1750

De productis ex infinitis factoribus ortis

Leonhard Euler

DE PRODVCTIS
EX INFINITIS FACTORIBVS ORTIS.

AVCTORE

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§. 1.

Cum in Analysi ad eiusmodi quantitates peruenitur, quae numeris nec rationalibus nec irrationalibus exponi possunt, expressiones infinitae ad eas quantitates denotandas adhiberi solent: quae eo magis idoneae sunt censendae, quo citius earum ope ad cognitionem et aestimationem quantitatum iis expressarum peruenitur. Huiusmodi igitur expressionum maximus et amplissimus est vsus ad valores quantitatum transcendentium, cuiusmodi sunt logarithmi, arcus circulares, aliaeque per quadraturas curuarum determinatae quantitates, repraesentandos earumque beneficio ad tam exactam cum logarithmorum, tum arcuum circularium, tum etiam plurium aliarum quantitatum transcendentium cognitionem pertigimus. Quin etiam istiusmodi expressiones infinitae insignem afferunt vtilitatem ad quantitates irrationales, et radices aequationum algebraicarum per numeros rationales vero proxime definiendas; quae si vsus spectetur veris expressionibus plerumque longe sunt anteferendae.

§. 2. Huiusmodi autem expressionum infinitarum nonnulla genera inter se maxime diuersa sunt constituenda, quorum primum in se complectitur omnes series infinitas, infinitis terminis ſignis + vel — iunctis conſtantes, quae doctrina nunc quidem iam tantopere eſt exculta, vt non ſolum plures habeantur methodi quasvis quantitates tam algebraicas quam tranſcendentes huiusmodi ſeriebus infinitis exprimendi, ſed etiam propoſita ſerie infinita inueſtigandi, cuiusmodi quantitas ea indicetur. Duplici enim modo expreſſiones infinitas cuiusque generis tractari oportet, quorum alter in conuerſione quantitatum vel algebraicarum vel tranſcendentium in expreſſiones infinitas conſiſtit; alter vero in indagatione illius quantitatis, quam propoſita exppreſſio infinita deſignat, viciſſim verſatur.

§. 3. Ad alterum genus expreſſionum infinitarum referri conuenit eas, quae ex innumerabilibus factoribus conſtant, cuiusmodi expreſſiones, quamquam iam complures ſunt inuentae ac cognitae, tamen nec modus ad eas perueniendi, nec via earum valores dignoſcendi vsquam eſt expoſita. Aeque autem dignae huius generis expreſſiones infinitae videntur, quae excolantur, ac priores ex infinito terminorum numero conſtantes, neque forte minus commodi Analyſi afferetur earum pertractatione. Praeterquam enim, quod iſtiusmodi expreſſiones naturam quantitatum quas referunt ſatis diſtincte ob oculos ponant, et ſaepe numero ad valores proximos inueniendos perquam ſunt accommodatae, inſignem praeſtant vſum ad logarithmos ipſarum quantitatum formandos, id quod in calculo ſaepiſſime ſummam affert vtilitatem. Sic ſi quantitas quaecunquae X transformata fuerit in iſtiusmodi expreſſionem $\frac{a}{\alpha}\cdot\frac{b}{\beta}\cdot\frac{c}{\gamma}\cdot\frac{d}{\delta}\cdot\frac{e}{\varepsilon}\cdot$ etc. ſtatim habebitur logarithmus quantitatis $X = l\frac{a}{\alpha} + l\frac{b}{\beta} + l\frac{c}{\gamma} + l\frac{d}{\delta} +$ etc. quae ſeries eo magis con-

vergit

vergit, quo propius factores illi ad vnitatem inclinant. Hanc ob caussam constitui in hac dissertatione theoriam huiusmodi expressionum infinitarum, quantum quidem obseruationes meae subsidii suppeditaverunt, inchoare, quo aliis facilius sit eam aliquando magis perficere.

§ 4. Primus eiusmodi expressionem infinitis factoribus contentam protulit Wallisius in Arithmetica infinitorum, vbi ostendit, si circuli diameter sit $= 1$. fore aream circuli $\frac{2.4.4.6.6.8.8.10.10.12.}{3.3.5.5.7.7.9.9.11.11.}$ etc. quam expressionem deduxit ex interpolatione seriei $\frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} +$ etc. cuius terminos intermidios demonstrauerat a circuli quadratura pendere. Cum igitur istae expressiones interpolationi serierum originem suam debeant, non incongruum fore visum est tractationem hanc de productis ex infinitis factoribus constantibus ab interpolationibus incipere. Cum enim in Tomo quinto Commentariorum nostrorum methodum tradidissem interpolationes per quadraturas curuarum perficiendi, simul constabit, cuiusmodi quantitatem transcendentem producta infinita hac ratione orta exhibeant.

§. 5. Considero igitur sequentem progressionem

$$\overset{1}{f+g}) + (f+g)\overset{2}{(}f+2g) + (f+g)\overset{3}{(}f+2g)(f+3g) + (f+g)\overset{4}{(}f+2g)(f+3g)\ (f+4g)$$

cuius quilibet terminus, cuius index est n, inuenitur ex praecedente hunc per $f+ng$ multiplicando: ostendi autem in dissertatione allegata huius seriei terminum, cuius index est n esse $= \frac{g^{n+1}\int dx(-lx)^n}{(f+(n+1)g)\int x^{f:g}\, dx(1-x)^n}$ vtraquae integratione ita peracta, vt integralia euanescant posito $x = 0$, tumquae facto $x = 1$. Quamobrem ista expressio simul indicabit, a quanam quadratura singuli ter-

 mini

mini intermedii pendeant. Quanquam enim ſi n ſit numerus fractus, non ita facile conſtat, qualem quadraturam $\int dx(-lx)^n$ contineat, tamen eodem loco oſtendi poſito $\frac{p}{q}$ loco n formulam $\int dx\,(-lx)^{\frac{p}{q}}$ congruere cum

$$\sqrt[q]{(1.2.3.\ldots\ldots p)}.\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right).\ \ldots\ldots\left(\frac{qp}{q}+1\right)$$
$$\int dx(x-xx)^{\frac{p}{q}}\int dx(x^2-x^3)^{\frac{p}{q}}\int dx(x^3-x^4)^{\frac{p}{q}}\int dx(x^4-x^5)^{\frac{p}{q}}\ldots\ldots\int dx(x^{q-1}-x^q)^{\frac{p}{q}}$$

cuius reductionis ope valor ipſius $\int dx(-lx)^{\frac{p}{q}}$ per quadraturas curuarum algebraicarum exprimi poteſt.

§ 6. Si nunc in ſerie aſſumta terminus, cujus index eſt $=\frac{1}{2}$, ponatur z, ex lege ſeriei termini, quorum indices ſunt $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, etc. ſequenti modo ſe habebunt:

$$\overset{\frac{1}{2}}{z} + \overset{\frac{3}{2}}{z(f+\tfrac{3}{2}g)} + \overset{\frac{5}{2}}{z(f+\tfrac{3}{2}g)(f+\tfrac{5}{2}g)} + \overset{\frac{7}{2}}{z(f+\tfrac{3}{2}g)(f+\tfrac{5}{2}g)(f+\tfrac{7}{2}g)} \text{ etc.}$$

Quoniam autem progreſſio aſſumta tandem cum Geometrica confunnditur, hi termini interpolati euadent tandem medii proportionales inter contiguos ſeriei terminos. Quare ſi ſinguli termini interpolati iam ab initio tanquam medii proportionales ſpectentur, ſequentes prodibunt approximationes ad terminum z, cuius index eſt $\frac{1}{2}$.

I. $z=\sqrt{(f+g)}$

II. $z=\sqrt{\dfrac{(f+g)(f+g)(f+2g)}{1(f+\frac{3}{2}g)(f+\frac{3}{2}g)}}$

III. $z=\sqrt{\dfrac{(f+g)(f+g)(f+2g)(f+2g)(f+3g)}{1(f+\frac{3}{2}g)(f+\frac{3}{2}g)(f+\frac{5}{2}g)(f+\frac{5}{2}g)}}$.

etc.

ex qua progreſſionis lege intelligitur terminum indicis $\frac{1}{2}$ vere eſſe $=(f+g)^{\frac{1}{2}}\sqrt{\dfrac{(f+g)}{(f+\frac{3}{2}g)}\dfrac{(f+2g)}{(f+\frac{3}{2}g)}\dfrac{(f+2g)}{(f+\frac{5}{2}g)}\dfrac{(f+3g)}{(f+\frac{5}{2}g)}}$
$(f+3g)$

$$\frac{(f+3g)}{(f+\frac{7}{2}g)}\frac{(f+4g)}{(f+\frac{7}{2}g)}\frac{(f+4g)}{(f+\frac{9}{2}g)}\frac{(f+5g)}{(f+\frac{9}{2}g)}\frac{(f+5g)}{(f+\frac{11}{2}g)}\frac{(f+6g)}{(f+\frac{11}{2}g)} \text{ etc.}$$

§. 7. Nunc igitur non ſolum certum eſt hac expreſſione infinita terminum ſeriei aſſumtae

$$\overset{1}{(f+g)}+(f+g)\overset{2}{(f+2g)}+(f+g)\overset{3}{(f+2g)}(f+3g)+ \text{ etc.}$$

cuius index eſt $=\frac{1}{2}$, exhiberi, ſed etiam eadem expreſſio inuenta ad quadraturas curuarum reducitur. Poſito enim $n=\frac{1}{2}$, ob $p=1$. et $q=2$. fit $\int dx(-lx)^{\frac{1}{2}}=\sqrt{1.2.}\int dx\sqrt{(x-xx)}$; quae expreſſio debito modo integrata dat radicem quadratam ex area circuli cuius diameter eſt $=1$: vel poſita $1:\pi$ ratione diametri ad peripheriam, erit $\int dx(-lx)^{\frac{1}{2}}=\sqrt{\frac{\pi}{4}}$. Hinc ergo idem terminus, cuius index $=\frac{1}{2}$, quem poſuimus z reperitur $=\frac{g\sqrt{\pi g}}{(2f+3g)\int x^{f:g}dx\sqrt{(1-x)}}$ $=\frac{\sqrt{\pi g}}{(2f+3g)\int y^{f+g-1}dy\sqrt{(1-y^g)}}$; intergrali hoc eodem tractato modo, quo ante ratione variabilis x eſt praeſcriptum. At per reductionem formularum huius modi integralium eſt $\int y^{f+g-1}dy\sqrt{(1-y^g)}=\frac{2fg}{(2f+g)(2f+3g)}\int\frac{y^{f-1}dy}{\sqrt{(1-y^g)}}=$ $\frac{2f}{2f+3g}\int y^{f-1}dy\sqrt{(1-y^g)}$. His ſubſtitutis reperitur

$$\frac{(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)(2f+7g)}{(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)(2f+6g)} \text{ etc.}$$

$$=\frac{2ff(2f+g)}{\pi g}(\int y^{f-1}dy\sqrt{(1-y^g)})^2=\frac{2ffg}{\pi(2f+fg)}\left(\frac{\int y^{f-1}dy}{\sqrt{(1-y^g)}}\right)^2.$$

Per hanc igitur aequationem innumerabiles quadraturae in factores infinitos, et viciſſim huiusmodi factorum infinitorum valores in quadraturas curuarum transformari poſſunt.

§. 8.

§. 8. Vt hanc aequalitatem exemplis illuſtremus, ſit $g = 1$, eritque $\int y^{f-1}dy\sqrt{(1-y)} = \frac{2.2.4.6.8.10\ldots(2f-2)}{3.5.7.9.11.13\ldots(2f+1)}$. Vnde fiet $\frac{2ff(2f+1)^2}{\pi} = \frac{2.2.4.4\ldots\ldots(2f-2)(2f-2)}{3.3.5.5.7.7\ldots\ldots(2f+1)(2f+1)} \cdot \frac{(2f+1)(2f+3)(2f+3)}{(2f+2)(2f+2)(2f+4)}$ etc. quae expreſſio ordinata ſeu ad continuitatem reducta dat $\pi = 4 \cdot \frac{2.\,4.\,4.\,6.\,6.\,8.\,8.\,10.\,10.}{3.\,3.\,5.\,5.\,7.\,7.\,9.\,9.\,11.}$ etc. quae eſt ipſa formula Wallisiana, prodiitque quicunque numerus integer affirmatiuus loco f ſubſtituatur. Haec eadem expreſſio autem prodit ſi ponatur $g = 2$. et $f =$ numero cuicunque impari integro.

§. 9. Cum igitur ſit $\frac{fg}{\pi}\left(\frac{\int y^{f-1}dy}{\sqrt{(1-y^g)}}\right)^2 =$

$$\frac{(2f+g)\,(2f+g)\,(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f\quad(2f+2g)\,(2f+2g)(2f+4g)(2f+4g)(2f+6g)}\ \text{etc.}$$

erit pari modo $\frac{hk}{\pi}\left(\frac{\int y^{h-1}dy}{\sqrt{(1-y^k)}}\right)^2 =$

$$\frac{(2h+k)(2h+k)(2h+3k)(2h+3k)(2h+5k)(2h+5k)}{2h\quad(2h+2k)(2h+2k)(2h+4k)(2h+4k)(2h+6k)}\ \text{etc.}$$

Quare illa expreſſione per hanc diuiſa obtinebitur ſequens aequatio libera a peripheria circuli π

$$\frac{fg(\int y^{f-1}dy:\sqrt{(1-y^g)})^2}{hk(\int y^{h-1}dy:\sqrt{(1-y^k)})^2} = \frac{2h(2f+g)^2(2h+2k)^2(2f+3g)^2(2h+4k)^2(2f+5g)^2}{2f(2h+k)^2(2f+2g)^2(2h+3k)^2(2f+4k)^2(2h+5k)^2}$$

etc. Quae radice quadrata extracta praebet hanc aequationem

$$\frac{\int y^{f-1}dy:\sqrt{(1-y^g)}}{\int y^{h-1}dy:\sqrt{(1-y^k)}} \cdot \sqrt{\frac{g}{k}} = \frac{2h(2f+g)(2h+2k)(2f+3g)(2h+4k)(2f+5g)}{2f(2h+k)(2f+2g)(2h+3k)(2f+4g)(2h+5k)}\ \text{etc.}$$

§. 10. Haec autem expreſſio infinita valorem conſtantem non habet, nam etiamſi in infinitum continuetur, tamen alium habet valorem, ſi numerus factorum capiatur par, alium ſi numerus impar. Quamobrem niſi ſit $k = g$,

quo

quo casu perinde est, vbi multiplicatio abrumpatur, bini factores coniunctim sunt accipiendi, quo facto binae obtinebuntur aequationes, prout numerus factorum capiatur par siue impar. Primo autem accurate euoluta expressione generali obtinebitur:

$$\frac{g\int y^{f-1}dy:\sqrt{(1-y^g)}}{k\int y^{h-1}dy:\sqrt{(1-y^k)}}=\frac{2h(2f+g)}{2f(2h+k)}\cdot\frac{(2h+2k)(2f+3g)}{(2f+2g)(2h+3k)}\cdot\frac{(2h+4k)(2f+5g)}{(2f+4g)(2h+5k)}\cdot\frac{(2h+6k)(2f+7g)}{(2f+6g)(2h+7k)}$$

etc. Sumendis autem alteris terminorum paribus erit

$$\frac{f\int y^{f-1}dy:\sqrt{(1-y^g)}}{h\int y^{h-1}dy:\sqrt{(1-y^k)}}=\frac{(2f+g)(2h+2k)}{(2h+k)(2f+2g)}\cdot\frac{(2f+3g)(2h+4k)}{(2h+3k)(2f+4g)}\cdot\frac{(2f+5g)(2h+5k)}{(2h+5k)(2f+6g)}\cdot\frac{(2f+7g)(2h+8k)}{(2h+7k)(2f+8g)}.$$

etc. in quibus expressionibus loca, vbi operationem abrumpere licet, punctis sunt distincta.

§. 11. Consideremus autem attentius casum, quo est $k=g$ quippe quo expressio infinita tanquam ex simplicibus factoribus constans concipi potest; eritque

$$\frac{\int y^{f-1}dy:\sqrt{(1-y^g)}}{\int y^{h-1}dy:\sqrt{(1-y^g)}}=\frac{2h(2f+g)(2h+2g)(2f+3g)(2h+4g)}{2f(2h+g)(2f+2g)(2h+3g)(2f+4g)}$$

quae expressio, quo minus cum praecedente ob easdem litteras confundatur, ponamus hic $2f=a$ et $2h=b$ atque $y=x^2$, quo substituto prodibit

$$\frac{\int x^{2f-1}dx:\sqrt{(1-x^{2g})}}{\int x^{b-1}dx:\sqrt{(1-x^{2g})}}=\frac{b(a+g)(b+2g)(a+3g)(b+4g)(a+5g)}{a(b+g)(a+2g)(b+3g)(a+4g)(b+5g)}.$$

etc. quae expressio cum priori. §. 9. data, quae facto pariter $y=x^2$, transit in hanc

$$\frac{4fg}{\pi}\left(\frac{\int x^{2f-1}dx}{\sqrt{(1-x^{2g})}}\right)^2=\frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)}$$

etc. comparata, insignes manifestabit proprietates, quarum veritas alias vix ostendi poterit.

§. 12. Statim enim patet si ponatur $a=2f$; et $b=2f+g$, illam expressionem infinitam in hanc transmutari; quamobrem etiam expressiones illis aequales, quadraturas cur-

curuarum continentes, hoc casu fient aequales, ex quo sequens emergit aequalitas: $\frac{\int x^{2f-1} dx : \sqrt{(1-x^{2g})}}{\int x^{2f+g-1} dx : \sqrt{(1-x^{2g})}} = \frac{4fg}{\pi} . (\int x^{2f-1} dx : \sqrt{(1-x^{2g})})^2$, si quidem ponatur post integrationem $x = 1$. Hinc igitur sequitur fore $\pi = 4fg \int \frac{x^{2f-1} dx}{\sqrt{(1-x^{2g})}} . \frac{\int x^{2f+g-1} dx}{\sqrt{(1-x^{2g})}}$: siue posito $2f = a$, erit π

$$= 2ag \frac{\int x^{a-1} dx}{\sqrt{(1-x^{2g})}} . \frac{\int x^{a+g-1} dx}{\sqrt{(1-x^{2g})}}$$

: quod sane est theorema maxime notatu dignum, cum eius beneficio productum duorum integralium, quorum saepissime neutrum exhiberi potest, assignari queat.

§. 13. Veritas huius theorematis quidem facile declaratur iis casibus, quibus altera formula integralis vel absolute integrationem admittit vel a circuli quadratura pendet. Ponamus enim $g = 1$, et $a = 1$; vtique erit $\pi = 2 \int \frac{dx}{\sqrt{(1-x^2)}} . \int \frac{x dx}{\sqrt{(1-x^2)}}$ nam $2 \int \frac{dx}{\sqrt{(1-x^2)}}$ posito post integrationem $x = 1$ dat ipsam quantitatem π; atque $\int \frac{x dx}{\sqrt{(1-xx)}} = 1 - \sqrt{(-xx)}$ facto $x = 1$ fit $= 1$. Simili modo si $a = 2$ manente $g = 1$ perspicitur fore $\pi = 4 \int \frac{x dx}{\sqrt{(1-xx)}} . \int \frac{xx dx}{\sqrt{(1-xx)}}$ nam est $\int \frac{x dx}{\sqrt{(1-xx)}} = 1$, et $\int \frac{xx dx}{\sqrt{(1-xx)}} = \frac{\pi}{4}$; quibus casibus theorematis veritas aliunde cognita, confirmatur.

§. 14. Reliqui autem casus, quibus neutra quantitas integralis vel actu vel per quadraturam circuli exhiberi potest, totidem praebent theoremata maxime abstrusae indaginis. Ita posito $g = 2$ et $a = 1$ fiet $\pi = 4 \int \frac{dx}{\sqrt{(1-x^4)}} . \int \frac{xx dx}{\sqrt{(1-x^4)}}$; vbi $\int \frac{xx dx}{\sqrt{(1-x^4)}}$ exhibet applicatam in curua elastiea rectangula, $\int \frac{dx}{\sqrt{(1-x^4)}}$ vero arcum elasticae abscissae x respondentem. Quocirca rectangulum ex arcu elasticae abcissae 1 respondente et applicata respondente aequabitur areae circuli, cuius diameter est abscissa illa 1; quae proprietas elasticae fortasse alia methodo vix ac ne vix quidem cognosci demonstrarique poterit.

§. 15. Antequam autem hunc elasticae casum relinquam, iuuabit vtrumque integrale per seriem ordinariam exprimere casu saltem quo $x = 1$. Cum enim sit $\frac{1}{\sqrt{(1-x^4)}} = \frac{(1+x^2)^{-\frac{1}{2}}}{\sqrt{(1-x^2)}}$ atque $(1+xx)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 +$ etc. singula membra a circuli quadratura pendebunt. Absoluta autem vtraque integratione pro casu $x = 1$ erit $\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}(1 - \frac{1}{4} + \frac{1\cdot 9}{4\cdot 16} - \frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36} +$ etc. atque $\int \frac{x^2 dx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}(\frac{1}{2} - \frac{1\cdot 3}{4\cdot 4} + \frac{1\cdot 9\cdot 5}{4\cdot 16\cdot 6} - \frac{1\cdot 9\cdot 25\cdot 7}{4\cdot 16\cdot 36\cdot 8} +$ etc.) Hinc autem approximando prodit tam prope $\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{5}{6}\frac{\pi}{2}$ et $\int \frac{xxdx}{\sqrt{(1-x^4)}} = \frac{3}{5}$.

§. 16. Si fuerit $a = 1$ erit $\pi = 2g \int \frac{dx}{\sqrt{(1-x^{2g})}} \cdot \int \frac{x^g dx}{\sqrt{(1-x^{2g})}}$ quae duae expressiones integrales ita sunt comparatae, vt si fuerit $\int \frac{x^g dx}{\sqrt{(1-x^{2g})}}$ applicata curuae cuiusdam abscissae x respondens, futura sit $\int \frac{dx}{\sqrt{(1-x^{2g})}}$ ipsa eiusdem curuae longitudo. Quamobrem si in hac curua sumatur abscissa $x = 1$, erit productum seu rectangulum ex applicata in longitudinem curuae ad aream circuli, cuius diameter est abscissa $x = 1$, vti se habet 2 ad numerum g; quae pro-

 positio

poſitio locum habet, dummodo g fuerit numerus affirmatiuus; valores negatiui enim ſponte excipiuntur.

§. 17. Si $a-1$ minor accipiatur quam g, ita vt numeri a et g ſint primi inter ſe, ſequentia habebuntur theoremata notatu digna; nam ſi $a+g-1 > 2g$ tum integratio ad formulam ſimpliciorem reduci poſſet.

$$\pi = 2\int\frac{dx}{\sqrt{(1-x^2)}}\cdot\int\frac{x\,dx}{\sqrt{(1-x^2)}}$$

$$\pi = 4\int\frac{dx}{\sqrt{(1-x^4)}}\cdot\int\frac{x^2\,dx}{\sqrt{(1-x^4)}}$$

$$\pi = 6\int\frac{dx}{\sqrt{(1-x^6)}}\cdot\int\frac{x^3\,dx}{\sqrt{(1-x^6)}}$$

$$\pi = 12\int\frac{x\,dx}{\sqrt{(1-x^6)}}\cdot\int\frac{x^4\,dx}{\sqrt{(1-x^6)}}$$

$$\pi = 8\int\frac{dx}{\sqrt{(1-x^8)}}\cdot\int\frac{x^4\,dx}{\sqrt{(1-x^8)}}$$

$$\pi = 24\int\frac{x^2\,dx}{\sqrt{(1-x^8)}}\cdot\int\frac{x^6\,dx}{\sqrt{(1-x^8)}}$$

$$\pi = 10\int\frac{dx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^5\,dx}{\sqrt{1-x^{10}}}$$

$$\pi = 20\int\frac{x\,dx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^6\,dx}{\sqrt{(1-x^{10})}}$$

$$\pi = 30\int\frac{x^2\,dx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^7\,dx}{\sqrt{(1-x^{10})}}$$

$$\pi = 40\int\frac{x^3\,dx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^8\,dx}{\sqrt{(1-x^{10})}}$$

$$\pi = 12\int\frac{dx}{\sqrt{(1-x^{12})}}\cdot\int\frac{x^6\,dx}{\sqrt{(1-x^{12})}}$$

$$\pi = 60\int\frac{x^4\,dx}{\sqrt{(1-x^{12})}}\cdot\int\frac{x^{10}\,dx}{\sqrt{(1-x^{12})}}$$

$$\pi = 14\int\frac{dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^7\,dx}{\sqrt{(1-x^{14})}}$$

$$\pi = 28\int\frac{x\,dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^8\,dx}{\sqrt{(1-x^{14})}}$$

$$\pi = 42\int\frac{x^2\,dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^9\,dx}{\sqrt{(1-x^{14})}}$$

$$\pi = 56\int\frac{x^3\,dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^{10}\,dx}{\sqrt{(1-x^{14})}}$$

$$\pi = 70\int\frac{x^4\,dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^{11}\,dx}{\sqrt{(1-x^{14})}}$$

§. 18. Hoc ipſo igitur inuento reductio etiam formularum integralium ad ſimpliciores inſigniter eſt promota. Cum enim adhuc duae iſtae formulae $\frac{\int x^m\,dx}{\sqrt{(1-x^{2g})}}$ et $\frac{\int x^{m+n}dx}{\sqrt{(1-x^{2g})}}$ ad ſe inuicem tantum reduci potuiſſent, ſi n erat multiplum exponentis $2g$; ita nunc reductio etiam ſuccedit, ſi n tantum ipſius g fuerit multiplum: caſu intellige, quo fit $x=1$. Quemadmodum autem ſi n eſt productum exponentis g per numerum parem, quotus, qui reſultat ex diuiſione alterius formulae per alteram, facile aſſignatur, ita e contrario, ſi n ſit factum ex g in numerum

merum imparem, tum productum formularum facillime assignatur.

§. 19. Haec omnia ergo huc redeunt, vt si cognitum fuerit integrale formulae $\int \frac{x^m dx}{\sqrt{(1-x^{2g})}}$ casu quo $x = 1$, eodem casu etiam huius formulae $\int \frac{x^{m+n} dx}{\sqrt{(1-x^{2g})}}$ integrale, si sit n multiplum ipsius g, exhiberi queat. Sit enim A integrale formulae $\int \frac{x^m dx}{\sqrt{(1-x^{2g})}}$ casu, quo est $x = 1$; integralia alterius formulae, ponendo g, $2g$, $3g$, etc. successiue loco n sequenti modo se habebunt.

$$\int \frac{x^m dx}{\sqrt{(1-x^{2g})}} = A$$

$$\int \frac{x^{m+g} dx}{\sqrt{(1-x^{2g})}} = \frac{\pi}{2(m+1)gA}$$

$$\int \frac{x^{m+2g} dx}{\sqrt{(1-x^{2g})}} = \frac{(m+1)A}{m+g+1}$$

$$\int \frac{x^{m+3g} dx}{\sqrt{(1-x^{2g})}} = \frac{(m+g+1)\pi}{2(m+1)(m+2g+1)gA}$$

$$\int \frac{x^{m+4g} dx}{\sqrt{(1-x^{2g})}} = \frac{(m+1)(m+3g+1)A}{(m+g+1)(m+3g+1)}$$

$$\int \frac{x^{m+5g} dx}{\sqrt{(1-x^{2g})}} = \frac{(m+g+1)(m+3g+1)\pi}{2(m+1)(m+2g+1)(m+4g+1)gA} \text{ etc.}$$

§. 20. Cum deinde haec formula generalis $\int x^{m+ig} dx (1-x^{2g})^{k-\frac{1}{2}}$ denotantibus i et k numeros integros quoscunque, reduci queat ad hanc formulam $\int \frac{x^{m+ig} dx}{\sqrt{(1-x^{2g})}}$, intel-

 ligitur

ligitur illius formulae latiſſime patentis $\int x^{m+ig}dx\,(1-x^{2g})^{\frac{k-1}{2}}$ integrale aſſignari poſſe ex integrali $\int\frac{x^m\,dx}{\sqrt{(1-x^{2g})}}$ cognito, caſu ſaltem, quo poſt integrationem fit $x=1$. Caſus autem, quibus i eſt numerus impar, praeter hoc integrale etiam circuli quadraturam π requirunt.

§. 21. Quemadmodum igitur per terminum indicis $\frac{1}{2}$ ſeriei ſupra §. 5. aſſumtae ad iſtas formularum integralium comparationes ſum deductus, ita operae pretium forte erit alios terminos intermedios ſimili modo inueſtigare. Quaeratur igitur terminus cuius index eſt $\frac{p}{q}$ qui ponatur $=z$, ex quo ſequentes ita ſe habebunt

$$\begin{array}{lll} \frac{p}{q} & \frac{p+q}{q} & \frac{p+2q}{q} \\ z+ & \frac{z(fq+g(p+q))}{q}+ & \frac{z(fq+(p+q)g)(fq+(p+2q)g)}{q^2},+\text{ etc.} \end{array}$$

Conſiderando nunc pari modo, quod haec progreſſio tandem in geometricam abeat, ſequentes orientur approximationes ad terminum z.

I. $z=1\,(f+g)^{\frac{p}{q}}$

II. $\frac{z(fq+(p+q)g}{q}=(f+g)^{\frac{q-p}{q}}\,(f+g)^{\frac{p}{q}}\,(f+2g)^{\frac{p}{q}}$

III. $z(f+\frac{(p+q)g}{q})(f+\frac{(p+2q)}{q}g)=(f+g)^{\frac{q-p}{q}}\,(f+g)^{\frac{p}{q}}\,(f+2g)^{\frac{q-p}{q}}\,(f+2g)^{\frac{p}{q}}\,(f+3g)^{\frac{p}{q}}$ Hinc igitur elicietur verus valor ipſius $z=\frac{(f+g)^{\frac{p}{q}}\,(f+g)^{\frac{q-p}{q}}}{1\,(f+\frac{(p+q)}{q}g)^{\frac{p}{q}}}$

$(f+$

$$\frac{(f+2g)^{\frac{p}{q}}}{(f+\frac{(p+q)}{q}g)^{\frac{(q-p)}{q}}}\cdot\frac{(f+2g)^{\frac{q-p}{q}}}{(f+\frac{(p+2q)}{q}g)^{\frac{p}{q}}}\cdot\frac{(f+3g)^{\frac{p}{q}}}{(f+\frac{(p+2q)}{q}g)^{\frac{q-p}{q}}}\cdot\frac{(f+3g)^{\frac{q-p}{q}}}{(f+\frac{(p+2q)}{q}g)^{p:q}}$$

etc. Vel paucis mutatis, vt factores infiniteſimi fiant $=1$, et expreſſio vbi libuerit abrumpi queat erit

$$\frac{z}{(f+\frac{p}{q}g)^{\frac{p}{q}}}=\frac{(f+g)^{\frac{p}{q}}}{(f+\frac{p}{q}g)^{\frac{p}{q}}}\cdot\frac{(f+g)^{\frac{q-p}{q}}}{(f+\frac{(p+q)}{q}g)^{\frac{q-p}{q}}}$$

$$\frac{(f+2g)^{\frac{p}{q}}}{(f+\frac{(p+q)}{q}g)^{\frac{p}{q}}}\cdot\frac{(f+2g)^{\frac{q-p}{q}}}{(f+\frac{(p+2q)}{q}g)^{\frac{q-p}{q}}}\cdot\frac{(f+3g)^{\frac{p}{q}}}{(f+\frac{(p+2q)}{q}g)^{\frac{p}{q}}}$$ etc. cuius expreſſionis lex, qua factores progrediuntur, ſponte elucet.

§. 22. Eiusdem autem termini intermedii z valor ope termini generalis huius ſeriei exprimi poteſt, fiet enim $z=$

$$\frac{g^{\frac{p+q}{q}}\int dx(-lx)^{\frac{p}{q}}}{(f+\frac{(p+q)}{q}g)\int x^{f:g}dx(1-x)^{\frac{p}{q}}}.$$ Quare ſi ponatur $\int dx(-lx)^{\frac{p}{q}}=$

$$=\sqrt[q]{(1.2.3\ldots\ldots p)(\frac{2p}{q}+1)(\frac{3p}{q}+1)(\frac{4p}{q}+1)\ldots(\frac{qp}{q}+1)}$$

$$\int dx(x-x^2)^{\frac{p}{q}}.\int dx(x^2-x^3)^{\frac{p}{q}}.\int dx(x^3-x^4)^{\frac{p}{q}}\ldots.\int dx(x^{q-1}-x^q)^{\frac{p}{q}}$$

$=\sqrt[q]{P}$: atque $x=y^g$, quo fit $\int x^{f:g}dx(1-x)^{\frac{p}{q}}=g\int y^{f+g-1}dy$ $(1-y^g)^{\frac{p}{q}}=\frac{ggp}{fq+(p+q)g}\int\frac{y^{f+g-1}dy}{(1-y^g)^{\frac{q-p}{q}}}=\frac{pfgg}{q(f+\frac{p}{q}g)(f+\frac{(p+q)}{q}g)}$

ſy

$\int \frac{y^{f-1}\,dy}{(1-y^g)^{\frac{q-p}{q}}}$. Ponatur porro $\int \frac{y^{f-1}\,dy}{(1-y^g)^{\frac{q-p}{q}}} = Q$, erit $z =$

$\frac{q(f+\frac{p}{q}g)\,P^{\frac{1}{q}}}{pfg^{\frac{q-p}{q}}\,Q}$.

§. 23. Substituta nunc loco z superiore expressione infinita, sumtisque potestatibus exponentis q, prodibit ista aequatio: $\frac{q^q P}{p^q f^p g^{q-p} Q^q} = \frac{f^{q-p}}{(f+\frac{p}{q}g)^{q-p}} \cdot \frac{(f+g)^p}{(f+\frac{p}{q}g)^p} \cdot$

$\frac{(f+g)^{q-p}}{(f+\frac{(p+q)}{q}g)^{q-p}} \cdot \frac{(f+2g)^p}{(f+\frac{(p+q)}{q}g)^p} \cdot \frac{(f+2g)^{q-p}}{(f+\frac{(p+2q)}{q}g)^{q-p}}$ etc.

Si igitur pari modo ponatur $\int \frac{y^{h-1}\,dy}{(1-y^g)^{\frac{q-p}{q}}} = R$, erit

$$\frac{p^q h^p g^{q-p} R^q}{q^q P} = \frac{(h+\frac{p}{q}g)^{q-p}}{h^{q-p}} \cdot \frac{(h+\frac{p}{q}g)^p}{(h+g)^p} \cdot \frac{(h+\frac{(p+q)}{q}g)^{q-p}}{(h+g)^{q-p}} \text{ etc.}$$

quae duae expressiones in se mutuo ductae dabunt $\frac{h^p R^q}{f^p Q^q}$

$$= \frac{f^{q-p}}{h^{q-p}} \frac{(h+\frac{p}{q}g)^q}{(f+\frac{p}{q}g)^q} \frac{(f+g)^q}{(h+g)^q} \frac{(h+\frac{(p+q)}{q}g)^q}{(f+\frac{(p+q)}{q}g)^q} \frac{(f+2g)^q}{(h+2g)^q}$$

$$\frac{(h+\frac{(p+2q)}{q}g)^q}{(f+\frac{(p+2q)}{q}g)^q} \text{ etc.}$$

§. 24.

§. 24. Si ergo vtrinque multiplicetur per $\frac{fp}{bp}$ atque radix poteſtatis q extrahatur reperietur $\frac{R}{Q} = \frac{f(b+\frac{p}{q}g)}{b(f+\frac{p}{q}g)}$ $\frac{(f+g)(b+\frac{(p+q)}{q}g)(f+2g)(b+\frac{(p+2q)}{q}g)}{(b+g)(f+\frac{(p+q)}{q}g)(b+2g)(f+\frac{(p+2q)}{q}g)}$ etc. $=$ $\frac{\int y^{b-1}dy(1-y^g)^{\frac{p-q}{q}}}{\int y^{f-1}dy(1-y^g)^{\frac{p-q}{q}}}$, in quibus integralibus cum ita fuerint accepta, vt euaneſcant poſito $y=0$, fieri debet $y=1$, quo facto habebitur per quadraturas valor expreſſionis infinitae propoſitae. Ope huius igitur expreſſionis infinitae altera quadratura ad alteram, ſiquidem ponatur $y=1$, reduci poterit.

§. 25. Vt autem hinc eiusmodi integralium comparationes deducamus, ſicuti ex priori caſu, quo erat $p=1$ et $q=2$, ponamus hic $p=1$ et $q=3$; fietque $P = \frac{10}{3}$ $\int dx(x-x^2)^{\frac{1}{3}} . \int dx(x^2-x^3)^{\frac{1}{3}}$ et $Q = \frac{\int y^{f-1}dy}{(1-y^g)^{\frac{2}{3}}}$. atque $R =$ $\frac{\int y^{b-1}dy}{(2-y^g)^{\frac{2}{3}}}$. Erit ergo $\frac{27P}{fg^2Q^3} = \frac{f \cdot (f \cdot (f+g)}{(f+\frac{1}{3}g)(f+\frac{1}{3}g)(f+\frac{1}{3}g)}$ $\frac{(f+g)(f+g)(f+2g)}{(f+\frac{4}{3}g)(f+\frac{4}{3}g)(f+\frac{4}{3}g)}$ etc. atque $\frac{R}{Q} = \frac{f(b+\frac{1}{3}g)(f+g)}{b(f+\frac{1}{3}g)(b+g)}$ $\frac{(b+\frac{4}{3}g)(f+2g)(b+\frac{7}{3}g)}{(f+\frac{4}{3}g)(b+2g)(f+\frac{7}{3}g)}$ etc. quae duae expreſſiones, cum in illa vna reuolutio ex tribus hic autem ex duobus factoribus conſtet, in ſe mutuo transformari nequeunt; quicquid etiam loco b ſubſtituatur.

§. 26.

§. 26. Sit igitur $S = \frac{\int y^{x-1}dy}{(1-y^g)^{\frac{2}{3}}}$ erit $\frac{S}{Q} = \frac{f(k+\frac{1}{3}g)}{k(f+\frac{1}{3}g)} \frac{(f+g)(k+\frac{4}{3}g)(f+2g)(k+\frac{7}{3}g)}{(k+g)(f+\frac{4}{3}g)(k+2g)(f+\frac{7}{3}g)}$ etc. quae expreſſio cum praecedente coniuncta dabit $\frac{RS}{Q^2} = \frac{f.f}{h.k.} \frac{(h+\frac{1}{3}g)(k+\frac{1}{3}g)}{(f+\frac{1}{3}g)(f+\frac{1}{3}g)} \frac{(f+g)(f+g)(h+\frac{4}{3}g)}{(h+g)(k+g)(f+\frac{4}{3}g)}$ etc. quae expreſſio in illam ipſi $\frac{27P}{fg^2Q^3}$ aequalem conuertetur, ponendo $h = f+\frac{1}{3}g$, et $k = f+\frac{2}{3}g$. Quamobrem habebitur iſta aequatio $\frac{27P}{fg^2} = Q$ RS, ſeu ſubſtitutis veris valoribus erit $90\int dx(x-x^2)^{\frac{1}{3}}.$ $\int dx(x^2-x^3)^{\frac{2}{3}} = fg^2\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{2}{3}}}.\int\frac{y^{f+\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{2}{3}}}.\int\frac{y^{f+\frac{2}{3}g-1}dy}{(1-y^g)^{\frac{2}{3}}}.$

§. 27. Antequam autem haec vlterius proſequamur conueniet valori ipſius P commodiorem formam generaliter tribui. Facto autem $x = z^q$, cum ſit $\int dx(x^n - x^{n+1})^{\frac{p}{q}} = \frac{npq}{(n+1)((n+1)p+q)}\int\frac{z^{np-1}dz}{(1-z^q)^{\frac{q-p}{q}}}$, poſt ſubſtitutionem prodibit

$$P = 1.2.3\ldots\ldots p.\, \frac{p^{q-1}}{q} \cdot \int\frac{z^{p-1}dz}{(1-z^q)^{\frac{q-p}{q}}} \cdot \int\frac{z^{2p-1}dz}{(1-z^q)^{\frac{q-p}{q}}} \cdot \int\frac{z^{3p-1}dz}{(1-z^q)^{\frac{q-p}{q}}} \ldots\ldots \int\frac{z^{(q-1)p-1}dz}{(1-z^q)^{\frac{q-p}{q}}}.$$

Ex qua expreſſione ſi extrahatur radix poteſtatis q, prodibit valor ipſius $\int dx(-lx)^{\frac{p}{q}}$.

§. 28.

§. 28. Posito nunc $p = 1$ et $q = 3$ prodibit $P = \frac{1}{3} \int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{z\,dz}{(1-z^3)^{\frac{2}{3}}}$. Facto autem $y = z^3$; obtinebitur sequens aequatio: $\int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{z\,dz}{(1-z^3)^{\frac{2}{3}}} = 3fg^2 \int \frac{z^{3f-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{3f+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{3f+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}$. Si nunc ponatur $3f = a$ orietur sequens aequatio notatu digna; $\int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{z\,dz}{(1-z^3)^{\frac{2}{3}}} = ag^2 \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}$. Quae cum superiore $\int \frac{dz}{\sqrt{(1-z^2)}} = ag \int \frac{z^{a-1}dz}{\sqrt{(1-z^{2g})}} \cdot \int \frac{z^{a+g-1}dz}{\sqrt{(1-z^{2g})}}$ comparata iam quodammodo indicat, quo modo sequentes huius generis aequationes se sint habiturae.

§. 29. Antequam autem per inductionem quicquam concludendi periculum faciam, casus nonnullos actu euoluam. Sit igitur $p = 2$ et $q = 3$ hincque reperietur $P = \frac{8}{9} \int \frac{z\,dz}{(1-z^3)^{\frac{1}{3}}} \cdot \int \frac{z^3\,dz}{(1-z^3)^{\frac{1}{3}}} = \frac{8}{9} \int \frac{dz}{(1-z^3)^{\frac{1}{3}}} \cdot \int \frac{z\,dz}{(1-z^3)^{\frac{1}{3}}}$; $Q = \int \frac{y^{f-1}dy}{(1-y^g)^{\frac{1}{3}}}$; $R = \int \frac{y^{h-1}dy}{(1-y^g)^{\frac{1}{3}}}$. Expressiones autem infinitae ita se habebunt; $\frac{27 \cdot P}{8f^2gQ^3} = \frac{f}{(f+\frac{2}{3}g)} \frac{(f+g)}{(f+\frac{2}{3}g)} \frac{(f+g)}{(f+\frac{2}{3}g)} \frac{(f+g)(f+2g)(f+2g)}{(f+\frac{5}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)}$ etc. et $\frac{R}{Q} = \frac{f(h+\frac{2}{3}g)(f+g)}{h(f+\frac{2}{3}g)(h+g)}$

(h+

$\frac{(h+\frac{5}{3}g)(f+2g)(h+\frac{8}{3}g)}{(f+\frac{5}{3}g)(h+2g)(f+\frac{8}{3}g)}$. Sit praeterea $S=\int\frac{y^{m-1}dy}{(1-y^g)^{\frac{1}{3}}}$ et $T=\int\frac{y^{n-1}dy}{(1-y^g)^{\frac{1}{3}}}$ erit $\frac{T}{S}=\frac{m(n+\frac{2}{3}g)(m+g)(n+\frac{5}{3}g)(m+2g)}{n(m+\frac{2}{3}g)(n+g)(m+\frac{5}{3}g)(n+2g)}$ etc. quae duae expressiones in se ductae dant $\frac{RT}{QS}=$

$$\frac{fm(h+\frac{2}{3}g)(n+\frac{2}{3}g)(f+g)(m+g)(h+\frac{5}{3}g)(n+\frac{5}{3}g)}{hn(f+\frac{2}{3}g)(m+\frac{2}{3}g)(h+g)(n+g)(f+\frac{5}{3}g)(m+\frac{5}{3}g)}\text{ etc.}$$

§. 30. Haec autem expressio ad illam, cui $\frac{27P}{8f^2gQ^3}$ aequale est inuentum, reduci non potest, nisi illa multiplicetur per $\frac{f}{f-\frac{1}{3}g}$, ita vt sit $\frac{27P}{8fg(f-\frac{1}{3}g)Q^3}=\frac{f\cdot}{(f-\frac{1}{3}g)}$ —— $\frac{f\cdot(f+g)(f+g)(f+g)(f+2g)}{(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)}$ etc. nunc enim fiet reductio ponendo $m=f$; $h=f-\frac{1}{3}g$, et $n=f+\frac{1}{3}g$. His igitur valoribus substitutis erit $\frac{27P}{8fg(f-\frac{1}{3}g)Q^3}=\frac{RT}{QS}$. Cum vero sit $S=Q$ et $R=\int\frac{y^{f-\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}}=\frac{f+\frac{1}{3}g}{f-\frac{1}{3}g}\int\frac{y^{f+\frac{2}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}}$ et $T=\int\frac{y^{f+\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}}$. obtinebitur haec aequatio; posito $y=z^3$;

$$\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}}=3fg(3f+g)\int\frac{z^{3f-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{3f+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{3f+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}.$$

Ac si ponatur $3f=a$ erit $\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot$

$\int zdz$

$$\int\frac{z\,dz}{(1-z^3)^{\frac{1}{3}}} = ag(a+g)\int\frac{z^{a-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}.$$

§. 31. Ponamus $p=1$ et $q=4$; habebiturque $\frac{4^4 P}{fg^3 Q^4} = \frac{f\,.\,f\,.\,f\,.\,(f+g)(f+g)(f+g)}{(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{5}{4}g)(f+\frac{5}{4}g)}$ etc. et $\frac{R}{Q} = \frac{f(h+\frac{1}{4}g)(f+g)(h+\frac{5}{4}g)(f+2g)}{h(f+\frac{1}{4}g)(h+g)(f+\frac{5}{4}g)(h+2g)}$ etc. Sit vero vt ante $S=\int\frac{y^{m-1}dy}{(1-y^g)^{\frac{q-p}{q}}}$; $T=\int\frac{y^{n-1}dy}{(1-y^g)^{\frac{q-p}{q}}}$; erit $\frac{RST}{Q^3} = \frac{f\,.\,f\,.\,f(h+\frac{1}{4}g)(m+\frac{1}{4}g)(n+\frac{1}{4}).(f+g)}{h\,.\,m\,.\,n\,(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(h+g)}$ etc. cuius expressionis 6 factores in illius quatuor sunt transmutandi, quod fiet ponendo $h=f+\frac{1}{4}g$; $m=f+\frac{2}{4}g$; et $n=f+\frac{3}{4}g$; quo facto habebitur $4^4P=fg^3QRST$. Quare cum sit $P=\frac{1}{4}\int\frac{dz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{zdz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{z^2dz}{(1-z^4)^{\frac{3}{4}}}$; si ponatur $y=z^4$ et $4f=a$ orietur ista aequatio:

$$\int\frac{dz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{z\,dz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{zzdz}{(1-z^4)^{\frac{3}{4}}} = ag^3\int\frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}$$

cuius cum praecedentibus casibus, quibus erat $p=1, q=2$; et $p=1, q=3$, connexio facile perspicitur.

 §. 32.

§. 32. Ex his igitur licebit omnes istius modi aequationes, quae orientur si ponatur $p = 1$, et $q =$ numero cuicunque affirmatiuo integro, formare; erit scilicet.

$$\text{I.}\ \int \frac{dz}{\sqrt{(1-z^2)}} = ag \int \frac{z^{a-1}dz}{\sqrt{(1-z^{2g})}} \cdot \int \frac{z^{a+g-1}dz}{\sqrt{(1-z^{2g})}}.$$

$$\text{II.}\ \int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}} = ag^2 \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}.$$

$$\text{III.}\ \int \frac{dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{z^2dz}{(1-z^4)^{\frac{3}{4}}} = ag^3 \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}$$

$$\text{III.}\ \int \frac{dz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{zdz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{z^2dz}{(1-z^5)^{\frac{4}{5}}} \int \frac{z^3dz}{(1-z^5)^{\frac{4}{5}}} = ag^4 \int \frac{z^{a-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+4g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}.\ \text{etc.}$$

§. 33. Quo etiam eas aequationes, quae oriuntur si p non $= 1$, colligere queamus, ponamus $p = 3$ et $q = 4$; quo posito, et reliquis manentibus vt supra, erit $\frac{4^4 P}{3^4 f^3 g Q^4} = \frac{f\,(f+g)\,(f+g)\,(f+g)}{(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}$ etc. vbi reliqua membra ex quaternis factoribus constantia ex his formantur singulos factores quantitate g augendo. Simili vero modo erit

erit $\frac{RST}{Q^3} = \frac{f.\ f.\ f.}{h.\ m.\ n} \frac{(h+\frac{3}{4}g)(m+\frac{3}{4}g)(n+\frac{3}{4}g)}{(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}$ etc. vbi seni factores vnam reuolutionem seu periodum constituunt. Ad comparationem autem instituendam necesse est vtramque seriem ita contemplari: $\frac{4^4 P}{3^4 f^2 g (f-\frac{1}{4}g) Q^4} = \frac{f.}{(f-\frac{1}{4}g)} \frac{f \quad (f+g)\ (f+g)}{(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}$ etc. $\frac{hRST}{fQ^3} = \frac{f.f.(h+\frac{3}{4}g)}{m.n.(f+\frac{3}{4}g)} \frac{(m+\frac{3}{4}g)(n+\frac{3}{4}g)(f+g)}{(f+\frac{3}{4}g)(f+\frac{3}{4}g)(h+g)}$ etc. quarum haec transmutatur in illam, ita vt fiat $\frac{4^4 P}{3^4 f g h (f-\frac{1}{4}g)} = QRST$, si fiat $h = f+\frac{1}{4}g$; $m = f-\frac{1}{4}g$; et $n = f+\frac{2}{4}g$.

§. 34. Cum igitur sit $P = \frac{3}{2}{}^4 \int \frac{z^2 dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^5 dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^8 dz}{(1-z^4)^{\frac{1}{4}}} = \frac{3^4}{32} \int \frac{dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{zzdz}{(1-z^4)^{\frac{1}{2}}}$. et $Q = \int \frac{y^{f-1} dy}{(1-y^g)^{\frac{1}{4}}}$; $R = \int \frac{y^{f+\frac{1}{4}g-1} dy}{(1-y^g)^{\frac{1}{4}}}$; $S = \int \frac{y^{f-\frac{1}{4}g-1} dy}{(1-y^g)^{\frac{1}{4}}} = \frac{f+\frac{2}{4}g}{f-\frac{1}{4}g} \int \frac{y^{f+\frac{3}{4}g-1} dy}{(1-y^g)^{\frac{1}{4}}}$ atque $T = \int \frac{y^{f+\frac{2}{4}g-1} dy}{(1-y^g)^{\frac{1}{4}}}$. Ex quibus posito $y = z^4$ et $4f = a$ sequens conficitur aequatio:

$$\frac{dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{zzdz}{(1-z^4)^{\frac{1}{4}}} = ag \frac{(a+g)(a+2g)}{1 \cdot 2} \int \frac{z^{a-1} dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+g-1} dz}{(1-z^{4g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+2g-1} dz}{(1-z^{4g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+3g-1} dz}{(1-z^{4g})^{\frac{1}{4}}}$$

§. 35.

§. 35. Hoc modo progrediendo reperientur sequentes aequationes, quando p non est $=1$ et quidem si $p=2$ inuenietur.

I. $\int \frac{dz}{(1-z^3)^{\frac{1}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{1}{3}}} = ag(a+g) \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}$

II. $\int \frac{dz}{(1-z^4)^{\frac{2}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{2}{4}}} \cdot \int \frac{zzdz}{(1-z^4)^{\frac{2}{4}}} = ag^2(a+g) \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{2}{4}}}$ Generaliter autem quicquid sit q, si ponatur $\frac{dz}{(1-z^q)^{\frac{q-2}{q}}} = X\,dz$ et $\frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-2}{q}}} = Y\,dz$ erit $\int X dz \int z X dz . \int z^2 X dz \ldots . \int z^{q-2} X dz = ag^{q-2}(a+g) \int Y dz . \int z^g Y dz \int z^{2g} Y dz \ldots . \int z^{(q-1)g} Y dz$.

§. 36. Simili modo si fit $p=3$, ac ponatur $\frac{dz}{(1-z^q)^{\frac{q-3}{q}}} = X\,dz$, et $\frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-3}{q}}} = Y\,dz$ prodibit sequens aequatio generalis, $\int X dz . \int z X dz . \int z^2 X dz \ldots . \int z^{q-2} X dz = ag^{q-3} \frac{(a+g)(a+2g)}{1 \cdot 2} \int Y dz . \int z^g Y dz . \int z^{2g} Y dz \ldots . \int z^{(q-1)g} Y dz$. Atque hinc omnes has formulas in vnam latissime patentem colligi licet. Sint enim p et q numeri quicunque integri affirmatiui, ac ponatur $\frac{dz}{(1-z^q)^{\frac{q-p}{q}}} = X\,dz$

$X\,dz$ et $\frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-p}{q}}} = Y\,dz$; habebitur $\int X\,dz.\int zX\,dz.$
$\int z^2 X\,dz \ldots . \int z^{q-2} X\,dz = ag^{q-p}\frac{(a+g)(a+2g)(a+3g)\ldots(a+(p-1)g)}{1.\quad 2.\quad 3.\quad \ldots\ldots \quad (p-1)}.$
$\int Y\,dz.\int z^g Y\,dz.\int z^{2g} Y\,dz.\ldots . \int z^{(q-1)g} Y\,dz.$

§. 37. Cum autem sit $\int z^{q-1} X\,dz = \frac{1}{p}$, si per hunc factorem vtrinque multiplicetur proueniet sequens aequatio satis elegans: $\frac{a(a+g)(a+2g)(a+3g)\ldots\ldots(a+(p-1)g)}{1.\quad 2.\quad 3.\quad 4.\quad \ldots\ldots \quad p} g^{q-p} = \frac{\int X\,dz}{\int Y\,dz}.$
$\frac{\int zX\,dz}{\int z^g Y\,dz}.\frac{\int z^2 X\,dz}{\int z^{2g} Y\,dz}.\frac{\int z^3 X\,dz}{\int z^{3g} Y\,dz}.\ldots . \frac{\int z^{q-1} X\,dz}{\int z^{(q-1)g} Y\,dz}$ quae expressio omnes hactenus inuentas in se complectitur; atque ob insignem ordinem est notatu digna.

§. 38. Progrediar nunc ad aliam methodum, cuius ope ad huiusmodi expressiones ex factoribus innumerabilibus constantes peruenire licet, quae magis ad analysin est accommodata. Obseruaui enim ex reductione formularum integralium ad alias istiusmodi expressiones obtineri posse. Sit enim proposita ista formula integralis $\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}}$, quae non difficulter transmutatur in hanc expressionem $\frac{x^m(1-x^{nq})^{\frac{p+q}{q}}}{m} + \frac{m+(p+q)n}{m}\int x^{m+nq-1}dx(1-x^{nq})^{\frac{p}{q}}$. Si ergo m et $\frac{p+q}{q}$ fuerint numeri affirmatiui, atque integralia ita capiantur, vt euanescant, posito $x=0$, tumque ponatur $x=1$, fiet $\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+q)n}{m}\int x^{m+nq-1}dx(1-x^{nq})^{\frac{p}{q}}$.

§. 39. Cum deinde simili modo sit $\int x^{m+nq-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+2q)n}{m+nq}\int x^{m+2nq-1}dx(1-x^{nq})^{\frac{p}{q}}$ erit quoque $\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{(m+(p+q)n)(m+(p+2q)n)}{m\,(m+nq)}\int x^{m+2nq-1}dx(1-x^{nq})^{\frac{p}{q}}$. Hac ergo

ergo reductione in infinitum continuata prodibit: $\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{(m+(p+q)n)(m+(p+2q)n)(m+(p+3q)n)\ldots\ldots(m+(p+\infty q)n)}{m\cdot(m+nq)\quad(m+2nq)\cdot\ldots\ldots(m+\infty nq)} \int x^{m+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}}$. Ac fimili modo eft $\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{(\mu+(p+q)n)(\mu+(p+2q)n)(\mu+(p+3q)n)\ldots\ldots(\mu+(p+\infty q)n}{\mu\quad(\mu+nq)\quad(\mu+2nq)\quad\ldots\ldots(\mu+\infty nq)} \int x^{\mu+\infty nq-1} dx(1-x^{nq})^{\frac{p}{q}}$; dummodo m, et nq et $\frac{p+q}{q}$ fint numeri affirmatiui, feu nihilo maiores.

§. 40. Quoniam autem fi m eft infinitum fit $\int x^m dx(1-x^{nq})^{\frac{p}{q}} = \int x^{m+\alpha}dx(1-x^{nq})^{\frac{p}{q}}$, quicunque numerus finitus loco α accipiatur, vti ex paragr. 38 colligitur, erit quoque $\int x^{m+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}} = \int x^{\mu+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}}$. Quam obrem fi praecedentium expreffionum altera per alteram diuidatur, proueniet ifta aequatio: $\frac{\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}}} = \frac{\mu(m+(p+q)n)(\mu+nq)(m+(p+2q)n)(\mu+2nq)(m+(p+3q)n)(\mu+3nq)}{m(\mu+(p+q)n)(m+nq)(\mu+(p+2q)n)(m+2nq)(\mu+(p+3q)n)(m+3nq)}$ etc. in infinitum, cuius expreffionis ope innumerabilia producta ex infinitis factoribus conftantia exhiberi poffunt, quorum valores per quadraturas curuarum affignari poterunt.

§. 41. Si altera formula integralis admittat integrationem, tum commoda expreffio infinita pro altero integrali habebitur. Sit enim $\mu = nq$, erit $\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{1}{(p+q)n}$, quo valore fubftituto prodibit $\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{1}{(p+q).n}\cdot\frac{nq(m+(p+q)n)2nq(m+(p+2q)n)3nq}{m(p+2q)n(m+nq)(p+3q)n(m+2nq)}$ etc. cuius ope pro innumerabilibus integralibus expreffiones per continuos factores in infinitum excurrentes inueniri poffunt; eo faltem cafu quo $x=1$; quippe qui plerumque potiffimum defideratur.

§. 42.

§. 42. Ponatur n loco nq, et prodibit: $\int x^{m-1}dx (1-x^n)^{\frac{p}{q}} = \frac{q}{(p+q)n} \cdot \frac{n(mq+(p+q)n)\, 2n(mq+(p+2q)n)\, 3n(mq+(p+3q)n)}{m(p+2q)n\,(m+n)\,(p+3q)n\,(m+2n)\,(p+4q)n}$ etc. quae in binos factores resoluta fit simplicior euaditque $\int x^{m-1}dx(1-x^n)^{\frac{p}{q}} = \frac{q}{(p+q)n} \cdot \frac{1(mq+(p+q)n)}{m(p+2q)} \cdot \frac{2(mq+(p+2q)n)}{(m+n)(p+3q)} \cdot \frac{3(mq+(p+3q)n)}{(m+2n)(p+4q)}$ etc. vnde sequentia exempla notabiliora deducuntur.

$$\int \frac{dx}{\sqrt{(1-xx)}} = 1 \cdot \frac{1\cdot 4}{1\cdot 3} \cdot \frac{2\cdot 8}{3\cdot 5} \cdot \frac{3\cdot 12}{5\cdot 7} \text{ etc.} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7} \text{ etc.}$$

$$\int \frac{xdx}{\sqrt{(1-xx)}} = 1 \cdot \frac{1\cdot 6}{2\cdot 3} \cdot \frac{2\cdot 10}{4\cdot 5} \cdot \frac{3\cdot 14}{6\cdot 7} \text{ etc.} = 1$$

$$\int \frac{x^2dx}{\sqrt{(1-xx)}} = 1 \cdot \frac{1\cdot 8}{3\cdot 3} \cdot \frac{2\cdot 12}{5\cdot 5} \cdot \frac{3\cdot 16}{7\cdot 7} \text{ etc.} = \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7} \text{ etc.}$$

$$\int \frac{dx}{\sqrt{(1-x^3)}} = \frac{2}{3} \cdot \frac{1\cdot 5\cdot 2\cdot 11\cdot 3\cdot 17\cdot 4\cdot 23\cdot 5\cdot 29}{1\cdot 3\cdot 4\cdot 5\cdot 7\cdot 7\cdot 10\cdot 9\cdot 13\cdot 11} \text{ etc.}$$

$$\int \frac{xdx}{\sqrt{(1-x^3)}} = \frac{2}{3} \cdot \frac{1\cdot 7\cdot 2\cdot 13\cdot 3\cdot 19\cdot 4\cdot 25\cdot 5\cdot 31}{2\cdot 3\cdot 5\cdot 5\cdot 8\cdot 7\cdot 11\cdot 9\cdot 14\cdot 11} \text{ etc.}$$

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{1}{2} \cdot \frac{1\cdot 6\cdot 2\cdot 14\cdot 3\cdot 22\cdot 4\cdot 30}{1\ 3\cdot 5\cdot 5\cdot 9\cdot 7\ 13\cdot 9} \text{ etc.} = \frac{1}{2} \cdot \frac{2\cdot 3\cdot 4\cdot 7\cdot 6\cdot 11\cdot 8\cdot 15}{1\cdot 3\cdot 5\cdot 5\cdot 9\cdot 7\cdot 13\cdot 9} \text{ etc.}$$

$$\int \frac{xxdx}{\sqrt{(1-x^4)}} = \frac{1}{2} \cdot \frac{1\cdot 10\cdot 2\cdot 18\cdot 3\cdot 26\cdot 4\cdot 34}{3\cdot 3\cdot 7\cdot 5\cdot 11\cdot 7\cdot 15\cdot 9} \text{ etc.}$$

$$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = \frac{1}{2} \cdot \frac{3\cdot 3\cdot 6\cdot 6\cdot 9\cdot 9\cdot 12\cdot 12}{1\cdot 5\cdot 4\cdot 8\cdot 7\cdot 11\cdot 10\cdot 14} \text{ etc.}$$

$$\int \frac{dx}{\sqrt[4]{(1-x^4)}} = \frac{1}{3} \cdot \frac{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 16\cdot 16}{1\cdot 7\cdot 5\cdot 11\cdot 9\cdot 15\cdot 13\cdot 19} \text{ etc.}$$

Praeterea hae expressiones notari merentur.

$$\int x^{m-1}dx(1-x^n)^{-\frac{m}{n}} = \frac{1}{n-m} \cdot \frac{n\cdot n\cdot 2n\cdot 2n\cdot 3n\cdot 3n}{m(2n-m)(m+n)(3n-m)(m+2n)(4n-m)} \text{ etc.}$$

$$\int x^{m-1}dx(1-x^n)^{\frac{m-n}{n}} = \frac{1}{m} \cdot \frac{n\cdot 2m\cdot 2n\,(2m+n)\,3n\,(2m+2n)\,4n\,(2m+3n)}{m(m+n)(m+n)(m+2n)(m+2n)(m+3n)(m+3n)(m+4n)} \text{ etc.}$$

§. 43. Cum autem pari modo sit $\int x^{\mu-1}dx(1-x^\nu)^{\frac{r}{s}} = \frac{s}{(r+s)\nu} \cdot \frac{1(\mu s+(r+s)\nu)\, 2(\mu s+(r+2s)\nu)\, 3(\mu s+(r+3s)\nu)}{\mu(r+2s)(\mu+\nu)(r+3s)(\mu+2\nu)(r+4s}$ etc. erit priorem expressionem per hanc diuidendo $\frac{\int x^{m-1}dx(1-x^n)^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^\nu)^{\frac{r}{s}}} =$

$\frac{(r+s)qv}{(p+q)sn} \cdot \frac{\mu(r+2s)(mq+(p+q)n)}{m(p+2q)(\mu s+(r+s)v)} \cdot \frac{(\mu+v)(r+3s)(mq+(p+2q)n)}{(m+n)(p+3q)(\mu s+(r+2s)v)}$ etc.
Haec igitur expreſſio infinita, quoties habet valorem finitum, toties ſummatio alterius integralis ad alterum reduci poterit. Huiusmodi autem caſus exiſtunt, quando factores numeratoris deſtruunt factores denominatoris, ita vt poſt deſtructionem finitus factorum numerus ſuperſit. Continentur enim in hac expreſſione omnes omnino reductiones formularum integralium ad alias.

§. 44. Quo autem plures iſtiusmodi expreſſiones inter ſe comparari queant, eam hoc modo accipere viſum eſt:

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^g)^h} = \frac{(h+1)g}{(c+1)b} \cdot \frac{f(h+2)(a+(c+1)b)}{a(c+2)(f+(h+1)g)} \cdot \frac{(f+g)(h+3)(a+(c+2)b)}{(a+b)(c+3)(f+(h+2)g)}$$

etc. Simili modo erit $\frac{\int x^{\alpha-1}dx(1-x^\beta)^\gamma}{\int x^{\zeta-1}dx(1-x^\eta)^\theta} = \frac{(\theta+1)\eta}{(\gamma+1)\beta} \cdot \frac{\zeta(\theta+2)(\alpha+(\gamma+1)\beta)}{\alpha(\gamma+2)(\zeta+(\theta+1)\eta)} \cdot$
$\frac{(\zeta+\eta)(\theta+3)(\alpha+(\gamma+2)\beta)}{(\alpha+\beta)(\gamma+3)(\zeta+(\theta+2)\eta)}$ etc. quae expreſſiones, etſi re non inter ſe differunt, tamen quoniam habent formam diuerſam, inter ſe comparari poterunt.

§. 45. Vt nunc ex his expreſſionibus eadem theoremata eliciamus, quae ſupra inuenimus, ſit $\theta = \gamma = h = c$; $\eta = \beta = g = b$; erit $\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^b)^c} = \frac{f(a+(c+1)b)(f+b)}{a(f+(c+1)b)(a+b)} \cdot$
$\frac{(a+(c+2)b)(f+2b)(a+(c+3)b)}{(f+(c+2)b)(a+2b)(f+(c+3)b)}$ etc. atque altera formula

$$\frac{\int x^{\alpha-1}dx(1-x^b)^c}{\int x^{\zeta-1}dx(1-x^b)^c} = \frac{\zeta(\alpha+(c+1)b)(\zeta+b)(\alpha+(c+2)b)(\zeta+2b)(\alpha+(c+3)b)}{\alpha(\zeta+(c+1)b)(\alpha+b)(\zeta+(c+2)b)(\alpha+2b)(\zeta+(c+3)b)}$$

etc. Harum expreſſionum productum ſi ponatur $= \frac{f}{a}$ oportet eſſe $\frac{(a+(c+1)b)(f+b)\zeta(\alpha+(c+1)b)}{(f+(c+1)b)(a+b)\alpha(\zeta+(c+1)b)} = 1$, hoc enim ſi fuerit, totarum expreſſionum infinitarum productum fiet $= \frac{f}{a}$. At hoc obtinetinebitur faciendo $\alpha = a + (c+1)b$; $\zeta = f + (c+1)b$; fietque $c = -\frac{1}{2}$, ita vt fit $\alpha = a + \frac{1}{2}b$; $\zeta = f + \frac{1}{2}b$, eritque ideo $\int \frac{x^{a-1}dx}{\sqrt{(1-x^b)}} \cdot \int \frac{x^{a+\frac{1}{2}b-1}dx}{\sqrt{(1-x^b)}} = \frac{f}{a} \int \frac{x^{f-1}dx}{\sqrt{(1-x^b)}} \cdot \int \frac{x^{f+\frac{1}{2}b-1}dx}{\sqrt{(1-x^b)}}$ seu si ponatur $x = z^2$; erit $\int \frac{z^{a-1}dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{a+b-1}dz}{\sqrt{(1-z^{2b})}} = \frac{f}{a} \cdot \int \frac{z^{f-1}dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{f+b-1}dz}{\sqrt{(1-z^{2b})}}$ positis a et f loco $2a$ et $2f$. Haec autem aequatio nil aliud est nisi Theorema supra inuentum §. 12. facto enim $f = b$ fit $\int \frac{z^{2b-1}dz}{\sqrt{(1-z^{2b})}} = \frac{1}{b}$ et $\int \frac{z^{b-1}dz}{\sqrt{(1-z^{2b})}} = \frac{\pi}{2b}$; vnde fiet $\pi = 2ab \int \frac{z^{a-1}dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{a+b-1}dz}{\sqrt{(1-z^{2b})}}$.

§. 46. Simili modo alia huius generis theoremata inveniri possunt; sit enim $g = b$; $h = c$; $\eta = \varepsilon = b$ et $\theta = \gamma$, quaeraturque casus, quo productum ambarum expressionum fiat $= 1$. Hoc autem obtinebitur si fit $\frac{f(\alpha+(c+1)b)\zeta(\alpha+(\gamma+1)b)}{a(f+(c+1)b)\alpha(\zeta+(\gamma+1)b)} = 1$; id quod fiet capiendo $\alpha = a + (c+1)b$; $f = a + (\gamma+1)b$; $\zeta = \alpha$. His igitur valoribus substitutis orietur sequens theorema non inelegans $\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{a-1}dx(1-x^b)^\gamma} \cdot \frac{\int x^{a+(c+1)b-1}dx(1-x^b)^\gamma}{\int x^{a+(\gamma+1)b-1}dx(1-x^b)^c} = 1$; siue si ponatur $c+1 = m$ et $\gamma + 1 = n$ habebitur $\int \frac{x^{a-1}dx}{(1-x^b)^{1-m}} \cdot \int \frac{x^{a+mb-1}dx}{(1-x^b)^{1-n}} = \int \frac{x^{a-1}dx}{(1-x^b)^{1-n}} \cdot \int \frac{x^{a+nb-1}dx}{(1-x^b)^{1-m}}$.

§. 47. Alio inſuper modo concinnum theorema elici poterit ponendo $\gamma = b$ et $\theta = c$, manente $\eta = \mathfrak{G} = g = b$; atque efficiendo vt productum expreſſionum integralium fiat $= \frac{f}{a}$, quod, quo eueniat, oportet eſſe $\frac{(a+(c+1)b)(f+b)\zeta}{(f+(b+1)b)(a+b)a} \frac{(\alpha+(b+1)b)}{(\zeta+(c+1)b)} = 1$. Hoc vero efficietur capiendo $\alpha = a + (c+1)b$; $\zeta = f + (b+1)b$, ex quo reperietur $c + b + 1 = 0$ ſeu $b = -1 - c$; quare ſumatur $c = -\frac{1}{2} + n$; et $b = -\frac{1}{2} - n$, atque ſequens prodibit theorema: $\frac{f}{a} = \frac{\int x^{a-1} dx (1-x^b)^{-\frac{1}{2}+n} . \int x^{a+(\frac{1}{2}+n)b-1} dx (1-x^b)^{-\frac{1}{2}-n}}{\int x^{f-1} dx (1-x^b)^{-\frac{1}{2}-n} . \int x^{f+(\frac{1}{2}-n)b-1} dx (1-x^b)^{-\frac{1}{2}+n}}$.

§. 48. Sint nunc omnes exponentes c, b, γ et θ inaequales, at $g = \mathfrak{G} = \eta = b$, quaeranturque caſus quibus productum ambarum expreſſionum fiat $= \frac{(b+1)(\theta+1)}{(c+1)(\gamma+1)}$. Hoc autem eueniet ſi reddatur haec forma $\frac{f(bb+2b)(a+(c+1)b)\zeta(b\theta+2b)}{a(bc+2b)(f+(b+1)b)\alpha(b\gamma+2b)} \frac{(\alpha+(\gamma+1)b)}{(\zeta+(\theta+1)b)} = 1$ quos factores ita expreſſi, vt ſinguli in ſequentibus membris quantitate b creſcant. Ponatur iam $\zeta + (\theta+1)b = bb + 2b$, ſeu $\zeta = b(1+b-\theta)$ et $\alpha + (\gamma+1)b = bc + 2b$, ſeu $\alpha = b(1+c-\gamma)$. Porro fiat $f + (b+1)b = b\theta + 2b$, ſeu $f = b(1+\theta-b)$ et $a + (c+1)b = b\gamma + 2b$ ſeu $a = b(1+\gamma-c)$. Denique debebit eſſe $\alpha = f$ et $\zeta = a$, quae duae aequationes requirunt vt ſit $c - \gamma = \theta - b$, ſiue $c + b = \gamma + \theta$. Vnde ſequens orietur Theorema: $\frac{(b+1)(\theta+1)}{(c+1)(\gamma+1)} = \frac{\int x^{b(1+\gamma-c)-1} dx (1-x^b)^c . \int x^{b(1+c-\gamma)-1} dx (1-x^b)^\gamma}{\int x^{b(1+\theta-b)-1} dx (1-x^b)^b . \int x^{b(1+b-\theta)-1} dx (1-x^b)^\theta}$ dummodo ſit $c + b = \gamma + \theta$.

§. 49. Alio autem inſuper modo expreſſio illa effici poteſt $= 1$, ponendo $\alpha = a + (c+1)b$ et $\zeta = f + (b + 1)$

$+1)b$; $f=b(\gamma+2)$; $a=b(\theta+2)$; ita vt ſit $\alpha=b(3+c+\theta)$ et $\zeta=b(3+\gamma+b)$. Porro autem debet eſſe $\zeta+(\theta+1)b=bb+2b$, et $\alpha+(\gamma+1)b=bc+2b$; quibus poſtulatur vt ſit $\gamma+\theta+2=0$. Ponatur ergo $\gamma=-1+n$ et $\theta=-1-n$. At ſi requiratur, vt productum ambarum expreſſionum ſit $=\frac{f(b+1)(\theta+1)}{a(c+1)(\gamma+1)}$, id obtinebitur ponendo $\alpha=a+(c+1)b$, $\zeta=f+(b+1)b$; $f=b(\gamma+1)$; $a=b(\theta+1)$ vnde erit $\alpha=b(2+c+\theta)$ et $\zeta=b(2+b+\gamma)$. Tandem vero debebit eſſe $\gamma+\theta+1=0$. Ponatur $\gamma=-\frac{1}{2}+n$ et $\theta=-\frac{1}{2}-n$; atque habebitur hoc theorema $\frac{b+1}{c+1}=\frac{\int x^{b(\frac{1}{2}-n)-1}dx(1-x^b)^c}{\int x^{b(\frac{1}{2}+n)-1}dx(1-x^b)^b}\cdot\frac{\int x^{b(\frac{3}{2}+c-n)-1}dx(1-x^b)^{-\frac{1}{2}+n}}{\int x^{b(\frac{3}{2}+b+n)-1}dx(1-x^b)^{-\frac{1}{2}-n}}$: in quo notandum eſt, exponentes $c, b, -\frac{1}{2}+n, -\frac{1}{2}-n$ numeros negatiuos quidem eſſe poſſe, ſed tales vt cum vnitate ad affirmatiuos tranſeant; alioquin enim integralia valorem finitum non obtinerent caſu $x=1$.

§. 50. Quemadmodum igitur non ſolum theorema ſupra inuentum circa duarum formularum integralium producta detexi hac methodo magis directa, ſed etiam alia noua elicui non minus notatu digna, ita, ſi pari modo tres eiusmodi expreſſiones in ſe inuicem ducantur, theoremata complura circa producta trium formularum integralium prodibunt; atque vltra ad quotcunque factorum numerum progredi licebit; ſed cum haec inquiſitio adeo prolixum calculum requirat, vt etiam litterae vix ſufficiant, cum ipſis theorematis praecipuis indicatis, tum via monſtrata contentus ero.

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