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Variae observationes circa series infinitas

Leonhard Euler

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VARIAE OBSERVATIONES

CIRCA

SERIES INFINITAS.

AVCTORE

Leonh. Euler.

Observationes, quas hic proferre constitui, plerumque circa eiusmodi series versantur, quae prorsus sunt diuersae ab iis, quae etiamnum tractari sunt solitae. Cum enim adhuc nullae aliae series sint consideratae, nisi quarum vel terminus generalis esset datus, vel lex saltem, qua ex datis aliquot terminis sequentes inuenire liceret; ita hic eiusmodi potissimum series sum contemplaturus, quae neque terminum generalem proprie sic dictum, neque legem continuationis agnoscant; sed quarum natura per alias condiciones determinetur. De huiusmodi ergo seriebus eo magis erit mirandum, si summari poterunt, cum ad methodos summandi adhuc cognitae necessario vel terminus generalis, vel lex progressionis requiratur; quibus deficientibus vix alia via patere videatur, qua ad summas cognoscendas pertingere queamus. Ad has autem observationes me peculiaris series a *Cel. Goldbach* mecum communicata deduxit, cuius summationem maxime admirandam *Viri Celeb.* permissu hic primo loco sum demonstraturus.

Theo-

Theorema I.

Huius seriei in infinitum continuatae

$$\frac{1}{2} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{25} + \frac{1}{32} + \frac{1}{35} + \text{etc.}$$

cuius denominatores unitate aucti dant omnes numeros, qui sunt potestates vel secundi vel altioris cuiusvis ordinis numerorum integrorum, cuiusque adeo terminus quisque exprimitur

hac formula $\frac{1}{m^n - 1}$, denotantibus m et n numeros integros unitate maiores; huius seriei autem summa est = 1.

Demonstratio.

Hoc est Theorema a *Celeb. Goldbach* primum mecum communicatum, quod me etiam ad sequentes propositiones manuduxit. Ex inspectione autem huius seriei facile intelligitur, quam irregulariter ea progrediatur, et propterea quisque in his rebus versatus maxime modum admirabitur, quo *Vir Celeb.* summam huius singularis seriei inuenit; sequenti vero modo mihi hoc Theorema demonstravit. Sit

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \text{etc.}$$

deinde cum sit

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.}$$

erit hanc seriem ab illa auferendo

$$x - 1 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \text{etc.}$$

ex denominatoribus ergo exclusi sunt omnes potestates binarii cum binario ipso; reliqui vero numeri omnes occurrunt.

Tom. IX.

X

Ab

Ab hac serie porro subtrahit hanc

$$\frac{1}{2} = \frac{1}{2} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \text{etc.}$$

et restabit

$$x - 1 - \frac{1}{2} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

denuoque subtrahit

$$\frac{1}{4} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \text{etc.}$$

restabitque

$$x - 1 - \frac{1}{2} - \frac{1}{4} = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

Atque simili modo omnes reliquos terminos successive tollendo reperietur tandem

$$x - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} - \text{etc.} = 1$$

seu

$$x - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \text{etc.}$$

cuius progressionis denominatores vnitare aucti dant omnes numeros, qui non sunt potestates. Quare si ista series ab initio assumpta

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{etc.}$$

subtrahatur, relinquetur

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{etc.}$$

cuius seriei igitur, in qua denominatores vnitare aucti dant omnes omnino potestates numerorum integrorum, summa est = 1. Q. E. I.

Theorema 2.

Huius seriei in infinitum continuatae

$$\frac{1}{2} + \frac{1}{7} + \frac{1}{12} + \frac{1}{17} + \frac{1}{22} + \frac{1}{27} + \text{etc.}$$

cuius

cuius denominatores unitate aucti dant omnes potestates pares, summa est $= 1/2$; atque huius seriei

$$\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \text{etc.}$$

in infinitum continuatae, cuius denominatores unitate aucti dant omnes potestates impares, summa aequalis est $1 - 1/2$. Quarum serierum prioris terminus quisque est

$$\frac{1}{(2m-2)^n - 1}, \text{ posterioris vero terminus quilibet hac con-}$$

tinetur formula $\frac{1}{(2m-1)^n - 1}$; retinentibus m et n praecedentes valores.

Demonstratio.

Consideretur sequens series, cuius summa ponatur x ;

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \text{etc.}$$

Iam cum sit

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.}$$

sublata hac serie ab illa prodibit sequens

$$x - 1 = \frac{1}{8} + \frac{1}{16} + \frac{1}{24} + \frac{1}{32} + \frac{1}{48} + \text{etc.}$$

2 qua subtrahatur

$$\frac{1}{3} = \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \text{etc.}$$

erit

$$x - 1 - \frac{1}{3} = \frac{1}{12} + \frac{1}{24} + \frac{1}{36} + \frac{1}{48} + \text{etc.}$$

simili modo ob

$$\frac{1}{9} = \frac{1}{18} + \frac{1}{27} + \frac{1}{36} + \text{etc.}$$

erit

$$x - 1 - \frac{1}{3} - \frac{1}{9} = \frac{1}{27} + \frac{1}{54} + \frac{1}{72} + \text{etc.}$$

X 2

Omni-

Omnibus ergo terminis hoc modo sublati prodibit

$$x = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729} - \frac{1}{2187} + \text{etc.}$$

cuius denominatores constituunt seriem naturalem numerorum imparium exceptis iis, qui unitate aucti sunt potestates, vti ex formatione huius seriei intelligitur. Cum vero fit

$$1/2 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \text{etc.}$$

atque

$$x = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \text{etc.}$$

$$\text{erit } x = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729} - \text{etc.} - 1/2.$$

Illo ergo pro x inuento valore sublato ab isto, in quo omnes omnino numeri impares occurrunt, restabit

$$0 = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \text{etc.} - 1/2,$$

seu ista

$$1/2 = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \frac{1}{729} + \text{etc.}$$

cuius seriei denominatores sunt ii numeri impares, qui unitate aucti dant omnes potestates pares. Huius ergo seriei summa est $1/2$, prout in propositione est assertum. Q. E. Vnum.

Cum vero praecedens Theorema sit

$$1 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \text{etc.}$$

vbi denominatores unitate aucti dant omnes numeros, qui sunt potestates tam pares quam impares, habebitur illa serie ab hac demta

$$1 - 1/2 = \frac{1}{8} + \frac{1}{24} + \frac{1}{72} + \frac{1}{216} + \text{etc.}$$

cuius denominatores adeo sunt ii numeri pares, qui unitate aucti dant omnes potestates impares. Q. E. Alterum.

Theo-

Theorema 3.

Posito π pro peripheria circuli, cuius diameter est 1, erit

$$\frac{\pi}{4} = 1 - \frac{1}{8} + \frac{1}{24} - \frac{1}{48} + \frac{1}{80} - \frac{1}{120} + \frac{1}{160} - \frac{1}{224} + \frac{1}{244} - \frac{1}{288} \text{ etc.}$$

cuius seriei denominatores sunt numeri pariter pares, unitate vel maiores vel minores quam potestates numerorum imparium. Illae autem fractiones, quarum denominatores unitate excedunt potestates, signum habent + reliquae signum -.

Demonstratio.

Cum sit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

cuius seriei eae fractiones, quarum denominatores unitate deficiunt a numeris pariter paribus, signum habent - 1, reliquae signum +. Ad illam seriem vero addatur haec geometrica

$$\frac{1}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \text{etc.}$$

erit

$$\frac{\pi}{4} + \frac{1}{4} = 1 + \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \text{etc.}$$

a qua subtrahatur

$$\frac{1}{4} = \frac{1}{3} + \frac{1}{23} + \frac{1}{123} + \text{etc.}$$

erit

$$\frac{\pi}{4} + \frac{1}{4} - \frac{1}{4} = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \text{etc.}$$

in qua serie nec 3, et 5 nec eorundem potestates amplius insunt, simili modo 7 eiusque potestates tollentur addendo hanc seriem

$$\frac{1}{4} = \frac{1}{7} - \frac{1}{49} + \text{etc.}$$

X 3

eritque

bus quadratis imparibus unitate deficiunt, eaeque fraction-
nes omnes habent signum idem —. Cum vero sit

$$\frac{1}{4} + \frac{1}{24} + \frac{1}{48} + \frac{1}{80} + \frac{1}{120} + \frac{1}{168} + \text{etc.} = \frac{1}{4}$$

habebitur loco harum fractionum omnium substituendo $\frac{1}{4}$
sequens forma

$$\frac{\pi}{4} = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} - \frac{1}{1024} + \text{etc.}$$

seu

$$\frac{\pi}{4} - \frac{1}{4} = \frac{1}{16} - \frac{1}{64} + \frac{1}{256} - \frac{1}{1024} + \text{etc.}$$

cuius seriei denominatores sunt numeri pariter pares vel
unitate maiores vel minores quam potestates numerorum
imparium non quadratae, ob quadratas iam exclusas,
atque prout unitate sunt vel maiores vel minores, fractio-
nes etiam habent signum + vel —. Q. E. D.

Corollarium 1.

Ad seriem ergo continuandam omnium numerorum
imparium, qui non sunt potestates sumendae sunt pote-
states exponentium imparium, eaeque unitate vel augen-
dae vel minuendae, quo prodeant numeri pariter pares, qui
erunt denominatores seriei inuentae: servata signorum re-
gula.

Corollarium 2.

Cum autem omnis numerus impar vel sit $4m-1$
vel $4m+1$, potestates autem exponentium imparium a
 $4m-1$ ortorum, si unitate augeantur, illae autem, quae
a $4m+1$ oriuntur si unitate minuantur, dent numeros
pariter pares; aequabitur $\frac{\pi}{4} - \frac{1}{4}$ seriei terminorum qui

omnes in hac forma: $\frac{1}{(4m-1)^{2n+1} + 1}$ continentur, dem-
ta.

ta serie terminorum in hac forma $\frac{1}{(4m+1)^{n+1-1}}$ contentorum; vbi loco m et n omnes numeri integri affirmatiui accipi debent praeter eos, qui vel $4m-1$ vel $4m+1$ faciunt potestates.

Corollarium 3.

Aequabitur ergo $\frac{\pi}{4} - \frac{3}{4}$ aggregato sequentium serierum infinitarum

$$\frac{\pi}{4} - \frac{3}{4} = \left\{ \begin{array}{l} \frac{1}{3^3+1} + \frac{1}{3^5+1} + \frac{1}{3^7+1} + \frac{1}{3^9+1} + \text{etc.} \\ - \frac{1}{5^3-1} - \frac{1}{5^5-1} - \frac{1}{5^7-1} - \frac{1}{5^9-1} + \text{etc.} \\ + \frac{1}{7^3+1} + \frac{1}{7^5+1} + \frac{1}{7^7+1} + \frac{1}{7^9+1} + \text{etc.} \\ + \frac{1}{11^3+1} + \frac{1}{11^5+1} + \frac{1}{11^7+1} + \frac{1}{11^9+1} + \text{etc.} \\ - \frac{1}{13^3-1} - \frac{1}{13^5-1} - \frac{1}{13^7-1} - \frac{1}{13^9-1} + \text{etc.} \\ + \frac{1}{15^3+1} + \frac{1}{15^5+1} + \frac{1}{15^7+1} + \frac{1}{15^9+1} + \text{etc.} \\ \text{etc.} \quad \text{etc.} \end{array} \right.$$

Corollarium 4.

Hac ergo serie eousque continuata, donec denominatores fiant maiores quam 100000 habebitur $\frac{\pi}{4} = \frac{3}{4} +$

$$\begin{array}{l} \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} + \frac{1}{1332} + \frac{1}{2188} - \frac{1}{2196} - \frac{1}{3224} + \frac{1}{3376} - \frac{1}{4912} \\ + \frac{1}{6860} - \frac{1}{9200} + \frac{1}{12168} + \frac{1}{16808} + \frac{1}{19684} - \frac{1}{24368} + \frac{1}{29792} - \frac{1}{35936} \\ + \frac{1}{42876} - \frac{1}{50852} + \frac{1}{59320} - \frac{1}{68920} - \frac{1}{78124} + \frac{1}{79568} - \frac{1}{91124}. \end{array}$$

Corollarium 5.

Cum omnes denominatores per 4 diuidi possint, erit

$$\pi = 3 + \frac{1}{2} - \frac{1}{31} + \frac{1}{62} + \frac{1}{86} + \frac{1}{332} + \frac{1}{547} - \frac{1}{549} - \frac{1}{701} + \frac{1}{844} \text{ etc.}$$

Quae

Quae series ideo notari meretur, quod eius duo primi termini iam dent Archimedis proportionem peripheriae circuli ad diametrum.

Theorema 5.

Retinente π priorem significationem, erit

$$\frac{\pi}{4} - 1/2 = \frac{1}{2^5} + \frac{1}{2^8} + \frac{1}{2^{13}} + \frac{1}{2^{18}} + \frac{1}{2^{25}} + \frac{1}{2^{32}} + \text{etc.}$$

cuius seriei haec est lex, ut numeri medii inter binos denominatores binario differentes, scilicet 27, 243, 343, etc. sint potestates exponentium imparium ortae a numeris imparibus, quae unitate auctae sint per 4 divisibiles seu numeri pariter pares.

Demonstratio.

Cum per Theorema tertium sit

$$\frac{\pi}{4} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \text{etc.}$$

fractionum signo $-$ affectarum denominatores sunt numeri pariter pares unitate deficientes a potestatibus numerorum imparium; fractionum vero signo $+$ affectarum denominatores sunt quoque pariter pares unitate superantes potestates numerorum imparium; atque praeterea sit per Theorema secundum

$$1 - 1/2 = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \text{etc.}$$

cuius seriei denominatores deficiunt unitate ab omnibus potentiis numerorum imparium; haec series complectetur omnes terminos illius signo $-$ affectos, et praeterea fractiones denominatores habentes impariter pares unitate deficientes a potestatibus numerorum imparium. Quare

Tom. IX.

Y

si haec

si haec series ad illam addatur prodibit

$$\frac{\pi}{4} - 12 = \frac{1}{26} + \frac{1}{28} + \frac{1}{242} + \frac{1}{244} + \frac{1}{342} + \frac{1}{344} + \text{etc.}$$

cuius binae fractiones erunt ita comparatae, ut prioris denominator sit numerus impariter par, posterioris binario maior pariter par, mediusque numerus inter binos huiusmodi denominatores sit potestas numeri imparis; quae ergo potestas unitate aucta dare debet numerum pariter parem. Q. E. D.

Corollarium I.

Quia hae potestates numerorum imparium ita sunt comparatae, ut unitate auctae fiant per 4 diuisibiles, erunt eae potestates imparium dimensionum, quae oriuntur a numeris huius formae $4m-1$, qui ipsi non sunt potestates.

Corollarium 2.

Si ergo omnes sumantur numeri huius formae $4m-1$, quae non sunt potestates, eorumque capiantur omnes potestates exponentium imparium; hae potestates unitate tam auctae quam minutae dabunt omnes fractionum seriei inuentae denominatores.

Corollarium 3.

Si binae fractiones in unam coalescant erit

$$\frac{\pi}{4} = 12 + \frac{2 \cdot 27}{26 \cdot 28} + \frac{2 \cdot 243}{242 \cdot 244} + \frac{2 \cdot 343}{342 \cdot 344} + \text{etc.}$$

Quae series formabitur sumendis omnibus fractionibus, quae

oriuntur ex hac forma $\frac{2(4m-1)^{2n+1}}{(4m-1)^{4n+2}-1}$ substituendo loco

loco m et n omnes numeros integros successive praeter eos ipsius m valores, qui reddant $4m-1$ potestatem.

Theorema 6.

Seriei huius

$$\frac{1}{15} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \frac{1}{624} + \text{etc.}$$

cuius denominatores unitate aucti dant omnia quadrata, quae simul sunt altiores potestates; huius inquam seriei in infinitum continuatae summa est $\frac{7}{4} - \frac{\pi^2}{6}$: denotante π peripheriam circuli, cuius diameter $= 1$.

Demonstratio.

Hoc quoque Theorema a *Cel. Goldbachio*, verum sine demonstratione accepi, atque iisdem quibus ante vestigiis insistens hanc inveni demonstrationem. Cum ante aliquot annos incidissem in huius seriei

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.}$$

summam $= \frac{\pi^2}{6}$, hanc ipsam seriem ita sum contemplatus

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc.}$$

Iam cum sit

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \text{etc.}$$

atque

$$\frac{1}{8} = \frac{1}{9} + \frac{1}{36} + \frac{1}{729} + \text{etc.}$$

similique modo

$$\frac{1}{24} = \frac{1}{25} + \frac{1}{625} + \text{etc.} \quad \text{et} \quad \frac{1}{35} = \frac{1}{36} + \text{etc.}$$

si loco harum serierum geometricarum substituantur summae prodibit

$$\frac{\pi^2}{6} = 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \frac{1}{99} + \text{etc.}$$

cuius seriei denominatores unitate aucti dant omnes numeros quadratos, praeter eos, qui simul fiat alius speciei potestates. Cum autem sumendis omnino quadratis unitate minutis sit

$$\frac{\pi^2}{4} = \frac{1}{2} + \frac{1}{8} + \frac{1}{18} + \frac{1}{32} + \frac{1}{50} + \frac{1}{72} + \frac{1}{98} + \frac{1}{128} + \text{etc.}$$

proveniet ab hac superiore seriem subtrahendo

$$\frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \text{etc.}$$

qui denominatores unitate aucti dant omnes numeros quadratos, qui simul sunt alius generis potestates. Q.E.D.

Constituunt haec sex theoremata alterum observationum istarum partem, quibus scilicet series additione subtractione terminorum ortae sunt consideratae. Sequentia vero theoremata circa series, quorum termini in se invicem multiplicantur, versabuntur; neque minus erunt admirabilia, quam praecedentia, cum in iis pariter lex progressionis tantopere sit irregularis. Discrimen autem in hoc potissimum erit positum, quod in theorematibus praecedentibus progressio terminorum secuta sit seriem potestatum, quae per se est maxime irregularis; in his autem termini progrediantur secundum numeros primos, quorum progressio non minus est abstrusa.

Theorema 7.

Factum continuum in infinitum ex his fractionibus
 $\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19}{3 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18}$ etc. *ubi numeratores sunt omnes numeri primi, denominatores vero unitate deficiunt a numeratoribus*
Hoc factum inquam aequale est summae huius seriei

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \text{etc.}$$

estque adeo infinitum.

Demon-

Demonstratio.

Nam fit

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

erit

$$\frac{1}{2}x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \text{etc.}$$

qua serie ab illa demta restat

$$\frac{1}{2}x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{etc.}$$

in qua nulli amplius denominatores pares insunt. Ab hac denuo auferatur ista series

$$\frac{1}{2} \cdot \frac{1}{2}x = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \text{etc.}$$

restabit

$$\frac{1}{2} \cdot \frac{1}{2}x = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

in cuius denominatoribus nec per 2 nec per 3 diuisibiles reperiuntur. Quo autem etiam numeri per 5 diuisibiles egrediantur, subtrahatur ista series

$$\frac{1 \cdot 2}{2 \cdot 3} \cdot \frac{1}{2}x = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \text{etc.}$$

restabitque

$$\frac{1 \cdot 2 \cdot 4}{2 \cdot 3 \cdot 5}x = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

Atque simili modo tollendis omnibus terminis tum per 7 tum per 11 etc. omnesque numeros primos diuisibilibus tandem reperiatur

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdot 22 \cdot \text{etc.}}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \text{etc.}} x = 1.$$

Quare cum fit

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

erit

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{etc.} = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \text{etc.}}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdot 22 \cdot \text{etc.}}$$

cuius expressionis numeratores constituunt progressionem numerorum primorum, denominatores vero unitate ab iis deficient. Q. E. D.

Corollarium I.

Expressiones ergo $\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \text{etc.}}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot \text{etc.}}$ valor est infinitus, et posito absolute infinito $=\infty$, erit istius expressionis valor $=1/\infty$, quod infinitum inter omnes infiniti potestates est minimum.

Corollarium 2.

Cum vero haec expressio $\frac{4 \cdot 9 \cdot 16 \cdot 25 \cdot 36 \cdot 49 \cdot \text{etc.}}{3 \cdot 8 \cdot 15 \cdot 24 \cdot 35 \cdot 48 \cdot \text{etc.}}$ finitum habeat valorem scilicet 2; sequitur infinities plures esse numeros primos, quam quadratos, in serie omnium omnino numerorum.

Corollarium 3.

Verum etiam hinc intelligitur infinities pauciores extare numeros primos, quam numeros integros; haec enim expressio $\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \text{etc.}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \text{etc.}}$ absolute infinitum habet valorem, cum similis a numeris tantum primis ortae valor sit logarithmus istius infiniti.

Theorema 8.

Si ex serie numerorum primorum sequens formetur expressio

$$\frac{2^n \cdot 3^n \cdot 5^n \cdot 7^n \cdot 11^n \cdot \text{etc.}}{(2^n - 1)(3^n - 1)(5^n - 1)(7^n - 1)(11^n - 1) \cdot \text{etc.}}$$

erit

erit eius valor aequalis summae huius seriei

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \text{etc.}$$

Demonstratio.

Sit

$$x = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

erit

$$\frac{1}{2^n} x = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{etc.}$$

vnde oritur

$$\frac{(2^n - 1)}{2^n} x = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.}$$

Porro est

$$\frac{(2^n - 1)}{2^n} \cdot \frac{1}{3^n} x = \frac{1}{3^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{etc.}$$

vnde fiet

$$\frac{(2^n - 1)(3^n - 1)}{2^n \cdot 3^n} x = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.}$$

Similibus ergo operationibus pro singulis numeris primis institutis omnes seriei termini praeter primum tollentur, reperieturque

$$1 = \frac{(2^n - 1)(3^n - 1)(5^n - 1)(7^n - 1)(11^n - 1) \text{ etc.}}{2^n \cdot 3^n \cdot 5^n \cdot 7^n \cdot 11^n \text{ etc.}} x$$

et loco x serie restituta fit

$$\frac{2^n \cdot 3^n \cdot 5^n \cdot 7^n \cdot 11^n \text{ etc.}}{(2^n - 1)(3^n - 1)(5^n - 1)(7^n - 1)(11^n - 1) \text{ etc.}} = 1$$

$$= 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

Q. E. D.

Corollarium 1.

Cum posito $n=2$ fit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = \frac{\pi^2}{6}$
denotante π peripheriam circuli, cuius diameter est 1,
erit

$$\frac{4 \cdot 9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdot \text{etc.}}{2 \cdot 8 \cdot 24 \cdot 48 \cdot 120 \cdot 168 \cdot \text{etc.}} = \frac{\pi^2}{6}$$

seu

$$\frac{\pi^2}{6} = \frac{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdot \text{etc.}}{1 \cdot 3 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot \text{etc.}}$$

Corollarium 2.

Cum praeterea posito $n=4$ fit

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = \frac{\pi^4}{95}$$

erit

$$\frac{\pi^4}{95} = \frac{4 \cdot 4 \cdot 9 \cdot 9 \cdot 25 \cdot 25 \cdot 49 \cdot 49 \cdot 121 \cdot 121 \cdot \text{etc.}}{3 \cdot 5 \cdot 8 \cdot 10 \cdot 24 \cdot 26 \cdot 48 \cdot 50 \cdot 120 \cdot 122 \cdot \text{etc.}}$$

Hac igitur expressione per illam diuisa prodit

$$\frac{\pi^2}{75} = \frac{4 \cdot 9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdot \text{etc.}}{5 \cdot 10 \cdot 26 \cdot 50 \cdot 122 \cdot 170 \cdot \text{etc.}}$$

Theorema 9.

Si quadrata numerorum primorum imparium omnium
resoluantur in duas partes unitate a se inuicem differentes,
harumque partium impares sumantur pro numeratoribus, pa-
res vero pro denominatoribus seriei ex factoribus compositae
valor huius expressionis erit

$$\frac{5 \cdot 13 \cdot 25 \cdot 61 \cdot 85 \cdot 145 \cdot \text{etc.}}{4 \cdot 12 \cdot 24 \cdot 60 \cdot 84 \cdot 144 \cdot \text{etc.}} = \frac{\pi^2}{2}$$

Demon-

Demonstratio.

Per theorematís præcedentis Coroll. 1. habemus

$$\frac{\pi^2}{6} = \frac{4 \cdot 9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdot 289 \cdot \text{etc.}}{3 \cdot 8 \cdot 24 \cdot 48 \cdot 120 \cdot 168 \cdot 288 \cdot \text{etc.}}$$

At in Coroll. 2. sequentem elicimus æquationem

$$\frac{\pi^2}{15} = \frac{4 \cdot 9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdot 289 \cdot \text{etc.}}{5 \cdot 10 \cdot 26 \cdot 50 \cdot 122 \cdot 170 \cdot 290 \cdot \text{etc.}}$$

Quarum expressionum si illa per hanc diuidatur, π ex calculo egredietur, habebiturque

$$\frac{5}{2} = \frac{5 \cdot 10 \cdot 26 \cdot 50 \cdot 122 \cdot 170 \cdot 290 \cdot \text{etc.}}{3 \cdot 8 \cdot 24 \cdot 48 \cdot 120 \cdot 168 \cdot 288 \cdot \text{etc.}}$$

cuius expressionis numeratores sunt unitate maiores quam quadrata numerorum primorum, denominatores vero unitate minores. Si ergo vtrunque per $\frac{5}{2}$ diuidatur singulaeque fractiones per 2 deprimantur, habebitur

$$\frac{5}{2} = \frac{5 \cdot 13 \cdot 25 \cdot 61 \cdot 85 \cdot 145 \cdot \text{etc.}}{4 \cdot 12 \cdot 24 \cdot 60 \cdot 84 \cdot 144 \cdot \text{etc.}}$$

vbi numeratores sunt unitate maiores quam denominatores respondentes, atque quisque numerator cum suo denominatore facit quadratum numeri primi imparis, ob sublatum diuisione quadratum numeri primi paris 2. Q. E. D.

Theorema 10.

Si π ut hætenus significet peripheriam circuli, cuius diameter est $= 1$, erit

$$\frac{\pi^3}{32} = \frac{30 \cdot 224 \cdot 440 \cdot 624 \cdot 728 \cdot \text{etc.}}{81 \cdot 225 \cdot 441 \cdot 625 \cdot 729 \cdot \text{etc.}}$$

cuius expressionis denominatores sunt quadrata numerorum imparium non primorum, numeratores vero unitate minores.

Demonstratio.

A Wallisio habetur sequens expressio pro π , nempe

$$\frac{\pi}{4} = \frac{8 \cdot 24 \cdot 48 \cdot 80 \cdot 120 \cdot 168 \cdot \text{etc.}}{9 \cdot 25 \cdot 49 \cdot 81 \cdot 121 \cdot 169 \cdot \text{etc.}}$$

Tom. IX

Z

quæ

Deinde est

$$\frac{1}{3} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} - \frac{1}{3^4} + \text{etc.}$$

qua sublata prodit

$$\frac{4}{3} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = 1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

in qua serie nulli amplius occurrunt denominatores vel per 3 vel per 5 diuisibiles. Simili modo tollentur omnes per 7 diuisibiles addendo

$$\frac{1}{17} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = \frac{1}{7} - \frac{1}{49} - \frac{1}{77} \text{ etc.}$$

prodibit autem

$$\frac{8}{7} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = 1 - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} \text{ etc.}$$

Perspicitur autem denominatores per numerum primum huius formae $4n-1$ diuisibiles tolli additione, unde iste nouus factor $\frac{4n}{4n-1}$ accedit: denominatores vero per numerum primum formae $4n+1$ diuisibiles tolli subtractione, unde nouus factor iste $\frac{4n}{4n+1}$ adiicietur. Horum ergo factorum successiue addendorum denominatores erunt numeri primi; numeratores vero numeri pariter pares unitate vel maiores vel minores quam denominatores. Hoc ergo modo si auferantur omnes termini seriei initio assumptae, prodibit eandem

$$\frac{\text{etc. } 24 \cdot 20 \cdot 16 \cdot 12 \cdot 12 \cdot 8 \cdot 4 \cdot 4}{\text{etc. } 23 \cdot 19 \cdot 17 \cdot 13 \cdot 11 \cdot 7 \cdot 5 \cdot 3} \cdot \frac{\pi}{4} = 1.$$

Ex qua oritur

$$\frac{\pi}{4} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \text{etc.}}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 20 \cdot 24 \cdot \text{etc.}} \quad \text{Q. E. D.}$$

Theorema 12.

Si omnes numeri primi impares in duas partes unitate a se inuicem differentes diuidantur atque partes pares

Z 2

su-

sumantur pro numeratoribus, impares vero pro denominatoribus, erit factum continuum.

$$\frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot \text{etc.}}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot \text{etc.}} = 2$$

Demonstratio.

Cum sit per Theorema praecedens $\pi = \frac{3 \cdot 5 \cdot 7 \cdot \text{etc.}}{4 \cdot 4 \cdot 8 \cdot \text{etc.}}$
erit

$$\frac{16}{\pi^2} = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 12 \cdot 12 \cdot 16 \cdot 16 \cdot \text{etc.}}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdot 13 \cdot 13 \cdot 17 \cdot 17 \cdot \text{etc.}}$$

At ex Coroll. 1. Theor. 8. si per $\frac{3}{4}$ multiplicetur habetur

$$\frac{\pi^2}{2} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdot 13 \cdot 13 \cdot \text{etc.}}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 12 \cdot 14 \cdot \text{etc.}}$$

quarum expressionum utraque per numeros primos impares formatur. Si ergo hae in se inuicem multiplicentur, prioris denominator destruet numeratorem posterioris, atque praeterea tam ex illius numeratore quam huius denominatore medietas terminorum auferetur. Prohibet scilicet

$$2 = \frac{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 20 \cdot 24 \cdot \text{etc.}}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdot \text{etc.}}$$

vbi numeratores sunt numeri pariter pares, denominatores vero impariter pares utrique unitate vel maiores vel minores quam numeri primi impares. Si ergo singulae fractiones per binarium deprimantur, numerator continebit numeros pares, denominator vero impares, atque bini respondententes et unitate a se inuicem different, et coniuncti numerum primum constituent. Habebitur igitur

$$2 = \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot \text{etc.}}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot \text{etc.}}$$

Q. E. D.

Theo-

Theorema 13.

Si omnes numeri impares non primi in duas partes diuidantur unitate a se inuicem distantes harumque pares pro numeratoribus impares vero pro denominatoribus accipiantur, erit

$$\frac{\pi}{4} = \frac{4. 8. 10. 12. 14. 16. 18. 20. 22. 24. \text{ etc.}}{5. 7. 11. 13. 15. 17. 19. 23. 25. \text{ etc.}}$$

Demonstratio.

Cum per Wallisii quadraturam circuli sit

$$\frac{\pi}{2} = \frac{2. 2. 4. 4. 6. 6. 8. 8. 10. 10. 12. 12. \text{ etc.}}{1. 3. 3. 5. 5. 7. 7. 9. 9. 11. 11. 13. \text{ etc.}}$$

cuius expressionis si singuli numeratores ad suos respondentes denominatores addantur, prodeunt omnes omnino numeri impares. At quoniam similis expressio, si ex numeris imparibus primis tantum formatur, aequatur binario, uti in praecedente Theoremate est demonstratum, quo erat

$$2 = \frac{2. 2. 4. 6. 6. 8. 10. 12. \text{ etc.}}{1. 3. 3. 5. 5. 7. 7. 9. 11. \text{ etc.}}$$

si illa expressio per hanc diuidatur, proveniet

$$\frac{\pi}{4} = \frac{4. 8. 10. 12. 14. 16. 18. 20. 22. 24. \text{ etc.}}{5. 7. 11. 13. 15. 17. 19. 23. 25. \text{ etc.}}$$

quae similiter ex numeris imparibus non primis formatur. Numeratores scilicet erunt numeri pares, denominatores vero impares unitate a numeratoribus distantes, atque singuli numeratores ad suos respondentes denominatores additi dabunt omnes numeros impares non primos. Q. E. D.

Theorema 14.

Denotante ut haecenus π circuli peripheriam cuius diameter est 1, dico fore

$$\frac{\pi}{2} = \frac{3. 5. 7. 11. 13. 17. 19. 23. 29. 31. \text{ etc.}}{2. 6. 6. 10. 14. 18. 18. 22. 30. 30. \text{ etc.}}$$

Z 3

cuius

cuius expressionis numeratores constituunt seriem numerorum primorum imparium, denominatores vero sunt numeri impariter pares unitate vel minores vel maiores quam numeratores respondentes.

Demonstratio.

Per Coroll. Theor. 8. si per $\frac{5}{4}$ multiplicetur est

$$\frac{\pi^2}{8} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdot 13 \cdot 13 \cdot \text{etc.}}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 12 \cdot 14 \cdot \text{etc.}}$$

in qua numeratores sunt numeri primi impares bis positi, denominatores vero numeri tam pariter pares quam impariter pares, unitate vel maiores vel minores quam ipsi numeri primi. Deinde Theoremate 11. demonstravimus esse

$$\frac{\pi}{4} = \frac{2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \text{etc.}}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 20 \cdot 24 \cdot \text{etc.}}$$

in qua expressione numeratores sunt numeri primi impares semel positi, denominatores vero numeri pariter pares unitate distantes a numeris primis, ita ut haec expressio inpraecedente sit contenta. Quare si illa expressio per hanc diuidatur prodibit

$$\frac{\pi}{2} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot \text{etc.}}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 18 \cdot \text{etc.}}$$

in qua numeri impares primi numeratores constituunt; denominatores vero sunt numeri impariter pares unitate vel maiores vel minores quam numeratores. Q. E. D.

Theorema 15.

Denotante π peripheriam circuli, cuius diameter est 1, erit

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \frac{1}{15} - \frac{1}{17} + \frac{1}{19} - \frac{1}{21} + \frac{1}{23} - \frac{1}{25} + \frac{1}{27} - \frac{1}{29} + \frac{1}{31} - \frac{1}{33} + \frac{1}{35} - \frac{1}{37} \text{ etc.}$$

cuius seriei denominatores sunt numeri impares omnes, ratio

Pro signorum autem hoc nititur fundamento. Numeris primis huius formae $4n-1$ datur signum $+$; numeris primis autem huius formae $4n+1$ signum $-$. Deinde numeris compositis id tribuitur signum, quod ipsis ratione compositionis ex primis cum suis signis secundum multiplicationis regulam competit.

Demonstratio.

Quemadmodum vtitis hic operationibus haec series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

conversa est in hanc expressionem $\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot \text{etc.}}{4 \cdot 6 \cdot 8 \cdot 12 \cdot \text{etc.}}$, ita vicissim methodus potest excogitari, quae hanc expressionem

$$\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \text{etc.}}{4 \cdot 6 \cdot 8 \cdot 12 \cdot 14 \cdot \text{etc.}}$$

in seriem

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

transmutare liceat. Atque si haec methodus ad expressionem theoremate praecedente inuentam

$$\frac{\pi^2}{6} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot \text{etc.}}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot \text{etc.}}$$

adhibeatur, ista expressio abibit in istam seriem propositam

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

cuius propterea summa est $\frac{\pi^2}{6}$. Ad idem a posteriore colligere licet, ponendo

$$x = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \text{etc.}$$

eritque

$$\frac{1}{3}x = \frac{1}{3} - \frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \frac{1}{27} - \frac{1}{33} + \frac{1}{39} - \text{etc.}$$

atque subtrahendo prodibit

$$\frac{2}{3}x = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

Deinde

Deinde cum simili modo fit

$$\frac{1}{5} \cdot \frac{2}{3} x = \frac{1}{5} - \frac{1}{25} + \frac{1}{125} - \frac{1}{625} \text{ etc.}$$

prodibit addendo.

$$\frac{6}{5} \cdot \frac{2}{3} x = 1 + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} \text{ etc.}$$

Atque similiter omnibus tollendis terminis praeter primum 1 inuenietur

$$x = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot \text{etc.}}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 18 \cdot \text{etc.}} = \frac{\pi}{2}.$$

Atque hinc simul ratio signorum seriei propositae colligitur eadem ipsa, quam descripsimus. Q. E. D.

Corollarium.

Summa ergo seriei propositae

$$1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} \text{ etc.}$$

duplo maior est quam summa huius seriei

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ etc.}$$

Quare cum ipsae fractiones sint vtrunque eadem, solis signis effectum est, ut altera alterius sit dupla.

Theorema 16.

Posito π ut haecenus pro peripheria circuli cuius diameter est 1, erit

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{6} + \frac{1}{10} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} + \frac{1}{18} + \frac{1}{20} \text{ etc.}$$

Fractionum autem affirmatiuarum denominatores sunt unitate minores quam numeri impares non potestates; fractionum autem negatiuarum denominatores sunt unitate maiores. Cuiusque autem fractionis signum congruit cum signo numeri imparis vel unitate maioris vel minoris non potestatis in praecedente Theoremate.

Demon-

Demonstratio.

Haec ipsa series oritur ex conuersione praecedentis secundum modum in Theorematis 1. 2. 3. vñtatis, quo continuo progressionēs geometricae vel adduntur vel demuntur, quoad solus primus terminus superfit. Q.E.D.

Theorema 17.

Si numeris imparibus primis huius formae $4n-1$ tribuatur signum $+$, reliquis huius formae $4n+1$ signum $-$, numeris vero compositis ea signa, quae ipsis per regulas multiplicationis ex primis competunt; erit

$\frac{\pi}{4} = 1 + \frac{1}{9} - \frac{1}{25} + \frac{1}{49} - \frac{1}{81} + \frac{1}{121} - \frac{1}{169} + \frac{1}{225} - \frac{1}{289} + \frac{1}{361} - \frac{1}{441} + \frac{1}{529} - \frac{1}{625} + \frac{1}{729} - \frac{1}{841} + \frac{1}{961} - \frac{1}{1089} + \frac{1}{1225} - \frac{1}{1369} + \frac{1}{1521} - \frac{1}{1681} + \frac{1}{1849} - \frac{1}{2025} + \frac{1}{2209} - \frac{1}{2401} + \frac{1}{2601} - \frac{1}{2809} + \frac{1}{3025} - \frac{1}{3249} + \frac{1}{3481} - \frac{1}{3721} + \frac{1}{3969} - \frac{1}{4225} + \frac{1}{4481} - \frac{1}{4741} + \frac{1}{5001} - \frac{1}{5269} + \frac{1}{5549} - \frac{1}{5841} + \frac{1}{6145} - \frac{1}{6461} + \frac{1}{6789} - \frac{1}{7129} + \frac{1}{7481} - \frac{1}{7845} + \frac{1}{8221} - \frac{1}{8609} + \frac{1}{8999} - \frac{1}{9401} + \frac{1}{9815} - \frac{1}{10241} + \frac{1}{10679} - \frac{1}{11129} + \frac{1}{11591} - \frac{1}{12065} + \frac{1}{12551} - \frac{1}{13049} + \frac{1}{13559} - \frac{1}{14081} + \frac{1}{14615} - \frac{1}{15161} + \frac{1}{15719} - \frac{1}{16289} + \frac{1}{16871} - \frac{1}{17465} + \frac{1}{18071} - \frac{1}{18689} + \frac{1}{19319} - \frac{1}{19961} + \frac{1}{20615} - \frac{1}{21281} + \frac{1}{21959} - \frac{1}{22649} + \frac{1}{23351} - \frac{1}{24065} + \frac{1}{24791} - \frac{1}{25529} + \frac{1}{26279} - \frac{1}{27041} + \frac{1}{27815} - \frac{1}{28601} + \frac{1}{29401} - \frac{1}{30215} + \frac{1}{31041} - \frac{1}{31879} + \frac{1}{32729} - \frac{1}{33591} + \frac{1}{34465} - \frac{1}{35351} + \frac{1}{36249} - \frac{1}{37159} + \frac{1}{38081} - \frac{1}{39015} + \frac{1}{39961} - \frac{1}{40919} + \frac{1}{41889} - \frac{1}{42871} + \frac{1}{43865} - \frac{1}{44871} + \frac{1}{45889} - \frac{1}{46919} + \frac{1}{47961} - \frac{1}{49015} + \frac{1}{50081} - \frac{1}{51159} + \frac{1}{52249} - \frac{1}{53351} + \frac{1}{54465} - \frac{1}{55591} + \frac{1}{56729} - \frac{1}{57889} + \frac{1}{59061} - \frac{1}{60245} + \frac{1}{61441} - \frac{1}{62649} + \frac{1}{63869} - \frac{1}{65101} + \frac{1}{66345} - \frac{1}{67601} + \frac{1}{68869} - \frac{1}{70149} + \frac{1}{71441} - \frac{1}{72745} + \frac{1}{74061} - \frac{1}{75389} + \frac{1}{76729} - \frac{1}{78081} + \frac{1}{79445} - \frac{1}{80819} + \frac{1}{82201} - \frac{1}{83591} + \frac{1}{85001} - \frac{1}{86419} + \frac{1}{87845} - \frac{1}{89281} + \frac{1}{90729} - \frac{1}{92189} + \frac{1}{93659} - \frac{1}{95141} + \frac{1}{96635} - \frac{1}{98141} + \frac{1}{99659} - \frac{1}{101189} + \frac{1}{102729} - \frac{1}{104281} + \frac{1}{105845} - \frac{1}{107419} + \frac{1}{109001} - \frac{1}{110591} + \frac{1}{112191} - \frac{1}{113801} + \frac{1}{115421} - \frac{1}{117051} + \frac{1}{118691} - \frac{1}{120341} + \frac{1}{122001} - \frac{1}{123671} + \frac{1}{125351} - \frac{1}{127041} + \frac{1}{128741} - \frac{1}{130451} + \frac{1}{132171} - \frac{1}{133901} + \frac{1}{135641} - \frac{1}{137391} + \frac{1}{139151} - \frac{1}{140921} + \frac{1}{142701} - \frac{1}{144491} + \frac{1}{146291} - \frac{1}{148101} + \frac{1}{149921} - \frac{1}{151751} + \frac{1}{153591} - \frac{1}{155441} + \frac{1}{157301} - \frac{1}{159171} + \frac{1}{161051} - \frac{1}{162941} + \frac{1}{164841} - \frac{1}{166751} + \frac{1}{168671} - \frac{1}{170601} + \frac{1}{172541} - \frac{1}{174491} + \frac{1}{176451} - \frac{1}{178421} + \frac{1}{180401} - \frac{1}{182391} + \frac{1}{184391} - \frac{1}{186401} + \frac{1}{188421} - \frac{1}{190451} + \frac{1}{192491} - \frac{1}{194541} + \frac{1}{196601} - \frac{1}{198671} + \frac{1}{200751} - \frac{1}{202841} + \frac{1}{204941} - \frac{1}{207051} + \frac{1}{209171} - \frac{1}{211301} + \frac{1}{213441} - \frac{1}{215591} + \frac{1}{217751} - \frac{1}{219921} + \frac{1}{222101} - \frac{1}{224291} + \frac{1}{226491} - \frac{1}{228701} + \frac{1}{230921} - \frac{1}{233151} + \frac{1}{235391} - \frac{1}{237641} + \frac{1}{239901} - \frac{1}{242171} + \frac{1}{244451} - \frac{1}{246741} + \frac{1}{249041} - \frac{1}{251351} + \frac{1}{253671} - \frac{1}{256001} + \frac{1}{258341} - \frac{1}{260691} + \frac{1}{263051} - \frac{1}{265421} + \frac{1}{267801} - \frac{1}{270191} + \frac{1}{272591} - \frac{1}{275001} + \frac{1}{277421} - \frac{1}{279851} + \frac{1}{282291} - \frac{1}{284741} + \frac{1}{287201} - \frac{1}{289671} + \frac{1}{292151} - \frac{1}{294641} + \frac{1}{297141} - \frac{1}{299651} + \frac{1}{302171} - \frac{1}{304701} + \frac{1}{307241} - \frac{1}{309791} + \frac{1}{312351} - \frac{1}{314921} + \frac{1}{317501} - \frac{1}{320091} + \frac{1}{322691} - \frac{1}{325301} + \frac{1}{327921} - \frac{1}{330551} + \frac{1}{333191} - \frac{1}{335841} + \frac{1}{338501} - \frac{1}{341171} + \frac{1}{343851} - \frac{1}{346541} + \frac{1}{349241} - \frac{1}{351951} + \frac{1}{354671} - \frac{1}{357401} + \frac{1}{360141} - \frac{1}{362891} + \frac{1}{365651} - \frac{1}{368421} + \frac{1}{371201} - \frac{1}{374001} + \frac{1}{376811} - \frac{1}{379631} + \frac{1}{382461} - \frac{1}{385301} + \frac{1}{388151} - \frac{1}{391011} + \frac{1}{393881} - \frac{1}{396761} + \frac{1}{399651} - 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\frac{1}{2461331} + \frac{1}{2468401} - \frac{1}{2475481} + \frac{1}{2482571} - \frac{1}{2489671} + \frac{1}{2496781} - \frac{1}{2503901} + \frac{1}{2511031} - \frac{1}{2518171} + \frac{1}{2525321} - \frac{1}{2532481} + \frac{1}{2539651} - \frac{1}{2546831} + \frac{1}{2554021} - \frac{1}{2561221} + \frac{1}{2568431} - \frac{1}{2575651} + \frac{1}{2582881} - \frac{1}{2590121} + \frac{1}{2597371} - \frac{1}{2604631} + \frac{1}{2611901} - \frac{1}{2619181} + \frac{1}{2626471} - \frac{1}{2633771} + \frac{1}{2641081} - \frac{1}{2648401} + \frac{1}{2655731} - \frac{1}{2663071} + \frac{1}{2670421} - \frac{1}{2677781} + \frac{1}{2685151} - \frac{1}{2692531} + \frac{1}{2700921} - \frac{1}{2708321} + \frac{1}{2715731} - \frac{1}{2723151} + \frac{1}{2730581} - \frac{1}{2738021} + \frac{1}{2745471} - \frac{1}{2752931} + \frac{1}{2760401} - \frac{1}{2767881} + \frac{1}{2775371} - \frac{1}{2782871} + \frac{1}{2790381} - \frac{1}{2797901} + \frac{1}{2805431} - \frac{1}{2812971} + \frac{1}{2820521} - \frac{1}{2828081} + \frac{1}{2835651} - \frac{1}{2843231} + \frac{1}{2850821} - \frac{1}{2858421} + \frac{1}{2866031} - \frac{1}{2873651} + \frac{1}{2881281} - \frac{1}{2888921} + \frac{1}{2896571} - \frac{1}{2904231} + \frac{1}{2911901} - \frac{1}{2919581} + \frac{1}{2927271} - \frac{1}{2934971} + \frac{1}{2942681} - \frac{1}{2950401} + \frac{1}{2958131} - \frac{1}{2965871} + \frac{1}{2973621} - \frac{1}{2981381} + \frac{1}{2989151} - \frac{1}{2996931} + \frac{1}{3004721} - \frac{1}{3012521} + \frac{1}{3020331} - \frac{1}{3028151} + \frac{1}{3035981} - \frac{1}{3043821} + \frac{1}{3051671} - \frac{1}{3059531} + \frac{1}{3067401} - \frac{1}{3075281} + \frac{1}{3083171} - \frac{1}{3091071} + \frac{1}{3098981} - \frac{1}{3106901} + \frac{1}{3114831} - \frac{1}{3122771} + \frac{1}{3130721} - \frac{1}{3138681} + \frac{1}{3146651} - \frac{1}{3154631} + \frac{1}{3162621} - \frac{1}{3170621} + \frac{1}{3178631} - \frac{1}{3186651} + \frac{1}{3194681} - \frac{1}{3202721} + \frac{1}{3210771} - \frac{1}{3218831} + \frac{1}{3226901} - \frac{1}{3234981} + \frac{1}{3243071} - \frac{1}{3251171} + \frac{1}{3259281} - \frac{1}{3267401} + \frac{1}{3275531} - \frac{1}{3283671} + \frac{1}{3291821} - \frac{1}{3300081} + \frac{1}{3308351} - \frac{1}{3316631} + \frac{1}{3324921} - \frac{1}{3333231} + \frac{1}{3341551} - \frac{1}{3349881} + \frac{1}{3358221} - \frac{1}{3366571} + \frac{1}{3374931} - \frac{1}{3383301} + \frac{1}{3391681} - \frac{1}{3400071} + \frac{1}{3408471} - \frac{1}{3416881} + \frac{1}{3425301} - \frac{1}{3433731} + \frac{1}{3442171} - \frac{1}{3450621} + \frac{1}{3459081} - \frac{1}{3467551} + \frac{1}{3476031} - \frac{1}{3484521} + \frac{1}{3493021} - \frac{1}{3501531} + \frac{1}{3510051} - \frac{1}{3518581} + \frac{1}{3527121} - \frac{1}{3535671} + \frac{1}{3544231} - \frac{1}{3552801} + \frac{1}{3561381} - \frac{1}{3570071} + \frac{1}{3578771} - \frac{1}{3587481} + \frac{1}{3596201} - \frac{1}{3604931} + \frac{1}{3613671} - \frac{1}{3622421} + \frac{1}{3631181} - \frac{1}{3640051} + \frac{1}{3648931} - \frac{1}{3657821} + \frac{1}{3666721} - \frac{1}{3675631} + \frac{1}{3684551} - \frac{1}{3693481} + \frac{1}{3702421} - \frac{1}{3711371} + \frac{1}{3720331} - \frac{1}{3729301} + \frac{1}{3738281} - \frac{1}{3747271} + \frac{1}{3756271} - \frac{1}{3765281} + \frac{1}{3774301} - \frac{1}{3783331} + \frac{1}{3792371} - \frac{1}{3801421} + \frac{1}{3810481} - \frac{1}{3819551} + \frac{1}{3828631} - \frac{1}{3837721} + \frac{1}{3846821} - \frac{1}{3855931} + \frac{1}{3865051} - \frac{1}{3874181} + \frac{1}{3883321} - \frac{1}{3892471} + \frac{1}{3901631} - \frac{1}{3910801} + \frac{1}{3919981} - \frac{1}{3929171} + \frac{1}{3938371} - \frac{1}{3947581} + \frac{1}{3956801} - \frac{1}{3966031} + \frac{1}{3975271} - \frac{1}{3984521} + \frac{1}{3993781} - \frac{1}{4003051} + \frac{1}{4012331} - \frac{1}{4021621} + \frac{1}{4030921} - \frac{1}{4040231} + \frac{1}{4049551} - \frac{1}{4058881} + \frac{1}{4068221} - \frac{1}{4077571} + \frac{1}{4086931} - \frac{1}{4096301} + \frac{1}{4105681} - \frac{1}{4115071} + \frac{1}{4124471} - \frac{1}{4133881} + \frac{1}{4143301} - \frac{1}{4152731} + \frac{1}{4162171} - \frac{1}{4171621} + \frac{1}{4181081} - \frac{1}{4190551} + \frac{1}{4200031} - \frac{1$

Corollarium.

Si ab huius theorematiss serie subtrahatur ista:

$$\frac{\pi}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \text{etc.}$$

prodibit

$$\frac{\pi}{8} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \frac{1}{11} + \frac{1}{12} - \text{etc.}$$

cuius denominatoris sunt vel ipsi numeri primi vel facta externis vel quinis vel etc. ii vero qui sunt formae $4n-1$ signum habent $+$ reliqui formae $4n-1$ signum $-$.

Theorema 18.

Si omnibus numeris primis tribuatur signum $-$, cuique vero numero composito id signum quod ipsi secundum multiplicationis regulas competit, atque ex omnibus numeris sequens formetur series

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \frac{1}{10} + \frac{1}{12} - \frac{1}{14} - \frac{1}{15} + \frac{1}{16} - \frac{1}{18} - \frac{1}{20} + \frac{1}{21} - \frac{1}{24} - \frac{1}{25} + \frac{1}{27} - \frac{1}{28} - \frac{1}{30} + \frac{1}{32} - \frac{1}{35} - \frac{1}{36} + \frac{1}{39} - \frac{1}{40} - \frac{1}{42} + \frac{1}{45} - \frac{1}{48} - \frac{1}{50} + \frac{1}{54} - \frac{1}{56} - \frac{1}{60} + \frac{1}{63} - \frac{1}{64} - \frac{1}{66} + \frac{1}{68} - \frac{1}{70} - \frac{1}{72} + \frac{1}{75} - \frac{1}{76} - \frac{1}{78} + \frac{1}{80} - \frac{1}{84} - \frac{1}{85} + \frac{1}{87} - \frac{1}{88} - \frac{1}{90} + \frac{1}{92} - \frac{1}{94} - \frac{1}{96} + \frac{1}{98} - \frac{1}{100} - \text{etc.}$$

erit eius summa in infinitum continuatae $= 0$.

Demonstratio.

Sit enim $x =$ summae istius seriei seu

$$x = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \frac{1}{10} + \frac{1}{12} - \frac{1}{14} - \frac{1}{15} + \frac{1}{16} - \frac{1}{18} - \frac{1}{20} + \frac{1}{21} - \frac{1}{24} - \frac{1}{25} + \frac{1}{27} - \frac{1}{28} - \frac{1}{30} + \frac{1}{32} - \frac{1}{35} - \frac{1}{36} + \frac{1}{39} - \frac{1}{40} - \frac{1}{42} + \frac{1}{45} - \frac{1}{48} - \frac{1}{50} + \frac{1}{54} - \frac{1}{56} - \frac{1}{60} + \frac{1}{63} - \frac{1}{64} - \frac{1}{66} + \frac{1}{68} - \frac{1}{70} - \frac{1}{72} + \frac{1}{75} - \frac{1}{76} - \frac{1}{78} + \frac{1}{80} - \frac{1}{84} - \frac{1}{85} + \frac{1}{87} - \frac{1}{88} - \frac{1}{90} + \frac{1}{92} - \frac{1}{94} - \frac{1}{96} + \frac{1}{98} - \frac{1}{100} - \text{etc.}$$

erit per operationes in posteribribus theorematiss adhibitass

$$\frac{1}{2}x = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14} - \frac{1}{15} + \frac{1}{16} - \frac{1}{18} - \frac{1}{20} + \frac{1}{21} - \frac{1}{24} - \frac{1}{25} + \frac{1}{27} - \frac{1}{28} - \frac{1}{30} + \frac{1}{32} - \frac{1}{35} - \frac{1}{36} + \frac{1}{39} - \frac{1}{40} - \frac{1}{42} + \frac{1}{45} - \frac{1}{48} - \frac{1}{50} + \frac{1}{54} - \frac{1}{56} - \frac{1}{60} + \frac{1}{63} - \frac{1}{64} - \frac{1}{66} + \frac{1}{68} - \frac{1}{70} - \frac{1}{72} + \frac{1}{75} - \frac{1}{76} - \frac{1}{78} + \frac{1}{80} - \frac{1}{84} - \frac{1}{85} + \frac{1}{87} - \frac{1}{88} - \frac{1}{90} + \frac{1}{92} - \frac{1}{94} - \frac{1}{96} + \frac{1}{98} - \frac{1}{100} - \text{etc.}$$

atque simili modo

$$\frac{1}{2} \cdot \frac{1}{3} x = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{15} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{21} - \frac{1}{24} - \frac{1}{25} - \frac{1}{27} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} - \frac{1}{35} - \frac{1}{36} - \frac{1}{39} - \frac{1}{40} - \frac{1}{42} - \frac{1}{45} - \frac{1}{48} - \frac{1}{50} - \frac{1}{54} - \frac{1}{56} - \frac{1}{60} - \frac{1}{63} - \frac{1}{64} - \frac{1}{66} - \frac{1}{68} - \frac{1}{70} - \frac{1}{72} - \frac{1}{75} - \frac{1}{76} - \frac{1}{78} - \frac{1}{80} - \frac{1}{84} - \frac{1}{85} - \frac{1}{87} - \frac{1}{88} - \frac{1}{90} - \frac{1}{92} - \frac{1}{94} - \frac{1}{96} - \frac{1}{98} - \frac{1}{100} - \text{etc.}$$

Denique hac operatione infinities repetita erit

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} x = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{15} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{21} - \frac{1}{24} - \frac{1}{25} - \frac{1}{27} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} - \frac{1}{35} - \frac{1}{36} - \frac{1}{39} - \frac{1}{40} - \frac{1}{42} - \frac{1}{45} - \frac{1}{48} - \frac{1}{50} - \frac{1}{54} - \frac{1}{56} - \frac{1}{60} - \frac{1}{63} - \frac{1}{64} - \frac{1}{66} - \frac{1}{68} - \frac{1}{70} - \frac{1}{72} - \frac{1}{75} - \frac{1}{76} - \frac{1}{78} - \frac{1}{80} - \frac{1}{84} - \frac{1}{85} - \frac{1}{87} - \frac{1}{88} - \frac{1}{90} - \frac{1}{92} - \frac{1}{94} - \frac{1}{96} - \frac{1}{98} - \frac{1}{100} - \text{etc.}$$

At cum sit per Theor. septimum

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} x = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{15} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{21} - \frac{1}{24} - \frac{1}{25} - \frac{1}{27} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} - \frac{1}{35} - \frac{1}{36} - \frac{1}{39} - \frac{1}{40} - \frac{1}{42} - \frac{1}{45} - \frac{1}{48} - \frac{1}{50} - \frac{1}{54} - \frac{1}{56} - \frac{1}{60} - \frac{1}{63} - \frac{1}{64} - \frac{1}{66} - \frac{1}{68} - \frac{1}{70} - \frac{1}{72} - \frac{1}{75} - \frac{1}{76} - \frac{1}{78} - \frac{1}{80} - \frac{1}{84} - \frac{1}{85} - \frac{1}{87} - \frac{1}{88} - \frac{1}{90} - \frac{1}{92} - \frac{1}{94} - \frac{1}{96} - \frac{1}{98} - \frac{1}{100} - \text{etc.}$$

facile intelligitur et nostrum ipsius x coefficientem esse infinite.

infinite magnum. Quare quo factum aequale esse pos-
sit x erit $x=0$, et hanc ob rem habebitur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} \text{ etc.}$$

cuius deuominatores, qui sunt vel ipsi primi vel facta ex
ternis, quinis, etc. habent signum - reliqui signum +. Q.E.D.

Corollarium I.

Modus ergo apparet, quo in progressionem harmonicam
signa in singulis terminis sint distribuenda, ut summa totius
seriei fiat $= 0$.

Corollarium 2.

Cum inuenerimus $x=0$, erit quoque $\frac{1}{2}x=0$, et hanc
ob rem habebimus quoque

$$0 = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} \text{ etc.}$$

in qua numeri tantum impares occurrunt; descriptamque ra-
tione signorum tenent legem.

Theorema 19.

Summa seriei reciprocae numerorum primorum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

*est infinite magna; infinites tamen minor, quam summa seriei
harmonicæ $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.}$ Atque illius sum-
mæ est huius summæ quasi logarithmus.*

Demonstratio.

Ponatur $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{etc.} = A$ atque
 $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = B$ et $\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \text{etc.} = C$.
atque ita porro omnes potestates peculiaribus litteris desi-
gnando; erit posito e pro numero cuius logarithmus hyper-
bolicus est 1

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$$A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \text{etc.}$$

$$e = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{etc.}$$

Nam est

$$A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \text{etc.} = l^{\frac{2}{1}} + l^{\frac{3}{2}} + l^{\frac{4}{3}} + l^{\frac{5}{4}} + \text{etc.}$$

ideoque

$$A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D$$

$$e = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot \text{etc.}}{1 \cdot 2 \cdot 4 \cdot 6 \cdot \text{etc.}} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

per Theor. 7. At non solum B, C, D, etc. habebunt valores finitos, sed etiam $\frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \text{etc.}$ valorem habet finitum. Quare quo

$$A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \text{etc.}$$

$$e \text{ fit} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} = \infty.$$

oportet ut A sit infinite magnum, cumque ideo eius respectu sequentes termini $\frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \text{etc.}$ evanescent, erit

$$A = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{etc.}$$

$$e = e = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{etc.}$$

Atque consequenter erit

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \text{etc.}$$

$$= l(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{etc.})$$

illius ergo seriei summa erit infinities minor quam huius, atque cum huius summa sit $= l\infty$ erit

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{etc.} = l.l\infty.$$

Q. E. D.

DE