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On the Vibration of Strings: An English Translation of Leonhard Euler's *Sur la Vibration des Cordes* (E140)*

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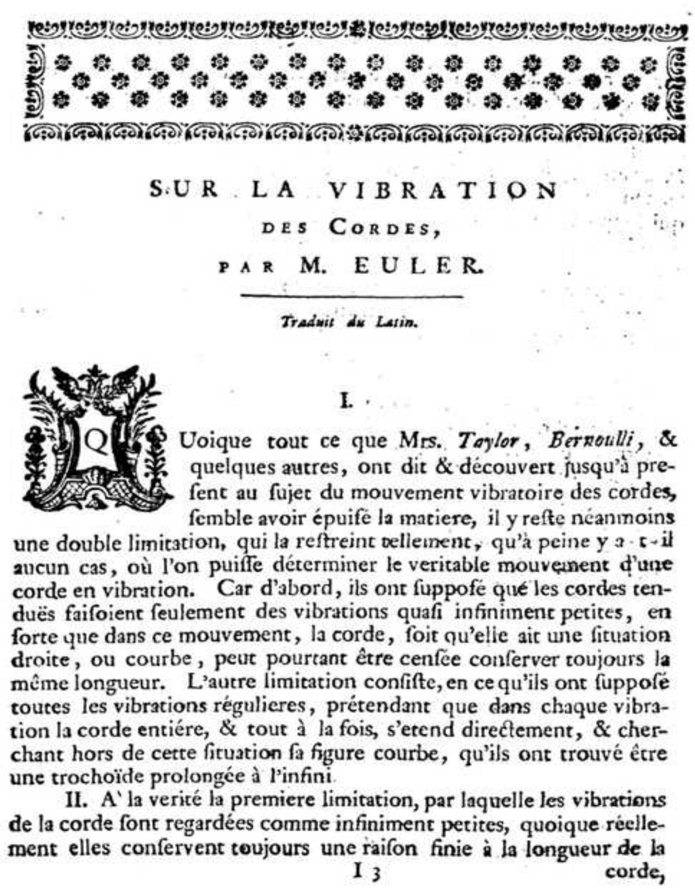

SUR LA VIBRATION
DES CORDES,
PAR M. EULER.

Traduit du Latin.

I.

QUoique tout ce que Mrs. *Taylor*, *Bernoulli*, & quelques autres, ont dit & découvert jusqu'à preſent au ſujet du mouvement vibratoire des cordes, ſemble avoir épuiſé la matiere, il y reſte néanmoins une double limitation, qui la reſtreint tellement, qu'à peine y a-t-il aucun cas, où l'on puiſſe déterminer le veritable mouvement d'une corde en vibration. Car d'abord, ils ont ſuppoſé que les cordes tenduës faiſoient ſeulement des vibrations quaſi infiniment petites, en ſorte que dans ce mouvement, la corde, ſoit qu'elle ait une ſituation droite, ou courbe, peut pourtant être cenſée conſerver toujours la même longueur. L'autre limitation conſiſte, en ce qu'ils ont ſuppoſé toutes les vibrations régulieres, prétendant que dans chaque vibration la corde entiére, & tout à la fois, s'etend directement, & cherchant hors de cette ſituation ſa figure courbe, qu'ils ont trouvé être une trochoïde prolongée à l'infini.

II. A' la verité la premiere limitation, par laquelle les vibrations de la corde ſont regardées comme infiniment petites, quoique réellement elles conſervent toujours une raiſon finie à la longueur de la

I 3 corde,

*Mémoires de l'académie des sciences de Berlin, IV (1750), p. 69-85. This translation has appeared in the author's senior thesis, presented to the Adelphi University Honors College in May 2023. The author is grateful to her thesis advisor, Prof. Robert Bradley, and Dean Nicole Rudolph of the Honors College for assistance in the translation.

(I) Although everything that Messrs. Taylor, Bernoulli[a], and some others have said and discovered up until now on the subject of vibratory movements of strings seems to have exhausted the subject, there nonetheless remains a double limitation, which restricts it so much that there is hardly any case where one may determine the true movement of a vibrating string. Because, first of all, they supposed that the taut strings only make nearly infinitely small vibrations, so that in this motion, this string, whether it be straight or curved, can yet be supposed to always maintain the same length. The other limitation consists in that they have supposed all vibrations to be regular[b], claiming that in each vibration, the whole string, and all at once, extends directly, and seeking out of this situation the curved figure, that they found to be a trochoid[c] extending to infinity.

(II) In truth the first limitation, by which the vibrations of the string are assumed to be infinitely small, though really they always maintain a finite ratio to the length of the string, hardly disturbs the conclusions that they drew, because in effect these vibrations are ordinarily so small that they can be taken as infinitely small, without it resulting in perceptible error. Besides, we have not yet pushed the Mechanics or the Analysis[d] far enough to be able to determine the movements under finite vibrations. With regard to the other limitation, which supposes all vibrations to be regular, one tries to defend it by saying that although they depart from this law at the start of the motion, they do not fail to subject themselves to uniformity after a short period of time, so that with each vibration, the string extends itself all at once and together in a straight line, taking on from this state the figure of a prolonged trochoid.

(III) It is effectively proven in a sufficient manner, that if a single vibration conforms to this rule, all of the following must also observe it. One sees at the same time, how the state of the following vibrations depend on the preceding ones, and may be determined by them; as reciprocally, by the state of the following ones, one can conclude the disposition of those which preceded it. That is why, if the following vibrations are regular, it will in no way be possible that the preceding departed from the rule; from which it follows just as clearly that if the vibration was irregular, the following ones can never achieve a perfect regularity. Now the first vibration depends on our good pleasure, since one may, before letting go of the string, give it an arbitrary shape, which means that the vibratory movement of the same string can vary infinitely, depending on whatever shape one gives to the string at the beginning of the motion.

[a]Euler is referring to Daniel Bernoulli (1700-1782).

[b]The term "regular" does not seem to have a standard definition in this context. Euler seems to be defining it here to mean that "the whole string, and all at once, extends directly...". It is not entirely clear what he means by this, but this definition seems to allude to the idea that the string is not restricted or damped in any way, and therefore the whole string is free to vibrate.

[c]Here, Euler writes the term "trochoid", instead of calling the wave "sinusoidal". We will consistently translate this literally, but it should be understood to modern readers as meaning a sine wave.

[d]Euler's distinction between "the Mechanics or Analysis" is roughly the distinction between the physics of the vibrating string problem, with finite vibrations, and the mathematics that would be needed to solve this problem. In paragraph (XIV) of this paper, Euler says "The proposed mechanical problem is reduced therefore to this analytical problem," when he derives a differential equation from physical principles, and then proceeds to use calculus to give the general solution.

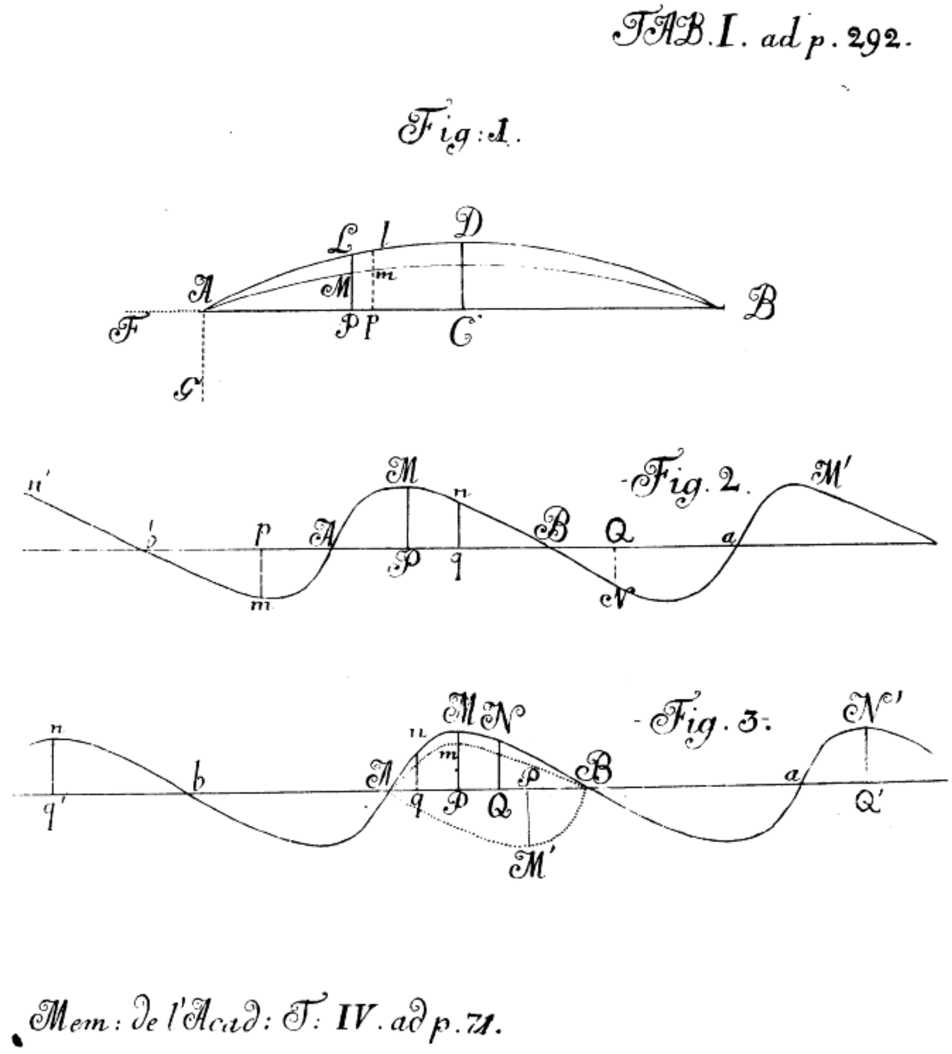

Figure 1: The Table of Figures in the original publication.

(IV) From this, therefore, the following question arises, in which all of this research is comprised.

If a string, of a given length and mass, is stretched by a force or a given weight, so that instead of being straight, one gives it an arbitrary shape which, however, differs only infinitely little from the straight line, and then one suddenly releases it; determine the total vibratory motion, by which it will be moved.

Mr. d'Alembert was the first to endeavor, with great success, to examine of this problem, so difficult as much in Mechanics as in Analysis, and he conveyed a very beautiful solution of it to our Academy[e]. However, as in these sublime discussions one often gains very considerable benefit from the comparison of several different solutions to the same problem, I don't hesitate to propose the one that I found for this question. Although it does not differ much from that of Mr. d'Alembert; however the great importance of this subject convinces me that I have added some rather interesting observations in the application of general formulas.

(V) I will start, therefore, by proposing the problem in a clear way, so that it shows what route one needs to take, as much in Analysis as in Mechanics, in order to arrive at the solution. Thus, let AB be the proposed string (see Figure 1), fixedly attached at extremities A and B, and stretched along the direction AF by an arbitrary force, as is ordinarily done in musical instruments. Let us suppose this string to be of equal thickness everywhere and let us denote its length $AB = a$, its mass, or its weight $= M$, and the stretching force AF is equal to a weight $= F$.

We then move this string from its natural state AB to an arbitrary curved state $ALlB$, which nevertheless only differs infinitely little from the straight, natural state AB, whereby the length $ALlB$ does not sensibly surpass the length AB; and that this shape $ALlB$ given first to the string, be known. One asks, supposing that the string is suddenly let go from this state, what motion will it acquire, and what will be the vibrations that it will make?

(VI) As soon as the string is released from its state $ALlB$, the force of tension will press it first toward the natural state AB, that all of its points will reach, either all at once, or at different moments. Consequently, the string will change shape continually[f], and all of the points will participate in the vibratory motion until the resistance has calmed all of the agitation. Now in order to know perfectly what this motion consists of it will suffice to have assigned, for each time, the state of the string, that is to say, its shape. Because, while on the one hand, one defines the change in the shape by the instantaneous succession, on the other, one determines at the same time the speed of each point on the string, and thus we arrive at the knowledge of all the motion. It will therefore not be necessary in this research to pay attention to the speeds of each of the points on the string, which considerably decreases the difficulty of this solution.

[e] "Our Academy" refers to the Berlin Academy.

[f] This could also be translated as "continuously", but did not wish don't to impose a modern conception of analysis.

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(VII) Since we have supposed that the length of the string suffers no change, while it successively takes on all of its shapes, such that $ALlB = AB$ [g], it follows from this that in taking the arbitrary ordinates PL and pl, normal to the axis AB, the arcs AL and Al will be equal to the abscissas AP and Ap, and consequently the ordinates PL and pl will be infinitely small with regard to the abscissas. Therefore, if we call the abscissa $AP = x$, the ordinate PL will be infinitely small in comparison to x, and the arc AL itself will be $= x$; from which we will have:

$$Pp = Ll = dx.$$

This explains why, when the string takes on various successive shapes, each of its points L perpetually moves itself along the direction of the ordinate LP, such that each ordinate LP represents the path by which the point L on the string approaches the natural state AB: but then, because of the received motion along the same direction normal to AB, it will move to the opposite side.

(VIII) After having made these remarks, let us suppose that at the end of the time t the string has arrived at the state $AMmB$, having left the original state $ALlB$, such that the point L is brought to M. By supposing, therefore, the arbitrary abscissa $AP = x$, which simultaneously expresses the length of the arc AM, is applied to this curve AMB corresponding to $PM = y$; and because this curve AMB depends on the elapsed time $= t$, y will be a function of two variables x and t, such that while supposing $t = 0$, the value of y provides the ordinate of the original curve ALB. Now it is clear that if one knows the nature of this function of x and t, which expresses the quantity of the ordinate y, one can, by means of this assign the shape to the string itself for an arbitrary time t; and furthermore one will easily conclude from the mutability, the motion of the entire string.

(IX) Thus y being a function of x and t, its differential will be of a form such that:

$$dy = p\,dx + q\,dt,$$

whose formula expresses not only the variablility of y along the curve AMB, but also with regard to the time that has elapsed. Indeed, if the time t is made constant, or $dt = 0$, the equation $dy = pdx$ will express the nature of the curve AMB; but if the abscissa x is supposed to be constant, or $dx = 0$, the equation $dy = qdt$ will define the motion of point L at all times that the motion of the string endures, because through it we can assign, for an arbitrary time t elapsed from the beginning, the place M, at which the point L will be taken. Now p and q will be new functions of x and t, whose differentials, while supposing x and t to be both variable, are

$$dp = r\,dx + s\,dt$$

and

$$dq = s\,dx + u\,dt.$$

[g] Although Euler does not say this, this paragraph refers to Figure 1.

Because it is known by the nature of differentials, that the element dt in dp and the element dx in dq, must have a common coefficient.

(X) As it is a matter at the present to determine the motion of the string by the acting forces, that is to say the accelerative force, by which the point M on the string is accelerated towards the axis $AB = P$, and it is clear that all of these forces, by which each of the elements of the string is pressed toward the axis AB, taken together must be equivalent to the force by which the string is actually stretched, and that we have supposed $AF = F$. Alternatively, if we conceive of forces opposite and equal to P, applied along ML at each point M of the string, they will therefore find themselves in equilibrium with the force which pulls the string $AF = F$, and by this property one will be able to determine the true accelerative force P, by which each element Mm of the string is actually moved.

(XI) The mass, or the weight of the whole string being $= M$, and being equally distributed along the whole length AB, the weight of the portion AP, or AM, will be $= \frac{Mx}{a}$, and consequently the small weight of the element $Mm = dx$ will be $= \frac{Mdx}{a}$, which being acted on along ML by the accelerative force $= P$, the motive force of this element will be $= \frac{Mdx}{a}P$, and the sum of all of the motive forces along the arc AM will be $= \frac{M}{a}\int Pdx$. But because the point A is assumed to be fixed, it is allowable to conceive of a certain force $AG = G$, which is applied to it in the direction AG normal to AB, and large enough so that the point A remains at rest. These things being assumed, the theory of equilibrium of forces applied to a perfectly flexible thread, will provide the following equation:

$$Fy - Gx + \frac{M}{a}\int dx \int Pdx = 0$$

where Fy and Gx are the moments of the forces F and G with regard to point M, and $\frac{M}{a}\int dx \int Pdx$ is the sum of all of the moments of the elementary forces with regard to the same point M.

(XII) Let us consider at present the curve AMB, which the string forms at this moment, whose nature will be expressed by the formulas given above if the time t taken as constant, or $dt = 0$, and consequently one will have $dy = p\,dx$ and $dp = r\,dx$. Hence, the equation that is given by the state of equilibrium being differentiated, and in substituting $p\,dx$ in place of dy, will give, when divided by dx,

$$Fp - G + \frac{M}{a}\int Pdx = 0.$$

Let us again differentiate this equation, substituting $r\,dx$ for dp, and dividing by dx, one will obtain

$$Fr + \frac{M}{a}P = 0.$$

Whereby we find the accelerative force P at the point M in the direction MP, namely $P = \frac{-Far}{M}$. This is why if the curve AMB were known, one could determine, by its nature, the accelerative force of each of its elements.

(XIII) Let us consider at present the motion of single point M, by which it approaches P, being acted on by the accelerative force P, and the abscissa $AP = x$, which must be supposed invariable. Now, because $dx = 0$, there is the instantaneous increment in the ordinate PM, $dy = q\,dt$ and $dq = u\,dt$, in the small time dt that the point M approaches P by the small space $= -q\,dt$, whose differential, supposing that the element of time dt is constant, will be

$$= -dq\,dt = -u\,dt^2 = -ddy.$$

However, from the acceleration which is born from the force P by the principles of mechanics, one deduces this equation:

$$P = \frac{-2ddy}{dt^2} = -2u,$$

if we describe, as is the custom, the element of time dt by the element of space applied to the speed, and that the speed itself be represented by the square root of the height due to this speed. Thus, since we have found $P = \frac{-Far}{M}$ as well as $P = -2u$, it will result that

$$2u = \frac{Far}{M}, \text{or } u = \frac{Far}{2M}.$$

(XIV) These two conditions, that we have brought forward by calculation, encompass the entirety of the proposed Question; and consequently, if an arbitrary time t having elapsed, one takes for an arbitrary point M on the string, the abscissa $AP = x$, and the ordinate $PM = y$, this is expressed by a function of x and t, such that in supposing $dy = p\,dx + q\,dt$, the character of the functions p and q will be drawn from these formulas:

$$dp = r\,dx + s\,dt$$

$$dq = s\,dx + \frac{Fa}{2M} r\,dt.$$

The proposed mechanical question is reduced therefore to this analytical problem: to find the functions r and s of x and t such that these differential formulas $r\,dx + s\,dt$ and $s\,dx + \frac{2M}{Fa} r\,dt$ become integrable[h]. For from the similar functions found for r and s, one will be able to assign the values $p = \int (r\,dx + s\,dt)$ and $q = \int (s\,dx + \frac{Fa}{2M} r\,dt)$, from which one will then infer the value of the ordinate itself,

$$y = \int (p\,dx + q\,dt).$$

(XV) This analytical problem considered in and of itself is extremely undetermined; thus, in order to accommodate any case which would present itself, one must make the

[h] i.e., each is an exact differential.

following remarks. First, in these integrations, it is necessary to adjust the constants so that in supposing $x = 0$, whatever value we attribute to t, we will always have $y = 0$. Then, we must do the same in the case of $x = a$. Thirdly, these precautions being taken, among the infinity of functions r and s that satisfy the conditions expressed above, one must choose, for each proposed case, those which, in supposing $t = 0$, make the resulting values of the ordinate y, give the arbitrary curve that one had given to the string at the beginning of the motion. That being performed, no indeterminate constant will remain in the solution, and the true motion of the string can be represented in an absolute manner.

(XVI) In order that the initial shape of the string can be drawn arbitrarily, the solution must have the greatest extent[i]. That is why, in the research beginning with the formulas

$$dp = r\,dx + s\,dt$$

and

$$dq = s\,dx + \frac{Fa}{2M} r\,dt,$$

one must discover in general all of the possible values for r and s which make these formulas jointly integrable. To this end, let us multiply these formulas in part by the constants m and n, and let us add these products, so that

$$m\,dp + n\,dq = dx(mr + ns) + dt(ms + \frac{Fa}{2M} nr).$$

This formula must still be integrable, whatever constant values are attributed to the letters m and n. Let us do it thusly,

$$m : n = \frac{Fa}{2M} n : m,$$

or

$$mm = \frac{Fa}{2M} nn,$$

from which it follows that $m = 1$ and $n = \pm\sqrt{\frac{2M}{Fa}}$, and one will have

$$dp \pm dq\sqrt{\frac{2M}{Fa}} = \left(dx \pm dt\sqrt{\frac{Fa}{2M}}\right)\left(r \pm s\sqrt{\frac{2M}{Fa}}\right).$$

(XVII) In short, let $\frac{Fa}{2M} = b$, and one will have

$$dp \pm dq\sqrt{\frac{1}{b}} = \left(dx + dt\sqrt{b}\right)\left(r \pm s\sqrt{\frac{1}{b}}\right),$$

[i] In other words, the greatest generality.

or

$$dp\sqrt{b} \pm dq = (dx \pm dt\sqrt{b})(r\sqrt{b} \pm s),$$

or also

$$dq \pm dp\sqrt{b} = (dx \pm dt\sqrt{b})(s \pm r\sqrt{b}).$$

Since, therefore, the formula $(dx \pm dt\sqrt{b})(s \pm r\sqrt{b})$ must be integrable, $s \pm r\sqrt{b}$ must be a function of $x \pm t\sqrt{b}$. Let us assume, in order to take account of both signs:

$$\begin{cases} x + t\sqrt{b} = v \\ x - t\sqrt{b} = u \end{cases}$$

and it will follow that

$$\begin{cases} x = \frac{v+u}{2} \\ t\sqrt{b} = \frac{v-u}{2} \end{cases}$$

and we will have these equations:

$$dq + dp\sqrt{b} = dv(s + r\sqrt{b})$$

$$dq - dp\sqrt{b} = du(s - r\sqrt{b})$$

where $s + r\sqrt{b}$ must be a function of v and $s - r\sqrt{b}$ must be a function of u; because otherwise the integration would not succeed.

(XVIII) This double integration thus being done, $q+p\sqrt{b}$ will become $=$ to a function of v and $q - p\sqrt{b} =$ to a function of u. Therefore, in order to give the greatest extent to the solution, let

V be an arbitrary function of $v = x + t\sqrt{b}$
U be an arbitrary function of $u = x - t\sqrt{b}$

and the given conditions will be satisfied by supposing that

$$q + p\sqrt{b} = V$$

$$q - p\sqrt{b} = U$$

from which is derived

$$p = \frac{V - U}{2\sqrt{b}}$$

$$q = \frac{V + U}{2}.$$

Therefore, since $dy = pdx + qdt$, one will have, by substituting for p and q as well as for dx and dt, the values we have found

$$dy = \frac{(dv + du)(V - U)}{4\sqrt{b}} + \frac{(dv - du)(V + U)}{4\sqrt{b}},$$

which, after simplification, gives

$$dy = \frac{Vdv - Udu}{2\sqrt{b}}, \text{and } y = \frac{1}{2\sqrt{b}}(\int Vdv - \int Udu).$$

(XIX) Now $\int Vdv$ will be a function of $v = x + t\sqrt{b}$ and $\int Udu$ a function of $u = x - t\sqrt{b}$, b being $= \frac{Fa}{2M}$, from which, if one uses the symbols f and ϕ in order to indicate the arbitrary functions of the quantities, in front of which we put them, we will have the following general expression for the ordinate y, which represents its quantity at an arbitrary time t, elapsed since the beginning, and for an arbitrary abscissa x[j]:

$$y = f : (x + t\sqrt{b}) + \phi : (x - t\sqrt{b}).$$

Now in order to retrace our steps, and to make use of the formula $dy = pdx + qdt$, one will have the following values p and q:

$$p = f' : (x + t\sqrt{b}) + \phi' : (x - t\sqrt{b})$$

$$q = \sqrt{b}(f' : (x + t\sqrt{b}) - \phi' : (x - t\sqrt{b})$$

and in place of the formulas $dp = rdx + sdt$ and $dq = sdx + brdt$, one will have, as the nature of the thing demands,

$$r = f''.(x + t\sqrt{b}) + \phi''.(x - t\sqrt{b})$$

$$s = \sqrt{b}.(f''.(x + t\sqrt{b}) - \phi''.(x - t\sqrt{b}))$$

provided that we denote the differential of the function $f : z$ by $dzf' : z$, and the differential of the function $f' : z$ by $dzf'' : z$[k].

(XX) Until now, the symbols f and ϕ in the equation

$$y = f : (x + t\sqrt{b}) + \phi : (x - t\sqrt{b})$$

signify arbitrary functions, which differ by means of their composition, and their relation is further determined by other conditions. Because, in assuming $x = 0$, one must always have $y = 0$, it must be that $f : (+t\sqrt{b}) + \phi(-t\sqrt{b}) = 0$, and consequently

[j] Here, Euler uses $f : (x+t\sqrt{b})$ to mean exactly what is today meant by $f(x+t\sqrt{b})$. Euler developed and began to use this notation style in the 1730s, and in his earlier writings uses $f : z$ to express the substitution of the expression z for the free variable in the function f.

[k] Euler uses f' and f'' as a way to denote the first and second derivative, respectively. This notation was later popularized by Joseph-Louis Lagrange (1736-1813), but in a slightly different sense. Euler uses $dzf' : z$ to simply denote the differential of $f : z$, as he states in paragraph XIX. In modern notation, this is equivalent to considering $d[f(x)] = g(x)\,dx$ for some function $g(x)$, and choosing to denote $g(x)$ as $f'(x)$. In a different sense, Lagrange will later conceptualize the derivative as an "operator", which takes the function f as an input and returns its derivative, denoted f'. Lagrange would have been familiar with this paper, as he also contributed to the discussion of the vibrating string later in the 18th century.

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$\phi(-t\sqrt{b}) = -f : (t\sqrt{b})$. Now then, because by assuming $x = a$, the value of y must similarly disappear, one will also have

$$f : (a + t\sqrt{b}) + \phi : (a - t\sqrt{b}) = 0;$$

and thus the nature of the functions f and ϕ must be defined in such a way that it will satisfy these conditions:

$$\phi : -t\sqrt{b} = -f : t\sqrt{b}$$

and

$$\phi : (a - t\sqrt{b}) = -f : (a + t\sqrt{b}).$$

(XXI) Because $f : z$ may be represented in general by the ordinate of a certain curve, for which the abscissa is z, let AMB (see Figure 2) be the curve for which the ordinates PM provide the functions of the abscissas AP which are designated by the symbol f : so that PM is $= f : t\sqrt{b}$; of which $\phi : -t\sqrt{b}$ must be negatively equal, one takes $Ap = AP$, such that $Ap = t\sqrt{b}$; and by assuming the curve Amb underneath the axis of the similar curve AMB, one will have $pm = -f : t\sqrt{b} = \phi : -t\sqrt{b}$. Therefore, the curve Amb, similar to the curve AMB, will express the nature of the function ϕ. Hence the curve AMB, existing in a similar manner beyond B, let $AB = a$ be continued underneath the axis, so that the portion BNa is similar and equal to the curve BnA; and by taking $BQ = Bq$, one will have $AQ = a + t\sqrt{b}$, $QN = f : (a + t\sqrt{b})$, and similarly because $Aq = a - t\sqrt{b}$, one will have $qn = f : (a - t\sqrt{b})$; from which, it appears that a curve of this form AMB, which is continued on both sides to infinity, by these similar and equal parts to itself Amb, BNa, and which are alternately located above and below, is suitable to represent the nature of either function f and ϕ.

(XXII) Having therefore described such a serpentine curve, be it regular, contained in a certain equation, or be it irregular, or mechanical, its arbitrary ordinate PM will provide the functions which we need for the solution to the problem. Indeed if one assumes an arbitrary abscissa $AP = z$, one will have the ordinate $PM = f : z$. From there, therefore, by assigning to the abscissa z the values $x + t\sqrt{b}$ and $x - t\sqrt{b}$, one will have $y = f : (x + t\sqrt{b}) + f : (x - t\sqrt{b})$; as a result of which we can assign, for an arbitrary time, the ordinate y in the vibrating string, which corresponds to an arbitrary abscissa. Now let us assume $t = 0$, in order to obtain the initial curve of the string, and we will have $AP = x$, and the ordinate in the vibrating string $y = f : x = 2PM$; or, because it is permissible to take halves of the above functions, such that

$$y = \frac{1}{2} f : (x + t\sqrt{b}) + \frac{1}{2} f : (x - t\sqrt{b})$$

the curve AMB itself will represent the shape given to the string at the beginning of the motion.

(XXIII) Reciprocally, therefore, if there is a given curve, or shape, that the string had received at the beginning, one is able to draw from it the determination of the shape of the string for an arbitrary time elapsed since the beginning. Because, by describing the axis $AB = a$, which is equal to the length of the string, the initial shape of the string AMB, that we repeat it on both sides in an inverted position, such that $Amb = AMB$ and $BNa = BnA$, and that one conceives of the continual repetition of this curve on both sides to infinity following the same rule. So, if this curve is used to express the derived functions, after an elapsed time $= t$, the ordinate which will correspond to the abscissa x, in this vibrating string will be

$$y = \frac{1}{2}f : \left(x + t\sqrt{b}\right) + \frac{1}{2}f : \left(x - t\sqrt{b}\right)$$

from which one can easily find the construction of the curve that the string forms at an arbitrary time.

(XXIV) But in order that this formula not appear to contain heterogeneous quantities, it must be noted that $t\sqrt{b}$ is represented by a straight line, and by consequence homogeneous to x. For, letting z be the height through which a heavy body falls in the time t, in giving the expression of time in the manner indicated above, one will have $t = 2\sqrt{z}$, and thus instead of t, one can write $2\sqrt{z}$ and reciprocally by the height z one will know the time t elapsed since the beginning of the movement. Therefore $t\sqrt{b}$ will be $= 2\sqrt{bz} = 2\sqrt{\frac{Faz}{2M}} = \sqrt{\frac{2Faz}{M}}$, and will consequently be expressed by a straight line. Now let us assume, in order to abbreviate, $\sqrt{\frac{2Faz}{M}} = v$, such that the value of v could be assigned for an arbitrary time, and after the time elapsed, during which a heavy body falls through the height $= z$, one will have

$$y = \frac{1}{2}f : (x + v) + \frac{1}{2}f : (x - v).$$

(XXV) If one gave, therefore, at the beginning, the shape AMB to the string $AB = a$, and its repetition then formed the serpentine curved line, $n'bAMBaN$, the shape that the string ought to have at the end of time t, during which a weighted body falls through the height $= z$, will be defined in this way. From this known height z, one finds the value $v = \sqrt{\frac{2Faz}{M}}$, and by proposing the arbitrary abscissa $AP = x$, one takes either $PQ = Pq = v$, by drawing the ordinates QN and qn at the points Q and q, one will have, because $QN = f : (x + v)$ and that $qn = f : (x - v)$, the ordinate which corresponds to the abscissa $AP = x$ of the string, $y = \frac{1}{2}QN + \frac{1}{2}qn$; or, if one takes $Pm = \frac{QN+qn}{2}$, m being the location of point M, and if one employs this construction for all of the points on the axis AB, the points m will give the present shape of the string AmB. In this manner, the shape that string takes in its vibrations will be easily described for an arbitrary time.

(XXVI) Let us look for the the shape of the string, after such time has elapsed that v is $= a$, or $z = \frac{Ma}{2F}$, and that will give

$$y = \frac{1}{2}f : (x + a) + \frac{1}{2}f : (x - a).$$

Now by the nature of the curve described $f:(x-a)$ will be $=-f:(a-x)$ and $f:(x+a)=-f:(a-x)$ from which will result $y=-f:(a-x)$ which indicates that the string will be entirely bent underneath the axis, and will take the shape $AM'B$ equal to the given shape AMB, but placed in an inverted position; such that by taking the abscissa $BP'=AP$, the ordinate will be $P'M'=PM$. And from there, reciprocally, if a time equal to t elapses again, from which $v=a$ results, the whole string will return to the state AMB, which was given to it at the beginning. What is also deduced from this, being the time elapsed since the beginning, from which it is seen that $v=2a$, it follows that

$$y=\frac{1}{2}f:(x+2a)+\frac{1}{2}f:(x-2a)$$

But in taking $PQ'=Pq'=2a$, by the nature of the curve $Q'N'$, will be $=PM=q'n'$, and by consequence $y=PM$, like at the beginning of the motion.

(XXVII) Whatever, therefore, the first given shape of the string may be, it takes that shape again in each of the vibrations, in as much as is permitted by the reduction caused by the resistance; which makes us see very clearly that there is no truth to the opinion reported above, being that the vibrations of the string, however irregular they were at first, return immediately after to uniformity in such a way that the shape degenerates into a prolonged trochoid. However it is no less clear that whatever the shape is of the string in vibration, the vibrations will not fail to be rather regular; because, by assuming $v=2a$, the string returns to its first state, it must be supposed to have made, during that time, two vibrations, and by consequence one will define from the value $v=a$, the time of one vibration, which will be equal to the time during which a weighted body falls from the height $\frac{Ma}{2F}$; or; if one expresses the length of the string $AB=a$ in thousandths of Rhineland feet[I], the time of one vibration expressed in seconds will be $=\frac{1}{125}\sqrt{\frac{Ma}{2F}}$, where the string will have as many vibrations in each second as this expression $125\sqrt{\frac{2F}{Ma}}$ will contain units, all as if the string would achieve its vibrations following the law of uniformity described by Taylor.

(XXVIII) Because the shape AMB given to the string at the beginning provides its first and largest displacement, likewise one vibration being completed, the string will find itself in the other largest displacement $AM'B$, that we have shown to be equal to the first one inverted. Let us therefore now see, if in the middle of the time taken by these two vibrations, the string stretches itself in a perfectly straight manner, and again takes the natural state, or not? Since the time of one vibration gives $v=a$, let us take for the moment the middle $v=\frac{1}{2}a$, and one will have the general formula

$$y=\frac{1}{2}f:\left(x+\frac{1}{2}a\right)+\frac{1}{2}f:\left(x-\frac{1}{2}a\right)$$

whose value will vanish if $f:(\frac{1}{2}a-x)$ is $=f:(\frac{1}{2}a+x)$, that is to say, if the shape ADB (see Figure 1), given to the string at the beginning is such that the abscissas $\frac{1}{2}a+x$

[I]The Rhineland foot is approximately 1.03 feet.

and $\frac{1}{2}a - x$ correspond to equal ordinates; this happens if the ordinate CD raised at a midpoint C of the length AB is a diameter[m] of the curve ADB, and that the part DB is similar and equal to the part DA. Every time, therefore, the initial curve has this property, just as many times the string extends itself in a straight line in the middle of each vibration, and as this can happen in countless ways, it is obvious that this condition itself does not require that the string always takes the shape of a prolonged trochoid in its vibrations.

(XXIX) Now, in order to consider the thing in general, the times of the vibrations do not depend on the shape that the vibrating string takes, but they are determined only by the quantities a, M, and F, of which the first a denotes the length of the string, M the weight of the string, and F the weight equal to the force that stretches.[n] However, there are singular cases in which the times of the vibrations may be reduced to half, to a third, to a quarter, or even to an arbitrary aliquot part of the whole length. Because, if the full length of the string were $Aa = a$, and that it were bent at the beginning, so that it made two parts AMB and Ba, which were perfectly similar and equal between themselves, it will then make its vibrations, as if it had only the half length AB, and consequently these vibrations will be two times quicker. Likewise, if the initial shape of the string had three parts similar and equal to $bABa$, as they are represented in the figure[o], the string would then make its vibrations, as if the length was three times less, and each vibration will become three times shorter; from which we understand well enough how the same vibrations may become four times, five times, etc. shorter.

(XXX) Having in this way given the general solution, let us understand it in a few more cases, in which the serpentine curve (see Figure 3) is a continuous curve, in which the parts are related by the law of continuity, in the manner that its nature may be contained by an equation[p]. And first it is certain that the curves, because they are cut by the axis in an infinity of points, will be transcendental. Assuming the length of the string $AB = a$, and an arbitrary abscissa $AP = u$ [q] and that $1 : \pi$, as the diameter of a circle is to the circumference, and it is obvious that the following equation, expressed by sines, yields a required curve:

$$PM = \alpha \sin\left(\frac{\pi u}{a}\right) + \beta \sin\left(\frac{2\pi u}{a}\right) + \gamma \sin\left(\frac{3\pi u}{a}\right) + \delta \sin\left(\frac{4\pi u}{a}\right) + \text{etc.}$$

Because if, instead of u, we take a, or $2a$, or $3a$, or $4a$. etc., the ordinate PM would vanish, and by assuming u to be negative, the ordinate itself would change to its negative. If therefore the curve AMB were the original shape of the string, at the end of time t, during which the weighted body descends from the height $= z$, by taking $v = \sqrt{\frac{2Faz}{M}}$,

[m] In other words, the curve ADB has a maximum.

[n] i.e. the tension

[o] Figure 3.

[p] Euler defined a continuous function to be a function whose nature is completely contained in one equation.

[q] The original publication states $AB = u$ here, which is believed to be a misprint.

as the abscissa x in the shape of the string will correspond to the ordinate y, such that one will have:

$$y = +\frac{1}{2}\alpha \sin \frac{\pi}{a}(x+v) + \frac{1}{2}\beta \sin \frac{2\pi}{a}(x+v) + \frac{1}{2}\gamma \sin \frac{3\pi}{a}(x+v) + \text{etc.}$$

$$+ \frac{1}{2}\alpha \sin \frac{\pi}{a}(x-v) + \frac{1}{2}\beta \sin \frac{2\pi}{a}(x-v) + \frac{1}{2}\gamma \sin \frac{3\pi}{a}(x-v) + \text{etc.}$$

(XXXI) Now as $\sin(a+b) + \sin(a-b)$ is $= 2\sin(a)cos(b)$, this equation will be transformed into this form

$$y = \alpha \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi v}{a}\right) + \beta \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi v}{a}\right) + \gamma \sin\left(\frac{3\pi x}{a}\right)\cos\left(\frac{3\pi v}{a}\right) + \text{etc.}$$

and the original shape of the string will be expressed by this equation,

$$y = \alpha \sin\left(\frac{\pi x}{a}\right) + \beta \sin\left(\frac{2\pi x}{a}\right) + \gamma \sin\left(\frac{3\pi x}{a}\right) + \text{etc.}$$

which returns to the same whenever v becomes either $2a$, or $4a$, or $6a$, etc. But if v is a, or $3a$, or $5a$, etc., the shape of the string will be

$$y = -\alpha \sin\left(\frac{\pi x}{a}\right) + \beta \sin\left(\frac{2\pi x}{a}\right) - \gamma \sin\left(\frac{3\pi x}{a}\right) + \text{etc.}$$

where it should be noted that if β is $= 0$, $\gamma = 0$, $\delta = 0$, etc. it results in the case that one commonly believes to be the only one which takes place in the vibration of strings, that is $y = \alpha \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi v}{a}\right)$, in which the curvature of the string is perpetually the sine curve, or a trochoid prolonged to infinity. But if only the term β, or γ, or δ, etc. is present, this forms the cases in which the time of the vibration is reduced, either by the double, or the triple, or the quadruple, etc.