \frac{1}{2q} - \frac{e^{\pi\sqrt{q}}\pi\sqrt{q}}{(e^{2\pi\sqrt{q}}-1)q}$$

$$\frac{1}{1+q} + \frac{1}{4+q} + \frac{1}{9+q} + \frac{1}{16+q} + \text{etc.} = \frac{(e^{2\pi\sqrt{q}}+1)\pi\sqrt{q}}{2(e^{2\pi\sqrt{q}}-1)q} - \frac{1}{2q}$$

$$\frac{1}{2+q} + \frac{1}{9+q} + \frac{1}{25+q} + \frac{1}{49+q} + \text{etc.} = \frac{(e^{\pi\sqrt{q}}-1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}+1)q}$$

$$\frac{1}{4+q} + \frac{1}{16+q} + \frac{1}{36+q} + \frac{1}{64+q} + \text{etc.} = \frac{(e^{\pi\sqrt{q}}+1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}-1)q} - \frac{1}{2q}$$

§. 18. Cum supra legem exhibuerim, qua summae potestatum omnium terminorum huius seriei

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.}$$

progrediuntur, inuestigabo nunc legem, quam potestates impares tantum inter se tenent, quo hae summae sine cognitione parium, quousque libuerit, continuari possint; sit itaque

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = A \pi$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = B \pi^3$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = C \pi^5$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = D \pi^7$$

$$1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \text{etc.} = E \pi^9$$

etc.

atque inuestiganda erit lex, qua coefficientes A, B, C, D etc. progrediuntur. Hunc in finem considero hanc seriem  $A \pi z + B \pi^3 z^3 + C \pi^5 z^5 + D \pi^7 z^7 + \text{etc.}$  cuius summa fit  $= s$ ; erit ergo his seriebus per respondentes potestates ipsius  $z$  multiplicatis respectiue:

$$1 z$$

$$s =$$



$$s = \frac{z}{1-zz} + \frac{z^2}{9-zz} + \frac{5z}{25-zz} - \frac{7z}{49-zz} + \text{etc. et}$$

$$\frac{zs}{z} = \frac{1}{1-z} + \frac{1}{1+z} - \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} + \frac{1}{5+z} \text{ etc.}$$

Cum autem ex §. 9. fit  $\frac{\pi \cos A \frac{(b-a)\pi}{2n}}{n \sin A \frac{(b-a)\pi}{2n} - n \sin A \frac{(b-a)\pi}{2n}} =$

$\frac{z}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.}$   
 fiat  $a=1-z$ ;  $n=2$ ; et  $b=1+z$ ; atque haec series transibit in illam; ex quo prodibit

$$\frac{zs}{z} = \frac{\pi}{2 \sin A \frac{(1-z)\pi}{2}} \text{ et } s = \frac{\pi z}{4 \sin A \frac{(1-z)\pi}{2}} \text{ siue } s =$$

$$4 \cos A \frac{\pi z}{2} = \frac{\pi z}{1 \cdot 2 \cdot 4} + \frac{\pi^2 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^4} - \frac{\pi^6 z^6}{1 \cdot 1 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2^6} + \text{etc.}$$

quae fractio cum, si actu diuidatur, ipsam assumptam seriem  $A\pi z + B\pi^5 z^5 + C\pi^9 z^9 + \text{etc.}$  reproducere debeat, erit:

$$A = \frac{1}{2}$$

$$B = \frac{A}{2 \cdot 4}$$

$$C = \frac{B}{2 \cdot 4} - \frac{A}{2 \cdot 4 \cdot 6 \cdot 8}$$

$$D = \frac{C}{2 \cdot 4} - \frac{B}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{A}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12}$$

$$E = \frac{D}{2 \cdot 4} - \frac{C}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{B}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} - \frac{A}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 20}$$

etc.

§. 19. Vel si ponatur:

$$I - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} = \frac{A\pi}{2^2}$$

$$II - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.} = \frac{B\pi^3}{2^4}$$

$$III - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.} = \frac{C\pi^5}{2^6}$$

$$IV - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.} = \frac{D\pi^7}{2^8}$$

etc.

coeffi-

coefficientes A, B, C, etc. hanc tenebunt legem :

$$A = 1$$

$$B = \frac{A}{1 \cdot 2}$$

$$C = \frac{B}{1 \cdot 2} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$D = \frac{C}{1 \cdot 2} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{A}{1 \cdot 2 \cdot \dots \cdot 6}$$

$$E = \frac{D}{1 \cdot 2} - \frac{C}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{B}{1 \cdot 2 \cdot \dots \cdot 6} - \frac{A}{1 \cdot 2 \cdot \dots \cdot 8}$$

Quodsi autem illae series retro continuentur, vt ad potestates affirmativas deueniatur, erunt omnium illarum serierum summae = 0; ita vt etiamsi in his formis vltius progredieremur, tamen alii valores non prodituri essent. Est scilicet

$$1 - 3 + 5 - 7 + 9 - \text{etc.} = 0$$

$$1 - 3^3 + 5^3 - 7^3 + 9^3 - \text{etc.} = 0$$

$$1 - 3^5 + 5^5 - 7^5 + 9^5 - \text{etc.} = 0$$

$$1 - 3^7 + 5^7 - 7^7 + 9^7 - \text{etc.} = 0$$

§. 20. Quemadmodum autem summae potestatum imparium peculiarem inter se tenent progressionis legem, ita etiam potestates pares simili proprietate gaudent, vt omnes ex se ipsis sine subsidio potestatum imparium definiri queant. Quam legem vt eruamus, simili vtamur operatione. Sit igitur

$$1 + \frac{1}{5^2} + \frac{1}{5^4} + \frac{1}{5^6} + \frac{1}{5^8} + \text{etc.} = A \pi^2$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = B \pi^4$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = C \pi^6$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = D \pi^8$$

ac inuestigetur summa huius seriei :

$$A \pi^2 z^2 + B \pi^4 z^4 + C \pi^6 z^6 + D \pi^8 z^8 + \text{etc.} = s,$$

$$\text{erit } s = \frac{z^2}{1-z^2} + \frac{z^4}{9-z^4} + \frac{z^6}{25-z^6} + \frac{z^8}{49-z^8} + \text{etc. vnde ex}$$

§. 17. fiet  $s = \frac{\pi z}{4 \cos. A \frac{\pi z}{2}}$ ; siue per seriem

$$s = \frac{\pi^2 z^2}{1 \cdot 2^3} - \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 2^5} + \frac{\pi^6 z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 2^7} - \text{etc.}$$

$$1 = \frac{\pi^2 z^2}{1 \cdot 2 \cdot 2^2} + \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^4} - \frac{\pi^6 z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2^6} + \text{etc.}$$

ex qua diuisione cum ipsa series assumpta oriri debeat, erit

$$A = \frac{1}{8}$$

$$B = \frac{A}{2 \cdot 4} - \frac{1}{2 \cdot 4 \cdot 6 \cdot 4}$$

$$C = \frac{B}{2 \cdot 4} - \frac{A}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{2}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 4}$$

$$D = \frac{C}{2 \cdot 4} - \frac{B}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{A}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 12} - \frac{7}{2 \cdot 4 \cdot \dots \cdot 14 \cdot 4}$$

etc.

§. 21. Lex haec facilius inspicietur, si ponatur

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = A \frac{\pi^2}{2^3}$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = B \frac{\pi^4}{2^5}$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} = C \frac{\pi^6}{2^7}$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.} = D \frac{\pi^8}{2^9}$$

etc.

Hic enim coefficientes A, B, C etc. sequentem tenebunt progressionem:

$$A = 1$$

$$B = \frac{A}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3}$$

$$C = \frac{B}{1 \cdot 2} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 6 \cdot 4 \cdot 5}$$

$$D = \frac{C}{1 \cdot 2} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{1}{1 \cdot 2 \cdot \dots \cdot 7}$$

etc.

Quodsi ergo fingatur series haec:  $s =$

$$A z + B z^3 + C z^5 + D z^7 + E z^9 + \text{etc. erit}$$

$$s = \frac{z - \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{z^7}{1 \cdot 2 \cdot \dots \cdot 7} + \text{etc.}}{1 - \frac{z^2}{1 \cdot 2} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{z^6}{1 \cdot 2 \cdot 3 \cdot \dots \cdot 6} + \text{etc.}}$$

atque

atque hinc  $s = \text{tang. } A. z$ , seu  $z = \text{tang. } s$ . Habebimus ergo  $d z = \frac{ds}{1+s^2}$  et  $d z + s s d z = d s$  cui aequationi cum satisfacere debeat valor hic

$$s = A z + B z^3 + C z^5 + D z^7 + E z^9 + \text{etc.}$$

substituantur valores loco  $d s$  et  $s s$ , eritque

$$\begin{aligned} \frac{ds}{dz} &= \frac{A + 3Bz^2 + 5Cz^4 + 7Dz^6 + 9Ez^8 + \text{etc.}}{1 + A^2z^2 + 2ABz^4 + 2ACz^6 + 2ADz^8 + \text{etc.}} \\ s s &= \frac{A^2z^2 + 2ABz^4 + 2ACz^6 + 2ADz^8 + \text{etc.}}{1 + B^2z^6 + 2BCz^8 + \text{etc.}} \\ \mathbf{I} &= \mathbf{I} \end{aligned}$$

Hinc itaque formatis aequationibus aliae sequentes prodibunt determinationes litterarum  $A, B, C, D, \text{etc.}$

$$A = \mathbf{I}$$

$$B = \frac{A^2}{3}$$

$$C = \frac{2AB}{5}$$

$$D = \frac{2AC + B^2}{7}$$

$$E = \frac{2AD + 2BC}{9}$$

$$F = \frac{2AE + 2BD + C^2}{11}$$

etc.

§. 22. Ab his seriebus potestatum parium pendent summae serierum sub hac forma generali contentarum

$$\mathbf{I} + \frac{\mathbf{I}}{2^n} + \frac{\mathbf{I}}{3^n} + \frac{\mathbf{I}}{4^n} + \frac{\mathbf{I}}{5^n} + \text{etc.}$$

denotante  $n$  numerum parem. Quodsi enim fuerit

$$\mathbf{I} + \frac{\mathbf{I}}{3^n} + \frac{\mathbf{I}}{5^n} + \frac{\mathbf{I}}{7^n} + \frac{\mathbf{I}}{9^n} + \text{etc.} = N \pi^n$$

$$\text{erit } \mathbf{I} + \frac{\mathbf{I}}{2^n} + \frac{\mathbf{I}}{3^n} + \frac{\mathbf{I}}{4^n} + \frac{\mathbf{I}}{5^n} + \text{etc.} = \frac{2^n N \pi^n}{2^n - \mathbf{I}}$$

vnde omnium harum serierum dummodo sit  $n$  numerus par,

par, summae per quadraturam circuli poterunt inueniri, atque ex iam inuentis summis similibus potestatum parium pro numeris imparibus solis. Verum ut hae summae directe inueniri queant, in legem peculiarem qua istae summae progrediuntur, inquiramus. Sit itaque

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = A \pi^2$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = B \pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = C \pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = D \pi^8$$

etc.

ac contemlor hanc seriem:  $s =$

$$A \pi^2 z^2 + B \pi^4 z^4 + C \pi^6 z^6 + D \pi^8 z^8 + E \pi^{10} z^{10} + \text{etc.}$$

quae substitutis loco  $A \pi^2$ ,  $B \pi^4$ ,  $C \pi^6$ , etc. seriebus quas denotant, additisque terminis homologis, prodibit

$$s = \frac{zz}{1-zz} + \frac{zz}{1-zz} + \frac{zz}{9-zz} + \frac{zz}{16-zz} + \frac{zz}{25-zz} + \text{etc.}$$

quae series per §. 17. summata dat  $s = \frac{1}{2} - \frac{\pi z}{2 \tan \frac{1}{2} \pi z}$  vel si tangens arcus  $\pi z$  per seriem exprimitur:

$$s = \frac{1}{2} - \frac{1}{2} \cdot \frac{1 - \frac{\pi^2 z^2}{1 \cdot 2} + \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\pi^6 z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}}{1 - \frac{\pi^2 z^2}{1 \cdot 2 \cdot 3} + \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{\pi^6 z^6}{1 \cdot 2 \cdot \dots \cdot 7} + \text{etc.}}$$

$$s = \frac{\frac{\pi^2 z^2}{1 \cdot 2 \cdot 3} - \frac{2\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{2\pi^6 z^6}{1 \cdot 2 \cdot \dots \cdot 7} - \frac{4\pi^8 z^8}{1 \cdot 2 \cdot \dots \cdot 9} + \text{etc.}}{1 - \frac{\pi^2 z^2}{1 \cdot 2 \cdot 4} + \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{\pi^6 z^6}{1 \cdot 2 \cdot \dots \cdot 7} + \frac{\pi^8 z^8}{1 \cdot 2 \cdot \dots \cdot 9} - \text{etc.}}$$

qua expressione euoluta, cum ipsam seriem assumtam  $A \pi^2 z^2 + B \pi^4 z^4 + C \pi^6 z^6 + D \pi^8 z^8 + \text{etc.}$  praebere debeat, sequentur hae coefficientium determinationes

$$A = \frac{1}{6}$$

$$B = \frac{A}{1 \cdot 2 \cdot 3} - \frac{2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$C = \frac{B}{1 \cdot 2 \cdot 3} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{3}{1 \cdot 2 \cdot \dots \cdot 7}$$

$$D = \frac{C}{1 \cdot 2 \cdot 3} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{A}{1 \cdot 2 \cdot \dots \cdot 7} - \frac{4}{1 \cdot 2 \cdot \dots \cdot 9}$$

etc.

§. 23. At pro iisdem his coefficientibus aliā lex progressionis potest exhiberi, cuius ope eos multo expeditius reperire licebit. Cum enim fit  $s = \frac{1}{2} - \frac{\pi z}{2 \operatorname{tang}. A \pi z}$  erit  $\operatorname{tang}. A \cdot \pi z = \frac{\pi z}{1-2s}$  et  $\pi z = A \operatorname{tang}. \frac{\pi z}{1-2s}$ ; ponatur  $\pi z = u$ , erit  $u = A \operatorname{tang}. \frac{u}{1-2s}$  et differentiando  $du = \frac{du-2sdu+2uds}{1-4s+4ss+uu}$  vel  $uu du + 4ss du = 2s du + 2u ds$ : cui aequationi satisfacit valor hic  $s = Au^2 + Bu^4 + Cu^6 + Du^8 + Eu^{10} + \text{etc.}$  quo substituto fiet

$$uu = uu$$

$$4ss = 4A^2u^4 + 8ABu^6 + 8ACu^8 + 8ADu^{10} + 8AEu^{12} + 4B^2u^8 + 8BCu^{10} + 8BDu^{12} + 4C^2u^{12}$$

$$2s = 2Au^2 + 2Bu^4 + 2Cu^6 + 2Du^8 + 2Eu^{10} + 2Fu^{12}$$

$$\frac{2uds}{du} = 4Au^2 + 8Bu^4 + 12Cu^6 + 16Du^8 + 20Eu^{10} + 24Fu^{12}$$

unde sequentes consequuntur determinationes:

$A = \frac{1}{6}$	$E = \frac{4AD + BC}{11}$
$B = \frac{2A^2}{5}$	$F = \frac{4AE + BD + 2C^2}{13}$
$C = \frac{4AB}{7}$	$G = \frac{4AF + 4BE + CD}{15}$
$D = \frac{4AC + 2B^2}{9}$	$H = \frac{4AG + 4BF + 4CE + 2D^2}{17}$
	etc.

§. 24. Ipsae autem huiusmodi serierum summae, quoque quidem eas supputari, sequentes sunt:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = \frac{2}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} \pi^2$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = \frac{2^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{1}{8} \pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = \frac{2^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \cdot \frac{1}{8} \pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = \frac{2^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \cdot \frac{3}{10} \pi^8$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \text{etc.} = \frac{2^9}{1 \cdot 2 \cdot \dots \cdot 11} \cdot \frac{5}{8} \pi^{10}$$

$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \text{etc.} = \frac{2^{12}}{1 \cdot 2 \cdot \dots \cdot 13} \cdot \frac{601}{215} \pi^{12}$$

$$1 + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}} + \frac{1}{5^{14}} + \text{etc.} = \frac{2^{13}}{1 \cdot 2 \cdot \dots \cdot 15} \cdot \frac{35}{2} \pi^{14}$$

$$1 + \frac{1}{2^{16}} + \frac{1}{3^{16}} + \frac{1}{4^{16}} + \frac{1}{5^{16}} + \text{etc.} = \frac{2^{15}}{1 \cdot 2 \cdot \dots \cdot 17} \cdot \frac{3617}{30} \pi^{16}$$

$$1 + \frac{1}{2^{18}} + \frac{1}{3^{18}} + \frac{1}{4^{18}} + \frac{1}{5^{18}} + \text{etc.} = \frac{2^{17}}{1 \cdot 2 \cdot \dots \cdot 19} \cdot \frac{43867}{42} \pi^{18}$$

$$1 + \frac{1}{2^{20}} + \frac{1}{3^{20}} + \frac{1}{4^{20}} + \frac{1}{5^{20}} + \text{etc.} = \frac{2^{19}}{1 \cdot 2 \cdot \dots \cdot 21} \cdot \frac{1222277}{110} \pi^{20}$$

$$1 + \frac{1}{2^{22}} + \frac{1}{3^{22}} + \frac{1}{4^{22}} + \frac{1}{5^{22}} + \text{etc.} = \frac{2^{21}}{1 \cdot 2 \cdot \dots \cdot 23} \cdot \frac{854513}{6} \pi^{22}$$

$$1 + \frac{1}{2^{24}} + \frac{1}{3^{24}} + \frac{1}{4^{24}} + \frac{1}{5^{24}} + \text{etc.} = \frac{2^{23}}{1 \cdot 2 \cdot \dots \cdot 25} \cdot \frac{1181820455}{546} \pi^{24}$$

In his expressionibus fractionum mediarum tantum lex non est manifesta, reliquae partes vero perspicue progrediuntur. Cum autem istas fractiones medias  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{5}$ ,  $\frac{5}{16}$ ,  $\frac{5}{8}$  etc. attentius effem contemplatus, easdem deprehendi occurrere in expressione generali, quam olim tradidi pro summa cuiuscunque seriei ex dato termino generali inveniendi, ita vt alterius expressionis ope altera possit confici.

§. 25. Operae pretium igitur erit in consensum harum duarum expressionum inter se tantopere diuersarum diligentius inquirere. Altera quidem expressio quam pro summatione serierum dedi, ita se habet: si seriei cuiuscunque terminus generalis, seu is qui respondet exponenti indefinito numerico  $x$  fuerit  $= X$ ; et summa seriei a termino primo vsque ad hunc  $X$  inclusue ponatur  $= S$ , erit  $S =$

$$\int X dx + \frac{X}{1 \cdot 2} + \frac{dX}{1 \cdot 2 \cdot 3 \cdot 2 dx} - \frac{d^2 X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 dx^3}$$

$$+ \frac{d^4 X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 6 dx^5} - \frac{d^6 X}{1 \cdot 2 \cdot 3 \cdot \dots \cdot 9 \cdot 10 dx^7} + \frac{d^8 X}{1 \cdot 2 \cdot \dots \cdot 11 \cdot 12 dx^9}$$

$$- \frac{691 d^{10} X}{1 \cdot 2 \cdot \dots \cdot 13 \cdot 210 dx^{11}} + \frac{35 d^{12} X}{1 \cdot 2 \cdot \dots \cdot 15 \cdot 2 dx^{13}} - \frac{3617 d^{14} X}{1 \cdot 2 \cdot \dots \cdot 17 \cdot 50 dx^{15}}$$

$$+ \frac{43567d^{17}X}{1 \cdot 2 \cdot \dots \cdot 19 \cdot 42 dx^{17}} - \frac{1222277d^{19}X}{1 \cdot 2 \cdot \dots \cdot 21 \cdot 110dx^{19}}$$

$$+ \frac{854513d^{21}X}{1 \cdot 2 \cdot \dots \cdot 23 \cdot 6 dx^{21}} - \frac{1181820455d^{23}X}{1 \cdot 2 \cdot \dots \cdot 25 \cdot 54dx^{23}} \text{ etc.}$$

in qua expressione apparet easdem omnino fractiones irregulares  $\frac{1}{2}$ ,  $\frac{1}{6}$ ,  $\frac{1}{6}$ ,  $\frac{5}{18}$ ,  $\frac{5}{6}$ , etc. inesse, quae ante in expressionibus summarum occurrerunt, hoc tantum discrimine, quod hic signa habeant alternantia, cum ibi omnia signo  $\rightarrow$  essent affectae. Atque hic ipse consensus mihi hanc praestitit vtilitatem, vt istam generalem summae S expressionem eousque continuare potuerim, cum hoc per eam legem, quae tum temporis mihi pro istorum terminorum progressionem inuenta erat, sine multo maiore labore praestare non potuissem.

§. 26. Tametsi autem haec tanti consensus mera observatio sufficere posset, ad consensum in sequentibus terminis, qui nondum constant, euincendum, tamen praestabit ex ipsa rei natura eandem conuenientiam eruere, vt ea non casu, sed necessario accidisse intelligatur. Hanc vero posteriorem expressionem sequenti modo sum affectus. Cum S denotet summam tot terminorum in serie quacunque, quot vnitates continentur in exponente x, vltimusque horum terminorum sit = X: manifestum est, si in S ponatur x - 1 loco x, tum prodire debere summam eandem S vltimo termino minutam, seu S - X. At posito x - 1 loco x quantitas S abibit in hanc:

$$S - \frac{ds}{1 \cdot dx} + \frac{dds}{1 \cdot 2 dx^2} - \frac{d^3s}{1 \cdot 2 \cdot 3 dx^3} + \frac{d^4s}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} - \text{etc.}$$

quae propterea aequalis est ipsi S - X; vnde habetur haec aequatio:

$$X = \frac{dS}{dx} - \frac{dds}{1 \cdot 2 dx^2} + \frac{d^3s}{1 \cdot 2 \cdot 3 dx^3} - \frac{d^4s}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} + \text{etc.}$$

K 2

Vt



Vt nunc ex hac aequatione S in X exprimatur, accipio hanc aequationem:

$$S = \int X dx + \alpha X + \frac{\epsilon dx}{dx} + \frac{\gamma ddx}{dx^2} + \frac{\delta d^3 X}{dx^3} + \text{etc.}$$

cuius, in illa facta substitutione habebitur

$$\begin{aligned} X = X + & \frac{\alpha dx}{dx} + \frac{\epsilon ddx}{dx^2} + \frac{\gamma d^3 X}{dx^3} + \frac{\delta d^4 X}{dx^4} \\ & - \frac{dX}{1 \cdot 2 dx} - \frac{\alpha ddx}{1 \cdot 2 dx^2} - \frac{\epsilon d^3 X}{1 \cdot 2 \cdot 3 dx^3} - \frac{\gamma d^4 X}{1 \cdot 2 \cdot 3 dx^4} \\ & + \frac{d^2 X}{1 \cdot 2 \cdot 3 dx^2} + \frac{\alpha d^3 X}{1 \cdot 2 \cdot 3 dx^3} + \frac{\epsilon d^4 X}{1 \cdot 2 \cdot 3 dx^4} \\ & - \frac{d^3 X}{1 \cdot 2 \cdot 3 \cdot 4 dx^3} - \frac{\alpha d^4 X}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} \\ & + \frac{d^5 X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 dx^5} \end{aligned}$$

§. 27. Ex hac aequalitate nascuntur sequentes coefficientium  $\alpha$ ,  $\epsilon$ ,  $\gamma$ ,  $\delta$ , etc. determinaciones.

$$\alpha = \frac{1}{1 \cdot 2}$$

$$\epsilon = \frac{\alpha}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3}$$

$$\gamma = \frac{\epsilon}{1 \cdot 2} - \frac{\alpha}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$\delta = \frac{\gamma}{1 \cdot 2} - \frac{\epsilon}{1 \cdot 2 \cdot 3} + \frac{\alpha}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$\epsilon = \frac{\delta}{1 \cdot 2} - \frac{\gamma}{1 \cdot 2 \cdot 3} + \frac{\epsilon}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\alpha}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

etc.

atque ex formulis tum temporis valores harum litterarum definiui, idque multo labore. Neque, quod hic euenit, aliter nisi sola obseruatione cognoui valores alternos  $\gamma$ ,  $\epsilon$ ,  $\gamma$ , etc. euanescere. At ex principiis nunc stabilitis idem luculenter ostendi poterit, si alia huius progressionis lex inuestigetur. Considero ad hoc istam seriem:

$$s = 1 + \alpha z + \epsilon z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5 + \text{etc.}$$

eritque ex praecedente coefficientium lege:

$$s = \frac{1}{1 - \frac{z}{1 \cdot 2} + \frac{z^2}{1 \cdot 2 \cdot 3} - \frac{z^3}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \text{etc.}}$$

quae aequatio abit in hanc  $s = \frac{z}{1 - e^{-z}}$  seu  $s = \frac{e^z z}{e^z - 1}$ . Hinc

oritur  $e^z s - s = e^z z$  et  $e^z = \frac{s}{s-z}$  atque  $z = 1s - 1(s-z)$ .

Differentiando autem habebitur  $dz = \frac{ds}{s} - \frac{ds+dz}{s-z}$  siue

$$s s dz - s z dz = s dz - z ds$$

cui aequationi satisfacere debet valor assumtus

$$s = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$$

substituatur itaque hic valor in hac aequatione

$$\frac{z ds}{dz} - s - s z + s s = 0$$

atque obtinebitur:

$$\begin{aligned} \frac{z ds}{dz} &= 1 + \alpha z + 2\beta z^2 + 3\gamma z^3 + 4\delta z^4 + 5\varepsilon z^5 \\ -s &= -1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \varepsilon z^5 \\ -sz &= -z - \alpha z^2 - \beta z^3 - \gamma z^4 - \delta z^5 \\ +s^2 &= 1 + 2\alpha z + 2\beta z^2 + 2\gamma z^3 + 2\delta z^4 + 2\varepsilon z^5 \\ &\quad + \alpha^2 + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta \\ &\quad + \beta^2 + 2\beta\gamma \end{aligned}$$

Hinc igitur colligitur fore:

$$\begin{aligned} \alpha &= \frac{1}{2} \\ \beta &= \frac{\alpha - \alpha^2}{3} \\ \gamma &= \frac{\beta - 2\alpha\beta}{4} \\ \delta &= \frac{\gamma - 2\alpha\gamma - 6\beta^2}{5} \\ \varepsilon &= \frac{\delta - 2\alpha\delta - 2\beta\gamma}{6} \\ \zeta &= \frac{\varepsilon - 2\alpha\varepsilon - 2\beta\delta - \gamma\gamma}{7} \\ \eta &= \frac{\zeta - 2\alpha\zeta - 2\beta\varepsilon - 2\gamma\delta}{8} \end{aligned}$$

§. 28. Cum igitur fit  $\alpha = \frac{1}{2}$  erit  $1 - 2\alpha = 0$ , qui valor cum in omnibus terminis sequentibus occurrat, erit:

$$\begin{aligned}\alpha &= \frac{1}{2} \\ \beta &= \frac{1}{12} \\ \gamma &= 0 \\ \delta &= -\frac{6\alpha}{5} \\ \epsilon &= -\frac{2\beta\gamma}{6} \\ \zeta &= -\frac{2\beta\delta - \gamma\eta}{7} \\ \eta &= -\frac{2\beta\epsilon - 2\gamma\delta}{8} \\ \theta &= -\frac{2\beta\zeta - 2\gamma\epsilon - \delta\delta}{9} \\ \iota &= -\frac{2\beta\eta - 2\gamma\zeta - 2\delta\epsilon}{10}\end{aligned}$$

etc.

cum nunc fit  $\gamma = 0$ , perspicuum est fore etiam  $\epsilon = 0$ , hincque porro  $\eta = 0$ ,  $\iota = 0$ , etc. ita vt omnes termini alterni incipiendo ab  $\gamma$  sint  $= 0$ , id quod ex praecedente lege tantum per obseruationes patebat, nunc vero id necessario euenire debere intelligitur. Hinc ergo manente  $\alpha = \frac{1}{2}$  erit vt sequitur

$$\begin{aligned}\beta &= \frac{1}{12} \\ \delta &= -\frac{6\alpha}{5} \\ \zeta &= -\frac{2\beta\delta}{7} \\ \theta &= -\frac{2\beta\zeta - \delta\delta}{9} \\ \kappa &= -\frac{2\beta\theta - 2\delta\zeta}{11}\end{aligned}$$

Quod si ergo ponatur  $\beta = \frac{A}{2}$ ;  $\delta = -\frac{B}{25}$ ;  $\zeta = \frac{C}{25}$ ;  $\theta = -\frac{D}{27}$ ,  $\kappa = \frac{E}{29}$ , etc. ita vt sit  $S = fX dx + \frac{X}{2} + \frac{Adx}{2dx} - \frac{Bd^2x}{2^5 dx^2} + \frac{Cd^5x}{2^5 dx^5} - \frac{Dd^7x}{2^7 dx^7} + \frac{Ed^9x}{2^9 dx^9} - \frac{Fd^{11}x}{2^{11} dx^{11}} + \text{etc.}$  tenebunt coefficientes A, B, C, D, etc; hanc legem

A =

$A = \frac{x}{6}$	$E = \frac{4AD + 4BC}{11}$
$B = \frac{2A^2}{5}$	$F = \frac{4AE + 4BD + 2C^2}{13}$
$C = \frac{4AB}{7}$	$G = \frac{4AF + 4BE + 4CD}{15}$
$D = \frac{4AC + 2B^2}{7}$	etc.

Obtinent ergo litterae A, B, C, D, etc. eos ipsos valores, quos ipsis supra in §. §. 22 et 23 tribuimus. Atque hinc de consensu coefficientium in his expressionibus maxime diuersis plene summus certi, neque eum casui amplius adscribere conueniet.

§. 29. Quanquam autem hoc modo satis expedite

summam huius seriei  $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.}$

assignare valemus, si  $n$  est numerus par, tamen ex his iisdem principiis nihil omnino concludere possumus ad summas inteniendas, si  $n$  sit numerus impar. Videri quidem possit, etiam has series a quadratura circuli ita pendere, vt summa earum sit  $= N \pi^n$  casibus scilicet quoque, quibus est  $n$  numerus impar: verum si has summas actu per approximationes sumamus, videbimus coefficientem  $N$  non fieri numerum rationalem, nisi sit  $n$  numerus par, id quod ex hac tabella clarius elucebit:

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \text{etc.} = 1,644934066 = \frac{\pi^2}{6}$
$1 + \frac{1}{2^4} + \frac{1}{3^4} + \text{etc.} = 1,202056903 = \frac{\pi^4}{96}$
$1 + \frac{1}{2^6} + \frac{1}{3^6} + \text{etc.} = 1,082323233 = \frac{\pi^6}{945}$
$1 + \frac{1}{2^8} + \frac{1}{3^8} + \text{etc.} = 1,036927755 = \frac{\pi^8}{720}$
$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \text{etc.} = 1,017343062 = \frac{\pi^{10}}{693}$
$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \text{etc.} = 1,008349329 = \frac{\pi^{12}}{648}$
$1 + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \text{etc.} = 1,004077356 = \frac{\pi^{14}}{648}$

I +

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$$1 + \frac{1}{2^9} + \frac{1}{3^9} + \text{etc.} = 1,002008392 = \frac{\pi^9}{29748,38}$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \text{etc.} = 1,000994575 = \frac{\pi^{10}}{93555}$$

$$1 + \frac{1}{2^{11}} + \frac{1}{3^{11}} + \text{etc.} = 1,000494188 = \frac{\pi^{11}}{294058,8}$$

$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \text{etc.} = 1,000246086 = \frac{\pi^{12}}{924041,897}$$

Neque vero vlla relatio inter summas potestatum imparium cernitur similis ei, quam summae potestatum parium inter se tenent.

§. 30. Videtur autem aliquid circa summas potestatum imparium concludi posse, si signa ponantur alternantia. Cum enim imparium potestatum prima  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \text{etc.}$  cognitam habeat summam scilicet  $\frac{1}{2}$ : valde verisimile viderur, etiam sequentium potestatum imparium summas a logarithmo binarii pendere, ac fortasse insuper a quadratura circuli. Sed antequam hic aliquid concludere queamus, inuestigemus summas potestatum parium: sitque

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{etc.} = A \pi^2$$

$$1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \text{etc.} = B \pi^4$$

$$1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \text{etc.} = C \pi^6$$

$$1 - \frac{1}{2^8} + \frac{1}{3^8} - \frac{1}{4^8} + \text{etc.} = D \pi^8$$

$$1 - \frac{1}{2^{10}} + \frac{1}{3^{10}} - \frac{1}{4^{10}} + \text{etc.} = E \pi^{10}$$

etc.

vbi valores litterarum A, B, C, D etc. facile ex cognitis valoribus pro seriebus iisdem, dum omnes termini accipiuntur affirmatiui, concludi possunt, sed praestabit peculiarem legem pro his elicere. Considero itaque sequentem seriem:

s =

$s = A\pi^2 z^2 + B\pi^4 z^4 + C\pi^6 z^6 + D\pi^8 z^8 + \text{etc.}$   
 quae substitutis ipsis seriebus abibit in hanc :

$s = \frac{zz}{1-zz} - \frac{zz}{4-zz} + \frac{zz}{9-zz} - \frac{zz}{16-zz} + \text{etc.}$   
 quae series per §. 17. summata dabit

$$s = \frac{\pi z}{2 \sin \frac{1}{2} \pi z} - \frac{1}{2} \text{ siue sinu expressio}$$

$$s = -\frac{1}{2} + \frac{1}{1 - \frac{\pi^2 z^2}{1 \cdot 2 \cdot 3}} + \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{\pi^6 z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} - \text{etc.}$$

Quodsi nunc ponatur terminus precedens in serie litterarum A, B, C, D, E, etc. seu ante primum A existens = Δ erit

$$\Delta = \frac{1}{2}$$

$$A = \frac{\Delta}{1 \cdot 2 \cdot 3} = \frac{1}{12}$$

$$B = \frac{A}{1 \cdot 2 \cdot 3} - \frac{\Delta}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$C = \frac{B}{1 \cdot 2 \cdot 3} - \frac{A}{1 \cdot 3 \cdot 3 \cdot 4 \cdot 5} + \frac{\Delta}{1 \cdot 2 \cdot \dots \cdot 7}$$

$$D = \frac{C}{1 \cdot 2 \cdot 3} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{A}{1 \cdot 2 \cdot \dots \cdot 7} - \frac{\Delta}{1 \cdot 2 \cdot \dots \cdot 9}$$

At valor ipsius Δ non est mere assumptus, sed reipsa summam seriei praecedentis exprimit, quae est

$$1 - 1 + 1 - 1 + 1 - 1 + \text{etc.} = \Delta \pi^0 = \frac{1}{2}$$

serierum vero reliquarum, quae hanc praecedunt, omnium summae sunt = 0 : scilicet

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = 0$$

$$1 - 2^4 + 3^4 - 4^4 + \text{etc.} = 0$$

$$1 - 2^6 + 3^6 - 4^6 + \text{etc.} = 0$$

etc.

§. 31. Ex his igitur sequitur summam cuiusvis seriei ex praecedentibus omnibus recte concludi posse hoc modo, si fuerit

$$1 - \frac{x}{2^n} + \frac{x^2}{3^n} - \frac{x^3}{4^n} + \text{etc.} = \alpha \pi^n$$

$$1 - \frac{x}{2^{n-2}} + \frac{x^2}{3^{n-2}} - \frac{x^3}{4^{n-2}} + \text{etc.} = \beta \pi^{n-2}$$

$$1 - \frac{x}{2^{n-4}} + \frac{x^2}{3^{n-4}} - \frac{x^3}{4^{n-4}} + \text{etc.} = \gamma \pi^{n-4}$$

$$1 - \frac{x}{2^{n-6}} + \frac{x^2}{3^{n-6}} - \frac{x^3}{4^{n-6}} + \text{etc.} = \delta \pi^{n-6}$$

$$\text{erit } \alpha = \frac{\beta}{1.2.3} - \frac{\gamma}{1.2.3.4.5} + \frac{\delta}{1.2...7} - \frac{\epsilon}{1.2...9} + \frac{\zeta}{1.2...11} - \text{etc.}$$

Vt igitur summam huius seriei inueniamus

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \text{etc.}$$

omnes, quae secundum hanc legem ipsam antecedunt considerari oportebit, quae sunt:

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \text{etc.} = A \pi^3$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{etc.} = A \pi$$

$$1 - 2 + 3 - 4 + \text{etc.} = \frac{\alpha}{\pi}$$

$$1 - 2^3 + 3^3 - 4^3 + \text{etc.} = \frac{\beta}{\pi^2}$$

$$1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \frac{\gamma}{\pi^5}$$

$$\text{eritque } B = \frac{A}{1.2.3} - \frac{\alpha}{1.2.3.4.5} + \frac{\beta}{1.2...7} - \frac{\gamma}{1.2...9} + \text{etc.}$$

At harum serierum omnium summae exhiberi possunt, est enim

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.} = 1/2$$

$$1 - 2 + 3 - 4 + \text{etc.} = \frac{1}{4} = \frac{2 \cdot 1}{\pi^2} (1 - \frac{1}{3^2} + \frac{1}{5^2} - \text{etc.})$$

$$1 - 2^3 + 3^3 - 4^3 + \text{etc.} = \frac{-1}{8} = \frac{-2 \cdot 1 \cdot 2 \cdot 3}{\pi^4} (1 - \frac{1}{3^4} + \frac{1}{5^4} - \text{etc.})$$

$$1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \frac{1}{4} = \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{\pi^6} (1 - \frac{1}{3^6} + \frac{1}{5^6} - \text{etc.})$$

$$1 - 2^7 + 3^7 - 4^7 + \text{etc.} = \frac{-17}{16} = \frac{-2 \cdot 1 \cdot 2 \cdot \dots \cdot 7}{\pi^8} (1 - \frac{1}{3^8} + \frac{1}{5^8} - \text{etc.})$$

etc.

Atque

Atque hinc erit

$$\begin{aligned}
 A &= \frac{1}{\pi} \\
 \alpha &= \frac{2 \cdot 1}{\pi} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} \right) \\
 \epsilon &= \frac{2 \cdot 1 \cdot 2 \cdot 3}{\pi} \left( 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} \right) \\
 \gamma &= \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{\pi} \left( 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} \right) \\
 \delta &= \frac{2 \cdot 1 \cdot 2 \cdot \dots \cdot 7}{\pi} \left( 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.} \right) \\
 \xi &= \frac{2 \cdot 1 \cdot 2 \cdot \dots \cdot 9}{\pi} \left( 1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.} \right) \\
 &\text{etc.}
 \end{aligned}$$

§. 32. Summas autem potestatum parium fractionum, quarum denominatores sunt numeri impares supra exhibuimus. Sit

$$\begin{aligned}
 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} &= P \pi^2 \\
 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} &= Q \pi^4 \\
 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} &= R \pi^6 \\
 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.} &= S \pi^8
 \end{aligned}$$

Erit per §. 21:  $P \pi^2 + Q \pi^4 + R \pi^6 + S \pi^8 + \text{etc.} = \frac{\pi}{4} \text{tang} A \cdot \frac{\pi}{2}$

At litterae  $\alpha$ ,  $\epsilon$ ,  $\gamma$ ,  $\delta$ , etc. sequentes obtinebunt valores.

$$\begin{aligned}
 \alpha &= \frac{2 \cdot 1}{\pi} P \pi^2 \\
 \epsilon &= \frac{2 \cdot 1 \cdot 2 \cdot 3}{\pi} Q \pi^4 \\
 \gamma &= \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{\pi} R \pi^6 \\
 \delta &= \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{\pi} S \pi^8 \\
 &\text{etc.}
 \end{aligned}$$

Ex lege ergo progressionis, si ponatur

$$\begin{aligned}
 1 - \frac{1}{3} + \frac{1}{3^3} - \frac{1}{3^5} + \frac{1}{3^7} - \frac{1}{3^9} + \text{etc.} &= A \pi = 1/2 \\
 1 - \frac{1}{5} + \frac{1}{5^3} - \frac{1}{5^5} + \frac{1}{5^7} - \frac{1}{5^9} + \text{etc.} &= B \pi^3
 \end{aligned}$$



#### §4 DE SERIEBUS QUIBUSDAM CONSIDERAT.

$$1 - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} - \frac{1}{6^5} + \text{etc.} = C \pi^5$$

$$1 - \frac{1}{2^7} + \frac{1}{3^7} - \frac{1}{4^7} + \frac{1}{5^7} - \frac{1}{6^7} + \text{etc.} = D \pi^7$$

etc.

Poterimus hos coefficientes A, B, C, D, etc. ita determinare, ut sit:

$$A = \frac{1}{2} \left( \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc.} \right)$$

$$B = \frac{A}{1 \cdot 2 \cdot 3} - \frac{2}{\pi} \left( \frac{P\pi^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{Q\pi^4}{4 \cdot 5 \cdot 6 \cdot 7} + \frac{R\pi^6}{6 \cdot 7 \cdot 8 \cdot 9} + \text{etc.} \right)$$

$$C = \frac{B}{1 \cdot 2 \cdot 3} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{2}{\pi} \left( \frac{P\pi^2}{2 \cdot 3 \cdot \dots \cdot 7} + \frac{Q\pi^4}{4 \cdot 5 \cdot \dots \cdot 9} + \frac{R\pi^6}{6 \cdot 7 \cdot \dots \cdot 11} + \text{etc.} \right)$$

$$D = \frac{C}{1 \cdot 2 \cdot 3} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{A}{1 \cdot 2 \cdot \dots \cdot 7} - \frac{2}{\pi} \left( \frac{P\pi^2}{2 \cdot 3 \cdot \dots \cdot 9} + \frac{Q\pi^4}{4 \cdot 5 \cdot \dots \cdot 11} + \text{etc.} \right)$$

§. 33. Antequam autem quicquam hinc concludere suscipiamus, exemplo vno doceamus regulam hanc inuentam recte se habere; ac valores veros litterarum inde prodire. Sumamus igitur primam formulam, et cum sit  $A = \frac{1}{2} \pi$  habebitur ista aequatio

$$\frac{1}{2} = \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc.}$$

Est vero ad veros valores appropinquando

$$1/2 = 0,693147180$$

$$P\pi^2 = 1,233700550$$

$$Q\pi^4 = 1,014678031$$

$$R\pi^6 = 1,001447077$$

$$S\pi^8 = 1,000155179$$

$$T\pi^{10} = 1,000017041$$

$$V\pi^{12} = 1,000001885$$

$$W\pi^{14} = 1,000000209$$

$$X\pi^{16} = 1,000000023$$

$$Y\pi^{18} = 1,000000002$$

Sumamus primum unitates integras, pro  $P\pi^2$ ,  $Q\pi^4$  etc. habe-

habebimus  $\frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 5} + \frac{1}{6 \cdot 7} + \frac{1}{8 \cdot 9} + \text{etc.}$  cuius seriei summa constat, quippe quae est  $= 1 - \frac{1}{2}$  seu  $0,306852810$ .

nunc sumamus eorundem terminorum fractiones annexas, quae per respectuos denominatores diuisae dabunt:

0,038950091  
 0,000733901  
 0,000134454  
 0,000021550  
 0,000004550  
 12  
 1

0,039720771 addatur  $1 - \frac{1}{2}$   
 0,306852819

0,346573590 At vero est  $\frac{1}{2} =$   
 0,346573590

unde aequalitas luculenter perspicitur.

§. 34. Cum igitur certo nunc constet de veritate propositionis §. 32. assertae, legem habemus, secundum quam summae serierum  $1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \text{etc.}$  denotante  $n$  numerum imparem quemcunque progrediuntur. Verum quia ex observatione tantum nobis constat esse

$$\frac{1}{2} = \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc. siue}$$

$$\frac{1}{2} = \left\{ \begin{array}{l} \frac{1}{3} (1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.}) \\ \frac{1}{15} (1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}) \\ \frac{1}{21} (1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.}) \\ \frac{1}{35} (1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.}) \\ \frac{1}{55} (+ \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{9^{10}} + \text{etc.}) \\ \text{etc.} \end{array} \right.$$

operae pretium erit in demonstrationem huius veritatis inquirere. Ponamus igitur

$$s = \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc.}$$

atque instituantur sequentes transformationes.

$$\frac{d \cdot \pi s}{d\pi} = \frac{P\pi^2}{2} + \frac{Q\pi^4}{4} + \frac{R\pi^6}{6} + \text{etc.}$$

$$\frac{d \cdot d \cdot \pi s}{d\pi^2} = P\pi + Q\pi^3 + R\pi^5 + \text{etc.}$$

Quia haec ultima series si per  $\pi$  multiplicetur, summam habet  $\frac{\pi}{4}$  tang. A.  $\frac{\pi}{2}$ , quae expressio locum habet, etiam si  $\pi$  quantitas fit variabilis, quemadmodum hic posuimus. Erit itaque

$$d \cdot d \cdot \pi s = \frac{d\pi^2}{4} \text{ tang. A. } \frac{\pi}{2} \text{ et hinc}$$

$$d \cdot \pi s = \frac{d\pi}{4} \int d\pi \text{ tang. A. } \frac{\pi}{2} \text{ ac denique}$$

$$s = \frac{1}{4\pi} \int d\pi \int d\pi \text{ tang. A. } \frac{\pi}{2}$$

cuius aequationis radix iam patet, quippe est  $s = -\frac{1}{2}$ .

§. 35. Consideremus primum formulam hanc  $\int d\pi$  tang. A.  $\frac{\pi}{2}$  quae abit in  $\int \frac{d\pi \sin A. \frac{\pi}{2}}{\cos A. \frac{\pi}{2}} = -2 \int \text{cof. A. } \frac{\pi}{2}$  hoc

vero integrali substituto habebimus  $s = \frac{1}{\pi} \int \frac{d\pi}{2} \int \text{cof. A. } \frac{\pi}{2}$ .

Ad hanc formulam integrandam pono tang. A.  $\frac{\pi}{2} = t$ , erit cof. A.  $\frac{\pi}{2} = \frac{1}{\sqrt{1+t^2}}$  et  $-2 \int \text{cof. A. } \frac{\pi}{2} = 2 \int \sqrt{1+t^2} = \frac{1}{2} \int (1+t^2)$

et  $\frac{d\pi}{2} = \frac{ds}{1+t^2}$ ; ex quo erit  $s = \frac{1}{2\pi} \int \frac{ds}{1+t^2} \int (1+t^2)$  atque ideo quaestio huc est reducta, ut definiatur integrale

$\int \frac{ds \int (1+t^2)}{1+t^2}$  tali adhibita constante ut integrale evanescat posito,  $t=0$ ; quo facto restitui oportet  $t = \text{tang. A. } \frac{\pi}{2}$  seu

ob  $\frac{\pi}{2} = \text{arctui } 90^\circ$ , erit  $t = \infty$ . Formula autem haec, quia est  $\int (1+t^2) = \frac{t}{1+t^2} + \frac{1}{2(1+t^2)^2} + \frac{t^6}{3(1+t^2)^3} + \frac{t^8}{4(1+t^2)^4}$

+ etc.

+ etc. transit in fequentem ita vt fit  $\int \frac{dt}{1+tt} l(1+tt) =$   
 $\int \frac{t dt}{(1+tt)^2} + \frac{1}{2} \int \frac{t^2 dt}{(1+tt)^3} + \frac{1}{3} \int \frac{t^3 dt}{(1+tt)^4} + \frac{1}{4} \int \frac{t^4 dt}{(1+tt)^5} +$  etc.  
 Per reductionem autem formularum integralium est generaliter.

$$\int \frac{t^{2m} dt}{(1+tt)^{m+1}} = \frac{-t^{2m-1}}{2m(1+tt)^m} + \frac{2m-1}{2m} \int \frac{t^{2m-2} dt}{(1+tt)^m}$$

Quare cum fit  $\int \frac{dt}{1+tt} = \frac{\pi}{2}$  erit

$$\begin{aligned} \int \frac{t dt}{(1+tt)^2} &= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \frac{t}{1+tt} \\ \int \frac{t^2 dt}{(1+tt)^3} &= \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{t}{1+tt} - \frac{1}{4} \cdot \frac{t^2}{(1+tt)^2} \\ \int \frac{t^4 dt}{(1+tt)^4} &= \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5}{4 \cdot 6} \cdot \frac{t^2}{(1+tt)^2} - \frac{1 \cdot 3 \cdot 5}{6(1+tt)^3} \\ \int \frac{t^6 dt}{(1+tt)^5} &= \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} \cdot \frac{t^2}{(1+tt)^2} - \frac{1 \cdot 3 \cdot 7}{6 \cdot 8} \cdot \frac{t^3}{(1+tt)^3} \\ &\quad - \frac{1}{8} \cdot \frac{t^4}{(1+tt)^4} \\ \int \frac{t^{10} dt}{(1+tt)^6} &= \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{t^2}{(1+tt)^2} \\ &\quad - \frac{1 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10} \cdot \frac{t^3}{(1+tt)^3} - \frac{1 \cdot 9}{8 \cdot 10} \cdot \frac{t^4}{(1+tt)^4} - \frac{1}{10} \cdot \frac{t^5}{(1+tt)^5} \text{ etc.} \end{aligned}$$

Ex his substitutis orietur  $\int \frac{dt}{1+tt} l(1+tt) =$

$$\begin{aligned} &+ \frac{\pi}{2} \left( \frac{1}{2 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 5} + \text{etc.} \right) \\ &- \frac{t}{1+tt} \left( \frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} \right) \\ &- \frac{t^2}{4(1+tt)^2} \left( \frac{1}{2} + \frac{5}{6 \cdot 3} + \frac{5 \cdot 7}{6 \cdot 8 \cdot 4} + \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 \cdot 5} + \text{etc.} \right) \\ &- \frac{t^3}{6(1+tt)^3} \left( \frac{1}{3} + \frac{7}{8 \cdot 4} + \frac{7 \cdot 9}{8 \cdot 10 \cdot 5} + \frac{7 \cdot 9 \cdot 11}{8 \cdot 10 \cdot 12 \cdot 6} + \text{etc.} \right) \\ &- \frac{t^4}{8(1+tt)^4} \left( \frac{1}{4} + \frac{9}{10 \cdot 5} + \frac{9 \cdot 11}{10 \cdot 12 \cdot 6} + \frac{9 \cdot 11 \cdot 13}{10 \cdot 12 \cdot 14 \cdot 7} + \text{etc.} \right) \\ &\text{etc.} \end{aligned}$$

§. 36. Queramus primum summam feriei huius,

$$\frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.}$$

ponamusque :

S =

$$s = \frac{x}{2 \cdot 1} + \frac{1 \cdot 3 \cdot x^2}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5 \cdot x^3}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot x^4}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.}$$

erit  $s = \int \frac{dx}{x\sqrt{1-x}} \ln x$ , vt euoluenti facile patebit. At est

$$\int \frac{dx}{x\sqrt{1-x}} = c - l(1 + \sqrt{1-x}) - l(1 - \sqrt{1-x}) \text{ hinc:}$$

que  $s = c - l(1 + \sqrt{1-x}) + l(1 - \sqrt{1-x}) - \ln x$ . vbi

constans  $c$  ita debet esse comparata, vt posito  $x = 0$  fiat

$s = 0$ . Fiat igitur  $x$  infinite paruum, erit  $\sqrt{1-x} =$

$1 - \frac{x}{2}$  et  $l(1 - \sqrt{1-x}) = l\frac{x}{2} = \ln x - \ln 2$  et  $l(1 + \sqrt{1-x})$

$= \ln 2$ ; vnde  $c = 2 \ln 2$ . Ponatur nunc  $x = 1$ , erit  $s = 2 \ln 2$

atque

$$\frac{x}{2 \cdot 1} + \frac{1 \cdot 3 \cdot x^2}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5 \cdot x^3}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot x^4}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} = 2 \ln 2$$

Ex hac autem serie reliquarum serierum summae ita determinabuntur vt fit:

$$\frac{1}{2} + \frac{5}{6 \cdot 3} + \frac{5 \cdot 7}{6 \cdot 3 \cdot 4} + \frac{5 \cdot 7 \cdot 9}{6 \cdot 3 \cdot 10 \cdot 5} + \text{etc.} = \frac{2 \cdot 4}{1 \cdot 3} \cdot 2 \ln 2 = \frac{2 \cdot 4}{1 \cdot 3 \cdot 2}$$

$$\frac{1}{3} + \frac{7}{8 \cdot 4} + \frac{7 \cdot 9}{8 \cdot 10 \cdot 5} + \frac{7 \cdot 9 \cdot 11}{8 \cdot 10 \cdot 12 \cdot 6} + \text{etc.} = \frac{2 \cdot 4 \cdot 6}{1 \cdot 1 \cdot 5} \cdot 2 \ln 2 = \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 2} = \frac{6}{5 \cdot 2}$$

$$\frac{1}{4} + \frac{9}{10 \cdot 5} + \frac{9 \cdot 11}{10 \cdot 12 \cdot 6} + \text{etc.} = \frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7} \cdot 2 \ln 2 = \frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 2} = \frac{6 \cdot 8}{5 \cdot 7 \cdot 2} = \frac{8}{7 \cdot 3}$$

etc.

Quibus summis substitutis proueniet  $\int \frac{dt}{1+t} l(1+tt)$

$$= + \frac{\pi}{2} \cdot 2 \ln 2$$

$$- \frac{t}{1+t} \cdot 2 \ln 2$$

$$- \frac{t^3}{(1+t)^2} \left( \frac{2}{3} \cdot 2 \ln 2 - \frac{1}{3 \cdot 1} \right)$$

$$- \frac{t^5}{(1+t)^3} \left( \frac{2 \cdot 4}{3 \cdot 5} \cdot 2 \ln 2 - \frac{4}{3 \cdot 5 \cdot 1} - \frac{1}{5 \cdot 2} \right)$$

$$- \frac{t^7}{(1+t)^4} \left( \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot 2 \ln 2 - \frac{4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 1} - \frac{6}{5 \cdot 7 \cdot 2} - \frac{2}{7 \cdot 3} \right)$$

$$- \frac{t^9}{(1+t)^5} \left( \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} \cdot 2 \ln 2 - \frac{4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 1} - \frac{6 \cdot 8}{5 \cdot 7 \cdot 9 \cdot 2} - \frac{8}{7 \cdot 9 \cdot 3} - \frac{3}{9 \cdot 4} \right)$$

etc.

§. 37. Quoniam vero ad institutum nostrum post integrationem absolutam poni debet  $t = \infty$ , fiet  $\int \frac{dt}{1+tt} l(1+tt) = \pi/2$  atque  $s = \frac{1}{2} \pi \int \frac{dt}{1+tt} l(1+tt) = \frac{\pi^2}{2}$ , qui est ille ipse valor quem praecedimus prodire debere (§. 34.). Reliqui enim termini in expressione, quam pro  $\int \frac{dt}{1+tt} l(1+tt)$  inuenimus, si ponatur  $t = \infty$  omnes euanescent, quia in denominatoribus singulorum terminorum  $t$  plures habet dimensiones quam in numeratoribus, atque insuper coefficientes numerici decrescunt. Nisi enim hoc eueniret, tuto concludere non possemus summam omnium terminorum, quorum quisque euanescit esse  $= 0$ . Nam si verbi gratia priores tantum coefficientium numericorum partes accipiantur, vt prodiret haec series:

$$\frac{t}{1+tt} + \frac{2t^3}{3(1+tt)^2} + \frac{2 \cdot 4t^5}{3 \cdot 5(1+tt)^3} + \frac{2 \cdot 4 \cdot 6 \cdot t^7}{3 \cdot 5 \cdot 7(1+tt)^4} \text{ etc.}$$

summa ipsius casu quo  $t = \infty$ , fit finita et  $= \frac{\pi}{2}$  etiam si singuli termini euanescant, quod si autem integri coefficientes capiuntur ob seriem eorum valde conuergentium, tota quoque series euadit  $= 0$ .

§. 38. Inquiramus nunc in summam huius seriei

$$1 - \frac{1}{2}s + \frac{1}{3}s^2 - \frac{1}{4}s^3 + \text{etc.} = B \pi^3 \text{ quae summa per §. 32.}$$

$$\text{erit } B \pi^3 = \frac{\pi^2 l 2}{1 \cdot 2 \cdot 3} - 2 \pi^2 \left( \frac{P \pi^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{Q \pi^4}{4 \cdot 5 \cdot 6 \cdot 7} + \frac{R \pi^6}{6 \cdot 7 \cdot 8 \cdot 9} + \text{etc.} \right)$$

Ad valorem huius quantitatis inueniendum fit  $s =$

$$\frac{P \pi^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{Q \pi^4}{4 \cdot 5 \cdot 6 \cdot 7} + \frac{R \pi^6}{6 \cdot 7 \cdot 8 \cdot 9} + \text{etc.}$$

$$\text{erit } \frac{d \cdot \pi^3 s}{d \pi} = \frac{P \pi^4}{2 \cdot 3 \cdot 4} + \frac{Q \pi^6}{4 \cdot 5 \cdot 6} + \frac{R \pi^8}{6 \cdot 7 \cdot 8} + \text{etc.}$$

$$\frac{dd \cdot \pi^3 s}{d \pi^2} = \frac{P \pi^3}{2 \cdot 3} + \frac{Q \pi^5}{4 \cdot 5} + \frac{R \pi^7}{6 \cdot 7} + \text{etc.}$$

$$\frac{d^2 \cdot \pi^3 s}{d \pi^3} = \frac{P \pi^2}{2} + \frac{Q \pi^4}{4} + \frac{R \pi^6}{6} + \text{etc.}$$

$$\frac{d^4 \cdot \pi^3 s}{d \pi^4} = P \pi + Q \pi^3 + R \pi^5 + \text{etc.} = \frac{1}{2} \text{ tang. } A \frac{\pi}{2}.$$

Regrediendo erit ergo.

$$\frac{d^2 \cdot \pi^2 s}{d\pi^2} = \frac{1}{4} \int d\pi \text{ tang. } A \cdot \frac{\pi}{2}.$$

$$\frac{dd \cdot \pi^2 s}{d\pi^2} = \frac{1}{4} \int d\pi \int d\pi \text{ tang. } A \cdot \frac{\pi}{2}.$$

$$\frac{d \cdot \pi^2 s}{d\pi} = \frac{1}{4} \int d\pi \int d\pi \int d\pi \text{ tang. } A \cdot \frac{\pi}{2}.$$

$$\pi^2 s = \frac{1}{4} \int d\pi \int d\pi \int d\pi \int d\pi \text{ tang. } A \cdot \frac{\pi}{2}.$$

Atque hinc habebitur summa seriei propositae  $B \pi^2 = \frac{\pi^2 l^2}{6} - \frac{1}{2\pi} \int d\pi \int d\pi \int d\pi \int d\pi \text{ tang. } A \cdot \frac{\pi}{2}$ , quae omnia integralia ita debent accipi, vt evanescant posito  $\pi = 0$ .

§. 39. Ponatur  $\frac{\pi}{2} = q$ , ita vt integrationibus absolutis  $q$  denotet quartam peripheriae partem circuli, cuius radius = 1 seu arcum 90. graduum. Sitque porro  $\sin. A \cdot q = y$  et  $\cos. A \cdot q = x = \sqrt{1 - yy}$ , erit  $\text{tang. } A \cdot \frac{\pi}{2} = \frac{y}{x}$ .

Quare ob  $\pi = 2q$ , erit summa nostrae seriei  $B \pi^2 = \frac{2qq l^2}{3} - \frac{4}{q} \int dq \int dq \int dq \int \frac{y dq}{x}$ . Ponamus tantisper  $\int \frac{y dq}{x} = u$  erit  $B \pi^2 = \frac{2qq l^2}{3} - \frac{4u}{q}$ , vbi in quantitate  $u$  invenienda omnes integrationes ita debent accipi, vt integralia singula evanescant posito  $q = 0$  et  $y = 0$ , integralibus vero absolutis erit  $y = 1$  et  $x = 0$ .

Est vero  $\int \frac{y dq}{x} = \int \frac{y dy}{1 - yy} = -l \sqrt{1 - yy} = l \frac{1}{x}$ .

et  $l \frac{1}{x} = \frac{yy}{2} + \frac{y^3}{4} + \frac{y^5}{6} + \frac{y^7}{8} + \frac{y^9}{10} + \text{etc.}$

Cum nunc fit  $u = \int dq \int dq \int dq l \frac{1}{x}$ , erit per reductionem integralium

$$u = q \int dq \int dq l \frac{1}{x} - \int q dq \int dq l \frac{1}{x}$$

atque porro

$$\int dq \int dq l \frac{1}{x} = q \int dq l \frac{1}{x} - \int q dq l \frac{1}{x}$$

$$\int q dq \int dq l \frac{1}{x} = \frac{qq}{2} \int dq l \frac{1}{x} - \frac{1}{2} \int qq dq l \frac{1}{x}$$

ergo

$$u = \frac{1}{2} qq \int dq l \frac{1}{x} - q \int q dq l \frac{1}{x} + \frac{1}{2} \int qq dq l \frac{1}{x}$$

ita

ita vt nunc tres formulas habeamus simpliciter differentiales, quas integrare debemus.

§. 40. Consideremus singulas has tres formulas seorsim, ac primo quidem hanc  $\int dq \, l \frac{x}{x}$ , quam etsi iam supra integrauimus, tamen eandem ex consideratione sinuum et cosinum denuo integremus, vt via facilior paretur ad reliquas integrandas. Est igitur:

$$\int dq \, l \frac{x}{x} = \int dq \left( \frac{y^2}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \frac{y^{10}}{10} + \text{etc.} \right)$$

Ad hoc integrale inueniendum consideretur membrum eius quodcunque  $\int y^{n+2} dq$ , et cum fit  $dq = \frac{dy}{x} = \frac{-dx}{y}$  et  $xx + yy = 1$ , erit

$$\int y^{n+2} dq = -\int y^{n+1} dx = -y^{n+1} x + (n+1) \int y^n x dy$$

$$\text{at est } \int y^n x dy = \int y^n x^2 dq = \int y^n dq - \int y^{n+2} dq$$

ob  $xx = 1 - yy$ : ex quo erit:

$$\int y^{n+2} dq = -y^{n+1} x + (n+1) \int y^n dq - (n+1) \int y^{n+2} dq$$

atque

$$\int y^{n+2} dq = \frac{-y^{n+1} x}{n+2} + \frac{n+1}{n+2} \int y^n dq$$

Hinc itaque integrale cuiusque membri reducitur ad integrale praecedentis, et quoniam integratione absoluta fit  $x = 0$ ; erit pro hoc casu

$$\int y^{n+2} dq = \frac{n+1}{n+2} \int y^n dq.$$

Ex hac ergo formula reperientur singulae integralis partes vt sequitur.

$$\int y^2 dq = \frac{1}{2} q$$

$$\int y^4 dq = \frac{1 \cdot 3}{2 \cdot 4} q$$

$$\int y^6 dq = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} q$$

$$\int y^8 dq = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} q$$

M 2

Hanc



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Hancobrem habebitur:  $\int dq l \frac{x}{x} = \frac{1}{2} q \left( \frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} \right)$  cuius seriei cum iam supra inuenta fit summa (§. 36.)  $= 2 l 2$  erit  $\int dq l \frac{x}{x} = q l 2$ .

§. 41. Progrediamur iam ad secundam formulam integram  $\int q dq l \frac{x}{x}$ , quae abit in

$$\int q dq l \frac{x}{x} = \int q dq \left( \frac{y^2}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.} \right)$$

cuius partem consideremus quamcunque  $\int y^{n+2} q dq = - \int y^{n+1} q dx = -y^{n+1} q x + \int y^{n+1} x dq + (n+1) \int y^n q x dy$   
 $= -y^{n+1} q x + \frac{y^{n+2}}{n+2} + (n+1) \int y^n q dq - (n+1) \int y^{n+2} q dq$

Pofito itaque  $y = 1$  et  $x = 0$ , erit

$$\int y^{n+2} q dq = \frac{1}{(n+2)^2} + \frac{n+1}{n+2} \int y^n q dq$$

Integralia igitur fingulorum membrorum ita se habebunt:

$$\int y^2 q dq = \frac{1}{2^2} + \frac{1}{2} \cdot \frac{q^2}{2}$$

$$\int y^4 q dq = \frac{1}{4^2} + \frac{3}{4 \cdot 2^2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{q^2}{2}$$

$$\int y^6 q dq = \frac{1}{6^2} + \frac{5}{6 \cdot 4^2} + \frac{3 \cdot 5}{4 \cdot 6 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{q^2}{2}$$

$$\int y^8 q dq = \frac{1}{8^2} + \frac{7}{8 \cdot 6^2} + \frac{5 \cdot 7}{6 \cdot 8 \cdot 4^2} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{q^2}{2}$$

etc.

Ex quo obtinebitur integrale  $\int q dq l \frac{x}{x} =$

$$+ \frac{qq}{4} \left( \frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} \text{ etc.} \right)$$

$$+ \frac{1}{2 \cdot 2^2} \left( \frac{1}{2} + \frac{3}{4 \cdot 2} + \frac{3 \cdot 5}{4 \cdot 6 \cdot 3} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} \right)$$

$$+ \frac{1}{2 \cdot 4^2} \left( \frac{1}{2} + \frac{5}{6 \cdot 3} + \frac{5 \cdot 7}{6 \cdot 8 \cdot 4} + \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10 \cdot 5} + \text{etc.} \right)$$

$$+ \frac{1}{2 \cdot 6^2} \left( \frac{1}{2} + \frac{7}{8 \cdot 4} + \frac{7 \cdot 9}{8 \cdot 10 \cdot 5} + \frac{7 \cdot 9 \cdot 11}{8 \cdot 10 \cdot 12 \cdot 6} + \text{etc.} \right)$$

etc.

Vel etiam hac forma  $\int q dq l \frac{x}{x} =$

$$\frac{qq}{4} \left( \frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} \right)$$

$$\begin{aligned}
 & + \frac{1}{2^2 \cdot 2} + \frac{1}{4^2 \cdot 4} + \frac{1}{6^2 \cdot 6} + \frac{1}{8^2 \cdot 8} + \text{etc.} \\
 & + \frac{3}{2^2 \cdot 4^2} + \frac{5}{4^2 \cdot 6^2} + \frac{7}{6^2 \cdot 8^2} + \frac{9}{8^2 \cdot 10^2} + \text{etc.} \\
 & + \frac{3 \cdot 5}{2^2 \cdot 4 \cdot 6^2} + \frac{5 \cdot 7}{4^2 \cdot 6 \cdot 8^2} + \frac{7 \cdot 9}{6^2 \cdot 8 \cdot 10^2} + \frac{9 \cdot 11}{8^2 \cdot 10 \cdot 12^2} + \text{etc.} \\
 & + \frac{3 \cdot 5 \cdot 7}{2^2 \cdot 4 \cdot 6 \cdot 8^2} + \frac{5 \cdot 7 \cdot 9}{4^2 \cdot 6 \cdot 8 \cdot 10^2} + \frac{7 \cdot 9 \cdot 11}{6^2 \cdot 8 \cdot 10 \cdot 12^2} + \text{etc.}
 \end{aligned}$$

hae autem series id ipsum, quod est in quaestione involvant, nempe summationem cuborum terminorum seriei harmonicae.

§. 42. Quod si sequamur priorem formam, omnes series summabiles §. 36, habebiturque

$$\begin{aligned}
 \int q dq l \frac{1}{x} &= \frac{qq}{2} l 2 \\
 & + \frac{1}{2^2} \left( \frac{2}{1} l 2 \right) \\
 & + \frac{1}{4^2} \left( \frac{2 \cdot 4}{1 \cdot 3} l 2 - \frac{4}{3 \cdot 2} \right) \\
 & + \frac{1}{6^2} \left( \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5} l 2 - \frac{4 \cdot 6}{3 \cdot 5 \cdot 2} - \frac{6}{5 \cdot 4} \right) \\
 & + \frac{1}{8^2} \left( \frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7} l 2 - \frac{4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 2} - \frac{6 \cdot 8}{5 \cdot 7 \cdot 4} - \frac{8}{7 \cdot 6} \right) \text{ etc.}
 \end{aligned}$$

quae si denuo series deorsum capiantur, dant.

$$\begin{aligned}
 \int q dq l \frac{1}{x} &= \frac{qq}{2} l 2 \\
 & + l 2 \left( \frac{1}{2} + \frac{2}{3 \cdot 4} + \frac{2 \cdot 4}{3 \cdot 5 \cdot 6} + \frac{2 \cdot 4 \cdot 6}{5 \cdot 6 \cdot 7 \cdot 8} + \text{etc.} \right) \\
 & - \frac{1}{6} \left( \frac{1}{4} + \frac{4}{5 \cdot 6} + \frac{4 \cdot 6}{5 \cdot 7 \cdot 8} + \frac{4 \cdot 6 \cdot 8}{5 \cdot 7 \cdot 9 \cdot 10} + \text{etc.} \right) \\
 & - \frac{1}{4 \cdot 5} \left( \frac{1}{6} + \frac{6}{7 \cdot 8} + \frac{6 \cdot 8}{7 \cdot 9 \cdot 10} + \frac{6 \cdot 8 \cdot 10}{7 \cdot 9 \cdot 11 \cdot 12} + \text{etc.} \right) \\
 & - \frac{1}{6 \cdot 7} \left( \frac{1}{8} + \frac{8}{9 \cdot 10} + \frac{8 \cdot 10}{9 \cdot 11 \cdot 12} + \frac{8 \cdot 10 \cdot 12}{9 \cdot 11 \cdot 13 \cdot 14} + \text{etc.} \right)
 \end{aligned}$$

At est  $\frac{1}{2} + \frac{2}{3 \cdot 4} + \frac{2 \cdot 4}{3 \cdot 5 \cdot 6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 8} + \text{etc.} = \frac{qq}{2}$   
 unde erit

$$\begin{aligned}
 \frac{1}{4} + \frac{4}{5 \cdot 6} + \frac{4 \cdot 6}{5 \cdot 7 \cdot 8} + \text{etc.} &= \frac{3}{2} \cdot \frac{qq}{2} - \frac{3}{2} \cdot \frac{1}{2} \\
 \frac{1}{6} + \frac{6}{7 \cdot 8} + \frac{6 \cdot 8}{7 \cdot 9 \cdot 10} + \text{etc.} &= \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{qq}{2} - \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{1}{2} - \frac{5}{4} \cdot \frac{1}{4} \\
 \frac{1}{8} + \frac{8}{9 \cdot 10} + \frac{8 \cdot 10}{9 \cdot 11 \cdot 12} + \text{etc.} &= \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} \cdot \frac{qq}{2} - \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2} - \frac{5 \cdot 7}{4 \cdot 6} \cdot \frac{1}{4} - \frac{7}{6} \cdot \frac{1}{8} \\
 & \text{etc.}
 \end{aligned}$$

M 3

Quam-

Hinc igitur erit:

$$u = \frac{q}{2.2} \left( \frac{qq}{6} - \frac{1}{2}z \right) \\ + \frac{1.3}{2.4.4} q \left( \frac{qq}{6} - \frac{1}{2}z - \frac{1}{4}z \right) \\ + \frac{1.3.5}{2.4.6.6} q \left( \frac{qq}{6} - \frac{1}{2}z - \frac{1}{2}z - \frac{1}{4}z - \frac{1}{6}z \right)$$

vel serie prima verticali actu summata

$$u = \frac{q^3}{6} l_2 \\ - \frac{q}{2.2} \left( \frac{1}{2}z \right) \\ - \frac{1.3}{2.4.4} q \left( \frac{1}{2}z + \frac{1}{4}z \right) \\ - \frac{1.3.5}{2.4.6.6} q \left( \frac{1}{2}z + \frac{1}{4}z + \frac{1}{6}z \right) \\ \text{etc.}$$

§. 44 Cum nunc seriei nostrae propositae  $1 - \frac{1}{2}z + \frac{1}{3}z - \frac{1}{4}z + \frac{1}{5}z - \text{etc.}$  summa sit  $= B \pi^3 = \frac{2qq l_2}{3} - \frac{4u}{q}$  fiet eadem summa =

$$+ \frac{1}{2.2} \cdot 1 \\ + \frac{1.3}{2.4.4} \left( 1 + \frac{1}{2}z \right) \\ + \frac{1.3.5}{2.4.6.6} \left( 1 + \frac{1}{2}z + \frac{1}{3}z \right) \\ + \frac{1.3.5.7}{2.4.6.8.8} \left( 1 + \frac{1}{2}z + \frac{1}{2}z + \frac{1}{4}z \right) \\ \text{etc.}$$

Vel cum sit  $\frac{1}{2.2} + \frac{1.3}{2.4.4} + \frac{1.3.5}{2.4.6.6} \text{ etc.} = l_2$  erit seriei propositae summa  $B \pi^3 = l_2 + \frac{1}{2}z \left( l_2 - \frac{1}{2.2} \right) + \frac{1}{3}z \left( l_2 - \frac{1}{2.2} - \frac{1.3}{2.4.4} \right) + \frac{1}{4}z \left( l_2 - \frac{1}{2.2} - \frac{1.3}{2.4.4} - \frac{1.3.5}{2.4.6.6} \right) + \text{etc.}$  siue haec eadem summa ita poterit exprimi vt sit

$$B \pi^3 = \frac{\pi^3}{6} l_2 - \frac{1}{2}z \left( \frac{1}{2.2} \right) \\ - \frac{1}{3}z \left( \frac{1}{2.2} + \frac{1.3}{2.4.4} \right) \\ - \frac{1}{4}z \left( \frac{1}{2.2} + \frac{1.3}{2.4.4} + \frac{1.3.5}{2.4.6.6} \right) \\ - \frac{1}{5}z \left( \frac{1}{2.2} + \frac{1.3}{2.4.4} + \frac{1.3.5}{2.4.6.6} + \frac{1.3.5.7}{2.4.6.8.8} \right) \text{etc.}$$

Quoniam autem vtcunque has series transmitemus, eas ad seriei simplicem, cuius summa constet, reducere non possumus, negotium hoc abruptamus, pluribus his expressionibus contenti, quas seriei propositae  $1 - \frac{1}{2}z + \frac{1}{3}z - \frac{1}{4}z + \frac{1}{5}z - \text{etc.}$  aequales esse inuenimus.

COM-