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Alexander Aycock

Johannes Gutenberg Universitat, Mainz, aaycock@students.uni-mainz.de

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Euler and a Proof of the Functional Equation for the Riemann zeta-function he could have given

Alexander Aycock, Johannes-Gutenberg University Mainz
Staudinger Weg 9, 55128 Mainz
aaycock@students.uni-mainz.de

1 Introduction

That Euler (1707–1783) discovered the functional equation for the famous ζ-function in his 1749 paper “Remarques sur un beau rapport entre les series des puissances tant directes que reciproques” [7] (E352: “Remarks on the beautiful relation between the series of the direct and reciprocal powers”) is now well-known, cf. the book [13] and the article [3]. More precisely, Euler found the functional equation for what is nowadays referred to as the Dirichlet η-function, which is defined as

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \quad \text{for} \quad \Re(s) > 0.$$  \hspace{1cm} (1)

In modern notation, the functional equation reads:

$$\eta(1-s) = 2^{s-1} \frac{\pi^{-s} \cos \left( \frac{\pi s}{2} \right) \Gamma(s)}{\Gamma(s)} \eta(s),$$ \hspace{1cm} (2)

where \(\Gamma(x)\) is the function defined by

$$\Gamma(x) := \int_{0}^{\infty} e^{-t} t^{x-1} dt \quad \text{for} \quad \Re(x) > 0.$$

In this paper, we argue that, although Euler’s version of the functional equation in [7] (see figure 1) was stated conjectural, and he did not offer a rigorous proof in his later papers\(^b\), he could have given a proof. To demonstrate this,

\(^a\)On page 22 in his book [13] Hardy credited Landau and Cahen as the first who mentioned that Euler had stated the functional equation of the Riemann ζ-function in his paper [7].

\(^b\)The same conjecture of (2) is also found in his paper “Exercitationes analyticae” [8] (E432: “Analytical Exercises”). But he did not give a proof there either.
we will explain how to derive (2) just using formulas and applying techniques found by Euler in his other works.

Furthermore, we will present an argument why Euler could not have proved the functional equation for the Riemann $\zeta$-function, defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for} \quad \Re(s) > 1, \quad (3)$$

although (2) is equivalent to it from the modern perspective.

2 Derivation of the Functional Equation

2.1 Euler’s Formulas needed for the Proof

Let us state Euler’s formulas that we will need for a proof of (2) in advance. One will be a representation for the $\Gamma$-function, the other a partial fraction decomposition for a transcendental function, which we will introduce first.

In § 12 (see figure 2) of his paper “De resolutione fractionum transcendentium in infinitas fractiones simplices” [10] (E592: “On the resolution of transcendental fractions into infinitely many simple fractions”), aside from many others, Euler stated the (infinite) partial fraction decomposition

$$\frac{\lambda \pi}{2 \sin \lambda \pi} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 - \lambda^2}. \quad (4)$$

Using the formula

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

and putting $i\lambda \pi = u$ or $\lambda = \frac{u}{i\pi}$ in (4), we will have:
Figure 2: Euler states the partial fraction decomposition of the function $\frac{\lambda\pi}{\sin\lambda\pi}$. The scan is taken from the original version of Euler’s paper [10].

\[
\frac{u}{e^u - e^{-u}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^2 + \frac{u^2}{\pi^2}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1 + \left(\frac{u}{\pi n}\right)^2}.
\] (5)

Next, we will introduce a key integral formula given by Euler. In § 8 of his paper “De valoribus integralium a termino variabilis $x = 0$ usque ad $x = \infty$ extensorum” [11] (E675: “On the values of integrals extended from $x = 0$” to $x = \infty”) we find the formula

\[
\int y^{n-1} \partial y e^{-ky} = \frac{\Delta}{k^n}, \quad \text{with} \quad \Delta = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1).
\]

Euler understood the integral to be taken from $y = 0$ to $\infty$ and $k$ to be an arbitrary complex number. Finally, $n$, even if not mentioned explicitly by Euler, is a natural number $> 1$. Therefore, in modern notation, this equation can be rewritten as

\[
\Gamma(x) = \int_0^{\infty} e^{-ku} u^{x-1} du.
\] (6)

Here, we have changed $n$ to $x$ for reasons of convenience to be seen later.

2.2 Derivation of the Functional Equation

Having given everything necessary in advance, let us present the proof of (2). We start from (6) and let $k = 2n + 1$, where $n$ is a natural number:

\[
\frac{\Gamma(x)}{(2n+1)^x} = \int_0^{\infty} e^{-(2n+1)u} u^{x-1} du.
\]
Taking the sum over both sides from $n = 0$ to $n = \infty$, we have
\[
\sum_{n=0}^{\infty} \frac{\Gamma(x)}{(2n+1)^x} = \Gamma(x) \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x} =: \Gamma(x) \lambda(x),
\]
such that
\[
\lambda(x) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x},
\]
(7)
for the left-hand side and
\[
\sum_{n=0}^{\infty} \int_{0}^{\infty} e^{-(2n+1)u} u^{x-1} du = \int_{0}^{\infty} \sum_{n=0}^{\infty} \left( e^{-(2n+1)u} \right) u^{x-1} du
\]
\[
= \int_{0}^{\infty} u^{x-1} e^{-u} du = \int_{0}^{\infty} u^{x-1} du = \frac{1}{e^u - e^{-u}}
\]
for the right-hand side, where the interchanging of the summation and integration in the first step is justified by familiar theorems from the theory of Lebesgue integration\footnote{Questions concerning the interchangeability of limiting procedures were not discussed by Euler and his contemporaries for the most part in their respective works. It was implicitly assumed to be possible.} and the geometric series was used in the second step. Therefore, we may now present our equation as follows:
\[
\lambda(x) \Gamma(x) = \int_{0}^{\infty} u^{x-2} \cdot \frac{udu}{e^u - e^{-u}}.
\]
(8)
Thus, using the mentioned partial fraction decomposition (5), we will have:
\[
\lambda(x) \Gamma(x) = \int_{0}^{\infty} u^{x-2} du \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1 + \left( \frac{u}{\pi n} \right)^2}.
\]
Making the substitution $v = \frac{u}{n\pi}$, we will get
\[
\lambda(x) \Gamma(x) = \int_{0}^{\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \pi^{-1} n^{-1} v^{x-1} \frac{v^{x-2} dv}{1 + v^2} = \pi^{-1} \eta(1 - x) \int_{0}^{\infty} \frac{v^{x-2} dv}{1 + v^2},
\]
where we used the definition of the $\eta$-function in the second step.

The remaining integral can be evaluated using a more general formula that Euler proved on various occasions, e.g., in his paper “Investigatio formulæ integralis $\int \frac{x^{m-1}dx}{(1+x^2)^n}$ casu, quo post integrationem statuitur $x = \infty$” [9] (E588: “Investigation into the integral formula $\int \frac{x^{m-1}dx}{(1+x^2)^n}$ in the case in which one puts $x = \infty$ after the integration”). We have:

$$\int \frac{v^{x-2}dv}{1+v^2} = -\frac{1}{2} \frac{\pi}{\cos \left( \frac{\pi x}{2} \right)}.$$

Therefore, substituting this into the previous equation, we have:

$$\Gamma(x)\lambda(x) = -\frac{1}{2} \frac{\pi}{\cos \left( \frac{\pi x}{2} \right)} \pi^{-1} \eta(1-x).$$

Recalling the definition of $\lambda(x)$ from (7) and noting the elementary relation

$$\lambda(x) = \frac{2^x - 1}{2^x - 2} \eta(x)$$

and solving the penultimate equation for $\eta(1-x)$ we arrive at:

$$\eta(1-x) = \frac{2^x - 1}{1 - 2^x - 1} \pi^{-x} \cos \left( \frac{\pi x}{2} \right) \Gamma(x) \eta(x),$$

which is (2), of course. Thus, we completed the proof and showed that it can be given using just formulas found by Euler.

The proof of (2) hinges on the partial fraction decomposition of $\frac{x}{\sin x}$, i.e. equation (4). Even though the method Euler introduced in [10] does not work in general ($\frac{1}{e^x - 1}$ being an example for this), equation (4) and thus (5) are nevertheless correct. Moreover, Euler gave the partial fraction decompositions of several trigonometric functions on various occasions; we mention his paper “De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera, in qua eadem summationes ex fonte maxime diverso derivatur” [5] (E61: “Another dissertation on the sums of the series arising from the powers of reciprocals of the natural numbers, in which the same summations

\[d\]A proof of (2) based on partial fraction decompositions is also found in [1], which in turn took a more general formula from Malmstên’s (1814-1886) paper “De integralibus quibusdam definitis seriesbusque infinitis” (“On certain definite integrals and infinite series”). Malmstên’s paper and all the results found therein are extensively discussed in [4].

\[e\]This is discussed in more detail, e.g., in [2].
Euler states the functional equation for what is now called the Dirichlet $\beta$-function, defined by (9), as a conjecture. The scan was taken from original version of his paper [7].

are derived from a completely different source” as an example. Thus, we can be rather certain that Euler was convinced about the correctness of his formulas that he derived in [10].

Moreover, since we did not use other formulas than those contained in Euler’s papers, we conclude that Euler could have given a proof of (2) that he found in [7] by using divergent series.

Verification of Euler’s Second Functional Equation

In his paper [7], Euler also conjectured the formula for what is nowadays referred to as the Dirichlet $\beta(s)$ function (see figure 3), defined by

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \quad \text{for} \quad \Re(s) > 0. \quad (9)$$

In modern notation, Euler’s conjecture reads

$$\beta(1 - s) = \left(\frac{\pi}{2}\right)^{-s} \sin\left(\frac{\pi s}{2}\right) \beta(s) \Gamma(s). \quad (10)$$

A proof can be given along the same lines as the one presented for the $\eta$-function. Indeed, Euler also stated the necessary partial fraction decomposition in § 29 of [10] (see figure 4).

In modern notation, this partial fraction decomposition reads

$$\frac{\pi}{2 \cos \alpha \pi} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)}{(2n + 1)^2 - 4\alpha^2}.$$

Using the identity
\[
\frac{\pi}{2} \cos \alpha \pi = \frac{-1}{1-\alpha^2} + \frac{5}{5-\alpha^2} + \frac{10}{31-\alpha^2} - \frac{16}{45-\alpha^2} + \text{ etc.}
\]

vel etiam haec:

\[
\frac{\pi}{e^{\theta \pi} - e^{-\theta \pi}} = \frac{2}{1+4\beta \beta} - \frac{6}{9+4\beta \beta} + \frac{10}{25+4\beta \beta} - \frac{14}{49+4\beta \beta} + \text{ etc.}
\]

Figure 4: Euler states the partial fraction decomposition for \(\frac{\pi}{2 \cos \alpha \pi}\). The scan is taken from the original version of his paper [10].

\[
\cos x = \frac{e^{ix} + e^{-ix}}{2}
\]

and after making the substitution \(u = i\alpha \pi\), we infer the formula

\[
\frac{1}{e^u + e^{-u}} = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \cdot \left( \frac{1}{1 + \left( \frac{2u}{(2n+1)\pi} \right)^2} \right). \tag{11}
\]

Next, we see, proceeding as above, that

\[
\beta(s) \Gamma(s) = \int_0^{\infty} \frac{x^{s-1}dx}{e^x + e^{-x}}.
\]

Here, we used – as it is common now – the letter \(s\) as a variable as in (9). Whereas we only need \(\text{Re}(s) > 0\) for the integral to converge, Euler often restricted his considerations to real values for \(s\). The usage of the letter \(s\) in this context probably traces back to Riemann’s famous paper “Über die Anzahl der Primzahlen unter einer gegebenen Größe” [5] (“On the number of primes below a given magnitude”). Thus, with this notation, using (11) and simplifying the result, we find

\[
\beta(s) \Gamma(s) = \left( \frac{\pi}{2} \right)^{s-1} \sum_{n=0}^{\infty} (-1)^n (2n+1)^{s-1} \int_0^{\infty} \frac{u^{s-1}du}{1 + u^2}. \tag{12}
\]

Referring to [9] again, we can evaluate the integral

In Euler’s paper [10], both the original and the Opera Omnia version (see also figure 4), the fraction on the left-hand side contains a \(-\) instead of a \(+\) between the exponential functions. But this seems to be typographical error and not a conceptual one.
Thus, inserting this into (12) and solving for $\beta(1 - s)$, we will obtain

$$\beta(1 - s) = \left(\frac{\pi}{2}\right)^{-s} \sin \left(\frac{\pi s}{2}\right) \beta(s) \Gamma(s).$$

This is (10), of course. Malmstèn (1814 - 1886) seems to be the first to have proved this functional equation; he gave a proof in his paper “De integralibus quibusdam definitis seriebusque infinitis” [12] (“On certain definite integrals and infinite series”). See also [4] for a discussion of the results contained in Malmstèn’s work from a modern perspective.

2.3 The Zeta-Function

The $\eta$-function (1) is, of course related to the Riemann $\zeta$-function (3) by

$$\eta(s) = (1 - 2^{1-s})\zeta(s), \tag{13}$$

with $\zeta(s)$ satisfying the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin \left(\frac{\pi s}{2}\right) \Gamma(1 - s) \zeta(1 - s). \tag{14}$$

Euler only considered the alternating series defined by $\eta(s)$ in [7]. Indeed, in many instances Euler preferred alternating series over their non-alternating counterparts. A reason for this, at least for the case at hand, might be as follows: Euler was led to (2) and (10) by applying his notion of a sum of a divergent series, which he had outlined in his paper “De seriebus divergentibus” [6] (E247: “On divergent series”). There, he demonstrated several ways to assign a finite value to the alternating factorial series

$$1! - 2! + 3! - 4! + \cdots.$$

Thus, it must have been natural to him to apply his theories and consider the series:

$$1\!-\!1+1\!-\!1+1\!-\!\cdots, \quad 1\!-\!2+3\!-\!4+5\!-\!\cdots, \quad 1\!-\!4+9\!-\!16+25\!-\!\cdots \text{ etc.,}$$

i.e., in modern formulation, the values $\eta(-k)$, $k$ being a natural number. For this purpose, he used what would later become called Abel summation (cf. [13], [14], [15], [16]).
and found the correct values this way. For the corresponding values of the \( \zeta \)-function on the other hand, the same procedure does not give the correct values; indeed, Euler would have found all the values to be infinite. Thus, Euler would have arrived at a contradiction considering (13) and would have to discharge his theory of divergent series.

On a related note, let us mention that starting from (6) we would find the formula

\[
\zeta(s) \Gamma(s) = \int_0^\infty \frac{u^{s-1} du}{e^u - 1}
\]

such that using the partial fraction decomposition of \( \frac{1}{e^u - 1} \), which is not contained in [10] and does not seem to appear in any of Euler’s works. The reason for this, as also discussed in [2], might be that Euler’s method to find the partial fraction decomposition of transcendental functions does not work in general and, as mentioned above, \( \frac{1}{e^u - 1} \) is an example for this. Moreover, even if we use the correct partial fraction decomposition for \( \frac{1}{e^u - 1} \) (see, e.g., [2]) in the last integral representation and proceed as in the cases of \( \beta(s) \) and \( \eta(s) \), we encounter a divergent integral, which would have to be regularized by methods unknown to Euler. Therefore, we see that proving (14) is not as easy and smooth as it was for the previous examples.

In total, considering the \( \zeta \)-function, in contrast to the \( \eta \)-function, would have led Euler to contradictions in his theory of divergent series and he did not have the necessary formulas and concepts at this disposal to resolve them. Hence we conclude that Euler could have proved (2) but not (14) – as strange as this might sound from a modern perspective.

## 3 Conclusion

In this note we presented how one could prove (2) and (10), both conjectured by Euler in his paper [17], using only formulas also obtained by him in his other works. We drew from his papers [9], [10] and [11], the second one being the most important, since it contains the partial fraction decompositions that allowed to infer (5) and (11). The two others contained formulas that we used to show that Euler either had evaluated or would have been able to evaluate the integrals occurring in the proof that we presented. Therefore, since all formulas were found by Euler himself and the proof we presented did not use any tools
that were not at Euler’s disposal, we concluded that Euler could have given a proof of (2) and (10) himself, if he had combined his results as we did.

Concerning the functional equation for the Riemann $\zeta$-function (14) on the other hand, we put forward an argument that Euler could not have proved it. For, he would have encountered difficulties with his notion of the sum of divergent series, if he had considered (3) instead of the $\eta$-function (1), which might also explain his preference for alternating series in many of his other works. Aside from this, we mentioned that Euler did not write down the necessary partial fraction decomposition for $\frac{1}{e^x-1}$ and had no general method to find it. Additionally, trying to prove (14) along the lines done in this note for (2) and (10), one would encounter infinite integrals, which would then require another peculiar treatment using techniques and concepts that were not established at Euler’s time.

Those concepts and techniques lie in the field of complex analysis. One would especially need the notion of analytic continuation and contour integration. In conclusion, we might say that Euler discovered everything possible about the $\zeta$-function, if it is considered as a function of a real variable, but left the case of a complex variable to his successors, especially Riemann.

References


