1750

Consideratio progressionis cuiusdam ad circuli quadraturam inveniendam idoneae

Leonhard Euler

E125

CONSIDERATIO PROGRESSIONIS CVIVSDAM AD CIRCVLI QVADRATVRAM INVENIENDAM IDONEAE.

AVCTORE
L. Eulero

§. 1.

Posita arcus cuiusdam in circulo, cuius radius ſit $= 1$, tangente $= t$, erit ipſe arcus $= \int \frac{dt}{1+tt}$; ſi iam loco differentialium dt ſubſtituantur particulae tangentis finitae quidem, ſed valde exiguae, atque integrationis loco actualis eiusmodi particularum additio perficiatur, expreſſio prodibit eo propius ad arcum propoſitum accedens, quo minores capiantur particulae tangentis t. Sic diuiſa tangente in n partes aequales, quarum quaelibet erit $\frac{t}{n}$, vicem differentialis dt ſubeunda, loco t ſucceſſiue poni debebunt valores $\frac{t}{n}, \frac{2t}{n}, \frac{3t}{n} \ldots$ vsque ad $\frac{nt}{n}$; quo facto arcus cuius tangens eſt t aequabitur huic progreſſioni $\frac{nt}{nn+tt} + \frac{nt}{nn+4tt} + \frac{nt}{nn+9tt} + \ldots \frac{nt}{nn+nntt}$ quae expreſſio eo minus a vero arcus valore differet, quo maior capiatur numerus n. Semper autem haec expreſſio nimis erit parua, niſi pro n ſumatur numerus reuera infinitus.

§. 2. Cum igitur ſumto pro n numero finito iſta progreſſio $\frac{nt}{n^2+t^2} + \frac{nt}{n^2+4t^2} + \frac{nt}{n^2+9t^2} + \ldots + \frac{nt}{n^2+n^2t^2}$ eo propius exprimat arcum cuius tangens eſt t, quo maior fuerit numerus n; perpetuo autem hoc modo valor pro-

prodeat nimis paruus, inueſtigabo, quantum iſta expreſſio quouis caſu a vera arcus longitudine deficiat. Quodſi enim defectus commode atque ad calculum accommodate exhiberi queat, per ſeriem vehementer conuergentem, iſta methodus cuiusque arcus longitudinem determinandi perquam facilis et idonea videtur.

§. 3. Ad hoc inueſtigandum ſingulos expreſſionis terminos methodo conſueta in progreſſionem geometricam reſoluo infinitam, vt ſequitur

$$\frac{nt}{n^2+t^2} = \frac{t}{n} - \frac{t^3}{n^3} + \frac{t^5}{n^5} - \frac{t^7}{n^7} + \text{etc.}$$
$$\frac{nt}{n^2+4t^2} = \frac{t}{n} - \frac{2^2t^3}{n^3} + \frac{2^4t^5}{n^5} - \frac{2^6t^7}{n^7} + \text{etc.}$$
$$\frac{nt}{n^2+9t^2} = \frac{t}{n} - \frac{3^2t^3}{n^3} + \frac{3^4t^5}{n^5} - \frac{3^6t^7}{n^7} + \text{etc.}$$
$$\frac{nt}{n^2+16t^2} = \frac{t}{n} - \frac{4^2t^3}{n^3} + \frac{4^4t^5}{n^5} - \frac{4^6t^7}{n^7} + \text{etc.}$$
$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$
$$\frac{nt}{n^2+n^2t^2} = \frac{t}{n} - \frac{n^2t^3}{n^3} + \frac{n^4t^5}{n^5} - \frac{n^6t^7}{n^7} + \text{etc.}$$

§. 4. Ponamus progreſſionis noſtrae oblatae

$$\frac{nt}{n^2+t^2} + \frac{nt}{n^2+4t^2} + \frac{nt}{n^2+9t^2} + \dots + \frac{nt}{n^2+n^2t^2}$$

valorem iam eſſe actu determinatum, eumque eſſe $= s$; ac transformatio facta ſequentem ſuppeditabit aequationem:

$$s = \begin{cases} + \frac{t}{n}\,(1^0 + 2^0 + 3^0 + \ldots + n^0) \\ - \frac{t^3}{n^3}\,(1^2 + 2^2 + 3^2 + \ldots + n^2) \\ + \frac{t^5}{n^5}\,(1^4 + 2^4 + 3^4 + \ldots + n^4) \\ - \frac{t^7}{n^7}\,(1^6 + 2^6 + 3^6 + \ldots + n^6) \\ + \frac{t^9}{n^9}\,(1^8 + 2^8 + 3^8 + \ldots + n^8) \\ - \frac{t^{11}}{n^{11}}\,(1^{10} + 2^{10} + 3^{10} + \ldots + n^{10}) \\ \qquad \text{etc. in infinitum} \end{cases}$$

§. 5. Quoniam in hac expreſſione coefficientes terminorum $\frac{t}{n}$, $\frac{t^3}{n^3}$, $\frac{t^5}{n^5}$, etc. ſunt ſummae progreſſionum poteſtatum parium ſeriei numerorum naturalium: ſummae hae autem ſe habent ſequenti modo

$$1^0 + 2^0 + \ldots + n^0 = n$$
$$1^2 + 2^2 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$
$$1^4 + 2^4 + \ldots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$
$$1^6 + 2^6 + \ldots + n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$$
$$1^8 + 2^8 + \ldots + n^8 = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30}$$
$$\text{etc.}$$

ſubſtituantur hi valores definiti loco indefinitorum, ac prodibit ſequens aequatio

$$s = \begin{cases} + t \\ - \frac{t^3}{3} - \frac{t^3}{2n} - \frac{t^3}{6n^2} \\ + \frac{t^5}{5} + \frac{t^5}{2n} + \frac{t^5}{3n^2} - \frac{t^5}{30n^4} \\ - \frac{t^7}{7} - \frac{t^7}{2n} - \frac{t^7}{2n^2} + \frac{t^7}{6n^4} - \frac{t^7}{42n^6} \\ + \frac{t^9}{9} + \frac{t^9}{2n} + \frac{2t^9}{3n^2} - \frac{7t^9}{15n^4} + \frac{2t^9}{9n^6} - \frac{t^9}{30n^8} \\ \qquad \text{etc. etc.} \end{cases}$$

cuius

cuius lex processus vlterioris pendet a coefficientibus formulae generalis series summandi. Praecipue autem ad continuandam hanc seriem notari conuenit coefficientes vltimorum terminorum in quaque expressione, quae hanc tenent progressionem: $\frac{1}{6}$; $\frac{1}{30}$; $\frac{1}{42}$; $\frac{1}{30}$; $\frac{5}{66}$; $\frac{691}{13 \cdot 210}$; $\frac{7}{6}$; $\frac{3617}{17 \cdot 30}$; $\frac{43867}{19 \cdot 42}$; $\frac{174611}{330}$; $\frac{854513}{6 \cdot 23}$; $\frac{236364091}{5 \cdot 546}$; quam hucusque produxisse sufficit.

§. 6. Disponantur termini inuentae expressionis secundum columnas a summo ad imum extensas, atque ad legem, qua singulae columnae progrediuntur, ordinentur; quo facto erit $s =$

$$
\begin{aligned}
&+ t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{t^{11}}{11} + \text{etc.} \\
&- \frac{t^2}{2n}\left(t - t^3 + t^5 - t^7 + t^9 - t^{11} + \text{etc.}\right) \\
&- \frac{t^2}{6n^2}\left(t - 2t^3 + 3t^5 - 4t^7 + 5t^9 - 6t^{11} + \text{etc.}\right) \\
&- \frac{t^4}{30n^4}\left(t - 5t^3 + 14t^5 - 30t^7 + 55t^9 - 91t^{11} + \text{etc.}\right) \\
&- \frac{t^6}{42n^6}\left(t - \frac{28}{3}t^3 + 42t^5 - 132t^7 + \frac{1001}{3}t^9 - 728t^{11} + \text{etc.}\right) \\
&\qquad\qquad\qquad \text{etc.}
\end{aligned}
$$

quae series omnes hanc tenent legem, vt potestas $\frac{t^m}{Nn^m}$ multiplicari debeat per istam seriem $t - \frac{(m+1)(m+2)}{2 \cdot 3} t^3 + \frac{(m+1)(m+2)(m+3)(m+4)}{2 \cdot 3 \cdot 4 \cdot 5} t^5 + \text{etc.}$

§. 7. Quanquam haec series ob m numerum integrum affirmatiuum in infinitum excurrit, tamen semper habet summam finitam, quae sequenti modo inuenietur. Ponatur tantisper seriei illius summa $= v$ erit $m\,v = \frac{m\,t}{1} - \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} t^3 + \frac{m(m+1)(m+2)(m+3)(m+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} t^5 - \text{etc.} =$

$= \frac{(1-t\sqrt{-1})^{-m}-(1+t\sqrt{-1})^{-m}}{2\sqrt{-1}}$. Haec autem expressio transmutatur in istam $mv = \frac{(1+t\sqrt{-1})^m-(1-t\sqrt{-1})^m}{2(1+tt)^m\sqrt{-1}}$. At binomiis his actu ad potestatem exponentis m euectis prodibit per aliam seriem $mv = \frac{1}{(1+tt)^m}\left(\frac{mt}{1} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}t^3 + \frac{m(m-1)(m-2)(m-3)(m-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}t^5 - \text{etc.}\right)$ quae ad nostrum institutum maxime est accommodata, cum sponte abrumpatur, quando m est numerus integer affirmatiuus.

§. 8. Series ergo v, per quam terminus quisque $\frac{t^m}{Nn^m}$ multiplicari debet, nunc transmutata est in hanc $\frac{1}{m(1+tt)^m}\left(\frac{mt}{1} - \frac{m(m-1)}{1\cdot 2}\frac{(m-2)}{3}t^3 + \text{etc.}\right)$; quamobrem habebitur $s =$

$$t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{t^{11}}{11} + \text{etc.}$$
$$- \frac{t^3}{2n(1+tt)}$$
$$- \frac{t^2}{2\cdot 6n^2(1+tt)^2}\cdot\frac{2t}{1}$$
$$- \frac{t^4}{4\cdot 30n^4(1+tt)^4}\left(\frac{4t}{1} - \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}t^3\right)$$
$$- \frac{t^6}{6\cdot 42n^6(1+tt)^6}\left(\frac{6t}{1} - \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}t^3 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}t^5\right)$$
$$- \frac{t^8}{8\cdot 30n^8(1+tt)^8}\left(\frac{8t}{1} - \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}t^3 + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5}t^5 - \text{etc.}\right)$$
$$- \frac{5t^{10}}{10\cdot 66n^{10}(1+tt)^{10}}\left(\frac{10t}{1} - \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}t^3 + \text{etc.}\right)$$
$$- \frac{691t^{12}}{12\cdot 13\cdot 210n^{12}(1+tt)^{12}}\left(\frac{12t}{1} - \frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3}t^3 + \text{etc.}\right.$$
$$- \frac{7t^{14}}{14\cdot 6n^{14}(1+tt)^{14}}\left(\frac{14t}{1} - \frac{14\cdot 13\cdot 12}{1\cdot 2\cdot 3}t^3 + \text{etc.}\right)$$
$$- \frac{3617t^{16}}{16\cdot 17\cdot 30n^{16}(1+tt)^{16}}\left(\frac{16t}{1} - \frac{16\cdot 15\cdot 14}{1\cdot 2\cdot 3}t^3 + \text{etc.}\right)$$
$$\text{etc.}$$

§. 9.

§. 9. Cum nunc huius expreſſionis prima ſeries $t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} +$ etc. illum ipſum circuli arcum denotet, cuius tangens eſt t, quem quaerere inſtituimus, ſit z iſte arcus atque manente $s = \frac{nt}{n^2 + t^2} + \frac{nt}{n^2 + 4t^2} + \frac{nt}{n^2 + 9t^2} + \ldots\ldots + \frac{nt}{n^2 + n^2 t^2}$, reperietur arcus $z = s + \frac{t}{2n(1+tt)} + \frac{1}{6} \frac{tt}{2nn(1+tt)^2} \cdot 2t + \frac{1}{30} \cdot \frac{t^4}{4n^4(1+tt)^4} (4t - 4t^3) + \frac{1}{42} \cdot \frac{t^6}{6n^6(1+tt)^6} (6t - 20t^3 + 6t^5) + \frac{1}{30} \cdot \frac{t^8}{8n^8(1+tt)^8} (8t - 56t^3 + 56t^5 - 8t^7) + \frac{5}{66} \cdot \frac{t^{10}}{10n^{10}(1+tt)^{10}} (10t - 120t^3 + 252t^5 - 120t^7 + 10t^9) + \frac{691}{13 \cdot 210} \frac{t^{12}}{12n^{12}(1+tt)^{12}} (12t - 220t^3 + 792t^5 - 792t^7 + 220t^9 - 12t^{11}) + \frac{7}{6} \cdot \frac{t^{14}}{14n^{14}(1+tt)^{14}} (14t - 364t^3 + 2002t^5 - 3432t^7 + 2002t^9 - 364t^{11} + 14t^{13}) +$ etc.

§. 10. Expreſſio haec commodiſſime accommodabitur ad caſum, quo eſt $t = 1$, cum alterni ſeriei termini euaneſcant, atque inſuper arcus z abeat in quartam ſemiperipheriae circuli partem, poſita ergo ſemiperipheria circuli $= \pi$, ita vt ſit $z = \frac{\pi}{4}$, ſumtoque quocunque numero integro affirmatiuo pro n erit $\frac{\pi}{4} = \frac{n}{n^2+1} + \frac{n}{n^2+4} + \frac{n}{n^2+9} + \frac{n}{n^2+16} + \ldots \frac{n}{n^2+n^2} + \frac{1}{4n} + \frac{1}{6} \cdot \frac{1}{2 \cdot 2n^2} - \frac{1}{42} \cdot \frac{1}{2^3 \cdot 6n^6} + \frac{5}{66} \cdot \frac{1}{2^5 \cdot 10 n^{10}} - \frac{7}{6} \cdot \frac{1}{2^7 \cdot 14 n^{14}} + \frac{43867}{19 \cdot 42} \cdot \frac{1}{2^9 \cdot 18 n^{18}} - \frac{854513}{6 \cdot 23} \cdot \frac{1}{2^{11} \cdot 22 n^{22}} +$ etc. Hinc igitur erit $\pi = \frac{4n}{n^2+1} + \frac{4n}{n^2+4} + \frac{4n}{n^2+9} + \ldots\ldots \frac{4n}{n^2+n^2} + \frac{1}{n} + \frac{1}{6} \cdot \frac{1}{1n^2} - \frac{1}{42} \cdot \frac{1}{2^2 \cdot 3n^6} + \frac{5}{66} \cdot \frac{1}{2^4 \cdot 5n^{10}} - \frac{7}{6} \cdot \frac{1}{2^6 \cdot 7n^{14}} + \frac{43867}{19 \cdot 42} \cdot \frac{1}{2^8 \cdot 9n^{18}} - \frac{854513}{6 \cdot 23} \cdot \frac{1}{2^{10}\, 11n^{22}} +$ etc. quae ſeries eo magis conuergit, quo maior numerus pro n accipiatur.

§. 11. Quamuis autem haec ſeries eo magis conuergere videatur, quo maior ſit numerus n tamen perpetuo ad certum vſque terminum tantum conuergit, poſt quem

 termini

termini crescent iterum; hancque ob causam non iuuat seriem eo vsque adhibere, quoad termini diuergere incipiant, sed expediet operationem ibi finire, vbi maxima obseruatur conuergentia. Namque si fractionum $\frac{1}{6}$; $\frac{1}{30}$; $\frac{1}{42}$; $\frac{1}{30}$; $\frac{5}{66}$; etc. ea quae indicem habet ν ponatur $= X$, atque sequens $= Y$ erit semper $\frac{Y}{X} > \frac{(\nu-1)(2\nu-3)}{2\pi^2}$, atque ν in infinitum crescente fiet $\frac{Y}{X} = \frac{\nu^2}{\pi^2}$. Ex quo apparet terminos istius seriei continuo magis crescere, atque nullam progressionem geometricam quantumuis conuergentem cum ea coniunctam eam reddere posse conuergentem. Hinc autem concluditur in serie paragr. praec. plures terminos accipi non licere quam ad summum $\frac{\pi n}{\sqrt{2}}$ hoc est proxime $2n$: etiamsi enim sumerentur plures termini, summa non ad veram propior accedens reperiretur.

§. 12. Ex hoc vero ipso subsidium ad valorem ipsius π propius inueniendum ope seriei paragraphi 10, consequitur. Ponamus enim seriei: $\frac{1}{6} \cdot \frac{1}{1n^2} - \frac{1}{42} \cdot \frac{1}{2^2 \cdot 3n^6} + \frac{5}{66} \cdot \frac{1}{2^4 \cdot 5n^{10}} -$ etc. iam actu esse additos μ terminos, ac sequentem terminum esse $= P$, eius loco sumatur ista expressio $\frac{\pi^4 n^4 P}{\pi^4 n^4 + 4\mu^4}$, isque loco omnium reliquorum addatur vel subtrahatur, prout terminus P habuerit signum $+$ vel -1. Est vero proxime $\pi^4 = 90,740909$, vnde loco termini P substitui poterit $\frac{P}{1 + \frac{16\mu^4}{363n^4}}$. Hocque modo eo propius ad verum valorem ipsius π accedetur, quo maior fuerit numerus μ: hoc est quo plures termini iam fuerint additi.

§. 13. His tamen non obstantibus series paragrapho decimo data semper dat valorem ipsius π nimis magnum, quic-

quicquid pro n ſubſtituatur; eo propius autem acceditur, quo maior numerus pro n ſubſtituatur. Sumto enim 1 pro n prodit $\pi = 3,1646+$ quae expreſſio iam in figura ſecunda a vero valore $3,14159265358979 32$ aberrat. Si ponatur $n = 3$, prodibit $\pi = 3,14159272 16+$ a vero valore in octaua figura diſcrepans. At ſi ponatur $n = 5$ reperietur per eandem methodum

$$\begin{array}{l} \pi = 3,1415926535900726 + \\ \quad\; \underline{3,1415926535897932} \\ \quad\; 0,0000000000002794 \end{array}$$

cuius numeri exceſſus in decima tertia demum figura conſpicitur. Haecque aberratio a veritate eo magis eſt notatu digna, quo minus vitium in ratiocinio inſtituto deprehendi poteſt. Ad quod accedit vt iſta formula aberratione hac non obſtante commode ad valorem ipſius π inueniendum inſeruire queat, ſubſtituendo ſcilicet maiores numeros loco n.

§. 14. Ex his exemplis quibus 1, 3, et 5 loco n ſubſtituimus per inductionem concludi poſſe videtur, valorem ipſius π in fractionibus decimalibus fere ad triplo plures figuras iuſtum repertum iri, quam n contineat vnitates, ſiquidem prima figura 3 computetur; prima autem hac figura non computata videtur numerus figurarum iuſtarum fore $= 2\frac{1}{2}.n$. Sic ſi ponatur $n = 2$ reperitur $\pi = 3,141635$ cuius quinta figura quaternario nimis eſt magna. Ac poſito $n = 4$ prodit $\pi = 3,1415926537 4+$ cuius decima figura binario maior eſt vera. Poſito autem $n = 6$ reperitur, $\pi = 3,1415926535897935 58+$ cuius figura demum decima ſexta a veritate recedit.

§. 15. Si nunc in caussam huius a veritate aberrationis calculi inquiramus, aliam detegere non valemus, nisi diuergentiam seriei §. 10. allatae; reliqua enim omnia prorsus se recte habere deprehenduntur. Namque si t vnitatem excedat, eo maior reperietur aberratio a veritate, quo minor accipiatur numerus n; id quod clarissime se manifestabit si t ponatur infinitum atque simul n = numero infinito. Ponamus enim $t = \infty$, quo casu in §. 9. abibit z in quartam peripheriae partem, eritque ideo $z = \frac{\pi}{2}$. Sit insuper $n = pt$, denotante p numerum quemcunque affirmatiuum siue integrum siue fractum, eritque ob $z = \frac{\pi}{2} = s + \frac{1}{2p}$ ac reliqui termini omnes negligi posse videntur, quod tamen in terminis infinitesimis perperam fit, quippe qui tandem ad finitam magnitudinem excrescere possunt.

§. 16. Interim tamen notari meretur errorem satis esse exiguum, nisi p sit numerus vnitate minor, atque quo maior valor ipsi p tribuatur eo minorem fore aberrationem a veritate. Cum enim hoc casu sit $s = \frac{p}{p^2+1} + \frac{p}{p^2+4} + \frac{p}{p^2+9} + \frac{p}{p^2+16} + \frac{p}{p^2+25} +$ etc. in infinitum; videatur huius seriei summa posse per quadraturam circuli definiri, quod tamen secus se habet. Per vltimam enim aequationem foret $s = \frac{\pi}{2} - \frac{1}{2p}$ seu $\frac{\pi}{2p} - \frac{1}{2pp} = \frac{1}{p^2+1} + \frac{1}{p^2+4} + \frac{1}{p^2+9} + \frac{1}{p^2+16} +$ etc. cuius quidem aequationis falsitas si $p = 0$ sponte elucet. At sumto $p = 1$ foret $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} +$ etc. $= \frac{\pi}{2} - \frac{1}{2}$. Vera autem summa per alias regulas reperitur $= \frac{\pi}{2} - 0,4941222793$ ita vt illa summa sit iusto minor, idque parte $0,0058777206$ sin autem ponatur $p = 2$, habebitur ista series

ſeries $\frac{1}{5} + \frac{1}{8} + \frac{1}{13} + \frac{1}{20} +$ etc. cuius ſumma per viam hanc erroneam prodit $= \frac{\pi}{4} - \frac{1}{8} = \frac{\pi}{4} - 0,125$; cum tamen conſtet veram ſummam eſſe $= \frac{\pi}{4} - 0,124994522075$, ita vt illius defectus tantum ſit $= 0,000005477924$. Multo autem adhuc minor erit aberratio ſi maiores numeri pro p accipiantur: ſic ſi $p = 3$, in nona demum figura accidet aberratio, atque quocunque numero pro p ſumto prodibit ſumma iuſta ad $3p$ figuras.

§. 17. Ex his ſatis perſpicitur, quam caute circa ſummationem ſerierum diuergentium verſari oporteat, praeſertim ſi eiusmodi ſeries diuergentes occurrant infinitae. Huiusque rei adhuc vnum exemplum afferre viſum eſt, ex quo neceſſitas ſummae circumſpectionis clarius elucebit. Propoſita ſit ſeries quaecunque $\overset{1}{a} + \overset{2}{b} + \overset{3}{c} + \overset{4}{d} + \overset{5}{e} + \overset{6}{f} + \overset{7}{g} + \overset{8}{h}$ + etc. cuius conſtat terminum quemcunque indicis x fore $= a + \frac{(x-1)}{1}(b-a) + \frac{(x-1)(x-2)}{1 \cdot 2}(c-2b+a) + \frac{(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3}(d-3c+3b-a) +$ etc. Ex hac forma definiantur omnes termini praecedentes verſus ſiniſtram in infinitum progredientes, eritque vt ſequitur

term. indicis $0 = a + (a-b) + (a-2b+c) + (a-3b+3c-d) +$ etc.

term. indic. $-1 = a + 2(a-b) + 3(a-2b+c) + 4(a-3b+3c-d) +$ etc

term. indic. $-2 = a + 3(a-b) + 6(a-2b+c) + 10(a-3b+3c-d) +$ etc

term. ind. $-3 = a + 4(a-b) + 10(a-2b+c) + 20(a-3b+3c-d) +$ etc

etc.

§. 18. Colligantur omnes hi termini antecedentes in infinitum, reperieturque omnium ſumma $= \frac{a}{1-1} + \frac{a-b}{(1-1)^2} + \frac{a-2b+c}{(1-1)^3} + \frac{a-3b+3c-d}{(1-1)^4} +$ etc. quae in ſeries innumerabiles ſecundum litteras a, b, c, d, e, etc. reſoluta abibit in hanc formam:

$$+ a \left(\frac{1}{1-1} + \frac{1}{(1-1)^2} + \frac{1}{(1-1)^3} + \frac{1}{(1-1)^4} + \text{etc.}\right)$$
$$- b \left(\frac{1}{(1-1)^2} + \frac{2}{(1-1)^3} + \frac{3}{(1-1)^4} + \frac{4}{(1-1)^5} + \text{etc.}\right)$$
$$+ c \left(\frac{1}{(1-1)^3} + \frac{3}{(1-1)^4} + \frac{6}{(1-1)^5} + \frac{10}{(1-1)^6} + \text{etc.}\right)$$
$$- d \left(\frac{1}{(1-1)^4} + \frac{4}{(1-1)^5} + \frac{10}{(1-1)^6} + \frac{20}{(1-1)^7} + \text{etc.}\right)$$

§. 19. Series hae ſingulae autem ſummationem admittunt; atque ſummis earum loco ſubſtitutis prodibit aggregatum omnium terminorum antecedentium verſus ſiniſtram in infinitum, vt ſequitur

$$+ a \cdot \frac{1}{(1-1)-1} = - a$$
$$- b \cdot \frac{1}{((1-1)-1)^2} = - b$$
$$+ c \cdot \frac{1}{((1-1)-1)^3} = - c$$
$$- d \cdot \frac{1}{((1-1)-1)^4} = - d$$
$$+ \text{etc.}$$

Ex quo videtur terminorum horum antecedentium ſumma fore $= - a - b - c - d -$ etc. Quare ſi ſeries quaecunque infinita $a + b + c + d + e +$ etc. etiam verſus ſiniſtram in infinitum continuaretur, foret totius ſeriei vtrinque in infinitum abeuntis ſumma ſemper $= 0$: ſi quidem ratiocinium hoc eſſet iuſtum.

§. 20. Neque vero hoc ratiocinium ſemper fallit, ſed in innumerabilibus ſeriebus veritati conſentaneum deprehenditur. Primo enim omnes progreſſiones geometricae hac gaudent proprietate vt in infinitum vtrinque progredientes ſummam habeant $= 0$. Scilicet ſeriei $n + n^2 + n^3 + n^4 +$ etc. ſumma eſt $= \frac{n}{1-n}$ partis autem praecedentis $1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} +$ etc. ſumma eſt $= \frac{n}{n-1}$, quae cum illa iuncta producit nihil. In infinitis autem

ſeriebus

ſeriebus aliis ratiocinium hoc maxime a veritate recedit, cuiusmodi eſt ſeries $1 + \frac{1}{9} + \frac{1}{25} +$ etc. quae antrorſum continuata ſui fit ſimilis et aequalis, ſcilicet $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} +$ etc. cuius adeo totius ſumma non fit o ſed potius duplo maior. Haec igitur propoſuiſſe non minoris vtilitatis eſſe arbitror, quam ſummo rigore demonſtratas veritates.

DE