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Variae observationes circa series infinitas

Leonhard Euler

VARIAE OBSERVATIONES
CIRCA
SERIES INFINITAS.

AVCTORE
Leonh. Euler.

Obseruationes, quas hic proferre constitui, plerumque circa eiusmodi series versantur, quae prorsus sunt diuersae ab iis, quae etiamnum tractari sunt solitae. Cum enim adhuc nullae aliae series sint consideratae, nisi quarum vel terminus generalis esset datus, vel lex saltem, qua ex datis aliquot terminis sequentes inuenire liceret; ita hic eiusmodi potissimum series sum contemplaturus, quae neque terminum generalem proprie sic dictum, neque legem continuationis agnoscant; sed quarum natura per alias conditiones determinetur. De huiusmodi ergo seriebus eo magis erit mirandum, si summari poterunt, cum ad methodos summandi adhuc cognitas necessario vel terminus generalis, vel lex progressionis requiratur; quibus deficientibus vix alia via patere videatur, qua ad summas cognoscendas pertingere queamus. Ad has autem obseruationes me peculiaris series a Cel. *Goldbach* mecum communicata deduxit, cuius summationem maxime admirandam *Viri Celeb.* permissu hic primo loco sum demonstraturus.

Theo-

Theorema I.

Huius ſeriei in infinitum continuatae

$$\tfrac{1}{3}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{15}+\tfrac{1}{24}+\tfrac{1}{26}+\tfrac{1}{31}+\tfrac{1}{35}+\text{etc.}$$

cuius denominatores vnitate aucti dant omnes numeros, qui ſunt poteſtates vel ſecundi vel altioris cuiusuis ordinis numerorum integrorum, cuiusque adeo terminus quisque exprimitur hac formula $\frac{1}{m^n-1}$, *denotantibus* m *et* n *numeros integros vnitate maiores; huius ſeriei autem ſumma eſt* $=1$.

Demonſtratio.

Hoc eſt Theorema a *Celeb. Goldbach* primum mecum communicatum, quod me etiam ad ſequentes propoſitiones manuduxit. Ex inſpectione autem huius ſeriei facile intelligitur, quam irregulariter ea progrediatur, et propterea quisque in his rebus verſatus maxime modum admirabitur, quo *Vir Celeb.* ſummam huius ſingularis ſeriei inuenit; ſequenti vero modo mihi hoc Theorema demonſtrauit. Sit

$$x=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{9}+\text{etc.}$$

deinde cum ſit

$$1=\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\text{etc.}$$

erit hanc ſeriem ab illa auferendo

$$x-1=1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{10}+\text{etc.}$$

ex denominatoribus ergo excluſi ſunt omnes poteſtates binarii cum binario ipſo; reliqui vero numeri omnes occurrunt.

Ab hac ſerie porro ſubtrahit hanc

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \text{etc.}$$

et reſtabit

$$x - 1 - \frac{1}{2} = 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \text{etc.}$$

denuoque ſubtrahit

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \text{etc.}$$

reſtabitque

$$x - 1 - \frac{1}{2} - \frac{1}{4} = 1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \text{etc.}$$

Atque ſimili modo omnes reliquos terminos ſucceſſiue tollendo reperietur tandem

$$x - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{9} - \text{etc.} = 1$$

ſeu

$$x - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \text{etc.}$$

cuius progreſſionis denominatores vnitate aucti dant omnes numeros, qui non ſunt poteſtates. Quare ſi iſta ſeries ab initio aſſumta

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{etc.}$$

ſubtrahatur, relinquetur

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \text{etc.}$$

cuius ſeriei igitur, in qua denominatores vnitate aucti dant omnes omnino poteſtates numerorum integrorum, ſumma eſt $= 1$. Q. E. I.

Theorema 2.

Huius ſeriei in infinitum continuatae

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35} + \frac{1}{63} + \text{etc.}$$

cuius

cuius denominatores vnitate aucti dant omnes poteſtates pares, ſumma eſt $= l2$; atque huius ſeriei

$$\tfrac{1}{8} + \tfrac{1}{24} + \tfrac{1}{26} + \tfrac{1}{48} + \tfrac{1}{80} + \text{etc.}$$

in infinitum continuatae, cuius denominatores vnitate aucti dant omnes poteſtates impares, ſumma aequalis eſt $1 - l2$. Quarum ſerierum prioris terminus quisque eſt $\frac{1}{(2m-2)^n - 1}$, poſterioris vero terminus quilibet hac continetur formula $\frac{1}{(2m-1)^n - 1}$; retinentibus m et n praecedentes valores.

Demonſtratio.

Conſideretur ſequens ſeries, cuius ſumma ponatur x;

$$x = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \tfrac{1}{10} + \tfrac{1}{12} + \tfrac{1}{14} + \text{etc.}$$

Iam cum ſit

$$1 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \text{etc.}$$

ſublata hac ſerie ab illa prodibit ſequens

$$x - 1 = \tfrac{1}{6} + \tfrac{1}{10} + \tfrac{1}{12} + \tfrac{1}{14} + \tfrac{1}{18} + \text{etc.}$$

a qua ſubtrahatur

$$\tfrac{1}{5} = \tfrac{1}{6} + \tfrac{1}{36} + \tfrac{1}{216} + \text{etc.}$$

erit

$$x - 1 - \tfrac{1}{5} = \tfrac{1}{10} + \tfrac{1}{12} + \tfrac{1}{14} + \tfrac{1}{18} + \text{etc.}$$

ſimili modo ob

$$\tfrac{1}{9} = \tfrac{1}{10} + \tfrac{1}{100} + \tfrac{1}{1000} + \text{etc.}$$

erit

$$x - 1 - \tfrac{1}{5} - \tfrac{1}{9} = \tfrac{1}{12} + \tfrac{1}{14} + \tfrac{1}{18} + \text{etc.}$$

 Omni-

Omnibus ergo terminis hoc modo ſublatis prodibit

$$x = 1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \text{etc.}$$

cuius denominatores conſtituunt ſeriem naturalem numerorum imparium exceptis iis, qui vnitate aucti ſunt poteſtates, vti ex formatione huius ſeriei intelligitur. Cum vero ſit

$$l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \text{etc.}$$

atque

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \text{etc.}$$

erit $x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \text{etc.} - l2.$

Illo ergo pro x inuento valore ſublato ab iſto, in quo omnes omnino numeri impares occurrunt, reſtabit

$$0 = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35}\ \text{etc.} - l2,$$

ſeu iſta

$$l2 = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35} + \text{etc.}$$

cuius ſeriei denominatores ſunt ii numeri impares, qui vnitate aucti dant omnes poteſtates pares. Huius ergo ſeriei ſumma eſt $l2$, prout in propoſitione eſt aſſertum. Q. E. Vnum.

Cum vero praecedens Theorema ſit

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \text{etc.}$$

vbi denominatores vnitate aucti dant omnes numeros, qui ſunt poteſtates tam pares quam impares, habebitur illa ſerie ab hac demta

$$1 - l2 = \frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \text{etc.}$$

cuius denominatores adeo ſunt ii numeri pares, qui vnitate aucti dant omnes poteſtates impares. Q. E. Alterum.

Theo-

Theorema 3.

Posito π *pro peripheria circuli, cuius diameter est* 1, *erit*

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \frac{1}{120} - \frac{1}{124} - \frac{1}{168} - \frac{1}{224} + \frac{1}{244} - \frac{1}{288} - \text{etc.}$$

cuius seriei denominatores sunt numeri pariter pares, vnitate vel maiores vel minores quam potestates numerorum imparium. Illae autem fractiones, quarum denominatores vnitate excedunt potestates, signum habent + *reliquae signum* − 1.

Demonstratio.

Cum sit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

cuius seriei eae fractiones, quarum denominatores vnitate deficiunt a numeris pariter paribus, signum habent − 1, reliquae signum +. Ad illam seriem vero addatur haec geometrica

$$\frac{1}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \text{etc.}$$

erit

$$\frac{\pi}{4} + \frac{1}{4} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

a qua subtrahatur

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \text{etc.}$$

erit

$$\frac{\pi}{4} + \frac{1}{4} - \frac{1}{4} = 1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

in qua serie nec 3, et 5 nec eorundem potestates amplius insunt, simili modo 7 eiusque potestates tollentur addendo hanc seriem

$$\frac{1}{8} = \frac{1}{7} - \frac{1}{49} + \text{etc.}$$

eritque

$$\tfrac{\pi}{4}+\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{8}=1-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\tfrac{1}{17}-\text{etc.}$$

Pari modo tollendo reliquos terminos, qui non ſunt poteſtates, ſimul enim poteſtates tolluntur, prodibit tandem

$$\tfrac{\pi}{4}+\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{12}-\tfrac{1}{12}+\tfrac{1}{16}-\tfrac{1}{16}+\tfrac{1}{20}-\tfrac{1}{20}+\tfrac{1}{24}-\tfrac{1}{28}-\text{etc.}=1$$

ſeu

$$\tfrac{\pi}{4}=1-\tfrac{1}{8}-\tfrac{1}{24}+\tfrac{1}{28}-\tfrac{1}{48}-\tfrac{1}{80}-\tfrac{1}{120}\ \text{etc.}$$

ob binos terminos ſeſe plerumque deſtruentes, ita vt ſoli ſuperſint, qui ſolitarii erant: ſolitariae autem erant eae fractiones, quarum denominatores, qui ſemper prodierant pariter pares, vnitate vel aucti vel minuti poteſtates numerorum imparium efficiebant. Signa vero horum terminorum legem ſeruant praeſcriptam. Q. E. D.

Theorema 4.

Denotante π vt ante peripheriam circuli cuius diameter eſt $=1$, erit

$$\tfrac{\pi}{4}-\tfrac{3}{4}=\tfrac{1}{28}-\tfrac{1}{124}+\tfrac{1}{244}+\tfrac{1}{344}\ \text{etc.}$$

cuius ſeriei denominatores omnes ſunt numeri pariter pares vnitate vel maiores vel minores, quam poteſtates numerorum imparium non quadratae; atque illae fractiones quarum denominatores vnitate ſuperant tales poteſtates, ſignum habent $+$; reliquae quarum denominatores deficiunt ab huiusmodi poteſtatibus non quadratis ſignum habent $-$.

Demonſtratio.

Per Theorema praecedens tertium inuenimus

$$\tfrac{\pi}{4}=1-\tfrac{1}{8}-\tfrac{1}{24}+\tfrac{1}{28}+\tfrac{1}{48}-\tfrac{1}{80}-\tfrac{1}{120}-\tfrac{1}{124}\ \text{etc.}$$

in qua ſerie primo occurrunt denominatores, qui ab omni-

bus

bus quadratis imparibus vnitate deficiunt, eaeque fractiones omnes habent signum idem —. Cum vero sit

$$\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}+\frac{1}{168}+\text{etc.}=\frac{1}{4}$$

habebitur loco harum fractionum omnium substituendo $\frac{1}{4}$ sequens forma

$$\frac{\pi}{4}=1-\frac{1}{4}+\frac{1}{28}-\frac{1}{124}+\frac{1}{244}+\frac{1}{344}\text{ etc.}$$

seu

$$\frac{\pi}{4}-\frac{3}{4}=\frac{1}{28}-\frac{1}{124}+\frac{1}{244}+\frac{1}{344}\text{ etc.}$$

cuius seriei denominatores sunt numeri pariter pares vel vnitate maiores vel minores quam potestates numerorum imparium non quadratae, ob quadratas iam exclusas, atque prout vnitate sunt vel maiores vel minores, fractiones etiam habent signum + vel —. Q. E. D.

Corollarium 1.

Ad seriem ergo continuandam omnium numerorum imparium, qui non sunt potestates, sumendae sunt potestates exponentium imparium, eaeque vnitate vel augendae vel minuendae, quo prodeant numeri pariter pares, qui erunt denominatores seriei inuentae: seruata signorum regula.

Corollarium 2.

Cum autem omnis numerus impar vel sit $4m-1$ vel $4m+1$, potestates autem exponentium imparium a $4m-1$ ortorum, si vnitate augeantur, illae autem, quae a $4m+1$ oriuntur si vnitate minuantur, dent numeros pariter pares; aequabitur $\frac{\pi}{4}-\frac{3}{4}$ seriei terminorum qui omnes in hac forma $\frac{1}{(4m-1)^{2n+1}+1}$ continentur, dem-

ta

ta ſerie terminorum in hac forma $\frac{1}{(4m+1)^{2n+1}-1}$ contentorum; vbi loco m et n omnes numeri integri affirmatiui accipi debent praeter eos, qui vel $4m-1$ vel $4m+1$ faciunt poteſtates.

Corollarium 3.

Aequabitur ergo $\frac{\pi}{4}-\frac{3}{4}$ aggregato ſequentium ſerierum infinitarum

$$\frac{\pi}{4}-\frac{3}{4}=\left\{\begin{array}{l}
\frac{1}{3^3+1}+\frac{1}{3^5+1}+\frac{1}{3^7+1}+\frac{1}{3^9+1}+\text{etc.}\\
-\frac{1}{5^3-1}-\frac{1}{5^5-1}-\frac{1}{5^7-1}-\frac{1}{5^9-1}+\text{etc.}\\
+\frac{1}{7^3+1}+\frac{1}{7^5+1}+\frac{1}{7^7+1}+\frac{1}{7^9+1}+\text{etc.}\\
+\frac{1}{11^3+1}+\frac{1}{11^5+1}+\frac{1}{11^7+1}+\frac{1}{11^9+1}+\text{etc.}\\
-\frac{1}{13^3-1}-\frac{1}{13^5-1}-\frac{1}{13^7-1}-\frac{1}{13^9-1}+\text{etc.}\\
+\frac{1}{15^3+1}+\frac{1}{15^5+1}+\frac{1}{15^7+1}+\frac{1}{15^9+1}+\text{etc.}\\
\qquad\qquad\text{etc.}\quad\text{etc.}
\end{array}\right.$$

Corollarium 4.

Hac ergo ſerie eousque continuata, donec denominatores fiant maiores quam 100000 habebitur $\frac{\pi}{4}=\frac{3}{4}+\frac{1}{28}-\frac{1}{124}+\frac{1}{244}+\frac{1}{344}+\frac{1}{1332}+\frac{1}{2188}-\frac{1}{2196}-\frac{1}{3124}+\frac{1}{3376}-\frac{1}{4912}+\frac{1}{6860}-\frac{1}{9260}+\frac{1}{12168}+\frac{1}{16808}+\frac{1}{19684}-\frac{1}{24388}+\frac{1}{29792}-\frac{1}{35936}+\frac{1}{42876}-\frac{1}{50652}+\frac{1}{59320}-\frac{1}{68920}-\frac{1}{78124}+\frac{1}{79508}-\frac{1}{91124}$.

Corollarium 5.

Cum omnes denominatores per 4 diuidi poſſint, erit $\pi=3+\frac{1}{7}-\frac{1}{31}+\frac{1}{61}+\frac{1}{86}+\frac{1}{333}+\frac{1}{547}-\frac{1}{549}-\frac{1}{781}+\frac{1}{844}$ etc.

Quae

Quae ſeries ideo notari meretur, quod eius duo primi termini iam dent Archimedis proportionem peripheriae circuli ad diametrum.

Theorema 5.

Retinente π priorem ſignificationem, erit

$$\frac{\pi}{4} - l2 = \frac{1}{26} + \frac{1}{28} + \frac{1}{242} + \frac{1}{244} + \frac{1}{342} + \frac{1}{344} + \text{etc.}$$

cuius ſeriei haec eſt lex, vt numeri medii inter binos denominatores binario differentes, ſcilicet 27, 243, 343, *etc. ſint poteſtates exponentium imparium ortae a numeris imparibus, quae vnitate auctae ſint per* 4 *diuiſibiles ſeu numeri pariter pares.*

Demonſtratio.

Cum per Theorema tertium ſit

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} \text{ etc.}$$

fractionum ſigno − affectarum denominatores ſunt numeri pariter pares vnitate deficientes a poteſtatibus numerorum imparium; fractionum vero ſigno + affectarum denominatores ſunt quoque pariter pares vnitate ſuperantes poteſtates numerorum imparium; atque praeterea ſit per Theorema ſecundum

$$1 - l2 = \frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \frac{1}{80} + \text{etc.}$$

cuius ſeriei denominatores deficiunt vnitate ab omnibus potentiis numerorum imparium; haec ſeries complectetur omnes terminos illius ſigno − affectos, et praeterea fractiones denominatores habentes impariter pares vnitate deficientes a poteſtatibus numerorum imparium. Quare

 ſi haec

ſi haec ſeries ad illam addatur prodibit

$$\frac{\pi}{4} - l2 = \frac{1}{26} + \frac{1}{28} + \frac{1}{242} + \frac{1}{244} + \frac{1}{342} + \frac{1}{344} + \text{etc.}$$

cuius binae fractiones erunt ita comparatae, vt prioris denominator ſit numerus impariter par, poſterioris binario maior pariter par, mediusque numerus inter binos huiusmodi denominatores ſit poteſtas numeri imparis; quae ergo poteſtas vnitate aucta dare debet numerum pariter parem. Q. E. D.

Corollarium 1.

Quia hae poteſtates numerorum imparium ita ſunt comparatae, vt vnitate auctae fiant per 4 diuiſibiles, erunt eae poteſtates imparium dimenſionum, quae oriuntur a numeris huius formae $4m-1$, qui ipſi non ſunt poteſtates.

Corollarium 2.

Si ergo omnes ſumantur numeri huius formae $4m-1$, quae non ſunt poteſtates, eorumque capiantur omnes poteſtates exponentium imparium; hae poteſtates vnitate tam auctae quam minutae dabunt omnes fractionum ſeriei inuentae denominatores.

Corollarium 3.

Si binae fractiones in vnam coaleſcant erit

$$\frac{\pi}{4} = l2 + \frac{2 \cdot 27}{26 \cdot 28} + \frac{2 \cdot 243}{242 \cdot 244} + \frac{2 \cdot 343}{342 \cdot 344} + \text{etc.}$$

Quae ſeries formabitur ſumendis omnibus fractionibus, quae oriuntur ex hac forma $\frac{2(4m-1)^{2n+1}}{(4m-1)^{4n+2}-1}$ ſubſtituendo

loco

loco m et n omnes numeros integros ſucceſſiue praeter eos ipſius m valores, qui reddant $4m-1$ poteſtatem.

Theorema 6.

Seriei huius

$$\frac{1}{15}+\frac{1}{63}+\frac{1}{80}+\frac{1}{255}+\frac{1}{624}+\textit{etc.}$$

cuius denominatores vnitate aucti dant omnia quadrata, quae ſimul ſunt altiores poteſtates; huius inquam ſeriei in infinitum continuatae ſumma eſt $\frac{7}{4}-\frac{\pi^2}{6}$: *denotante* π *peripheriam circuli, cuius diameter* $=1$.

Demonſtratio.

Hoc quoque Theorema a *Cel. Goldbachio*, verum ſine demonſtratione accepi, atque iisdem quibus ante veſtigiis inſiſtens hanc inueni demonſtrationem. Cum ante aliquot annos incidiſſem in huius ſeriei

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{24}+\text{etc.}$$

ſummam $=\frac{\pi^2}{6}$, hanc ipſam ſeriem ita ſum contemplatus

$$\frac{\pi^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\text{etc.}$$

Iam cum ſit

$$\frac{1}{3}=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\text{etc.}$$

atque

$$\frac{1}{8}=\frac{1}{9}+\frac{1}{81}+\frac{1}{729}+\text{etc.}$$

ſimilique modo

$$\frac{1}{24}=\frac{1}{25}+\frac{1}{625}+\text{etc. et }\frac{1}{35}=\frac{1}{36}+\text{etc.}$$

ſi loco harum ſerierum geometricarum ſubſtituantur ſummae prodibit

$$\frac{\pi^2}{6}=1+\frac{1}{3}+\frac{1}{8}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\frac{1}{99}\text{ etc.}$$

cuius feriei denominatores vnitate aucti dant omnes numeros quadratos, praeter eos, qui fimul fint alius fpeciei poteftates. Cum autem fumendis omnino quadratis vnitate minutis fit

$$\frac{3}{4} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \frac{1}{63} + \frac{1}{80} + \text{etc.}$$

proueniet ab hac fuperiorem feriem fubtrahendo

$$\frac{7}{4} - \frac{\pi^2}{6} = \frac{1}{16} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \text{etc.}$$

qui denominatores vnitate aucti dant omnes numeros quadratos, qui fimul fint alius generis poteftates. Q.E.D.

Conftituunt haec fex theoremata alterum obferuationum iftarum partem, quibus fcilicet feries additione fubtractioneue terminorum ortae funt confideratae. Sequentia vero theoremata circa feries, quorum termini in fe inuicem multiplicantur, verfabuntur; neque minus erunt admirabilia, quam praecedentia, cum in iis pariter lex progreffionis tantopere fit irregularis. Discrimen autem in hoc potiffimum erit pofitum, quod in theorematis praecedentibus progreffio terminorum fecuta fit feriem poteftatum, quae per fe eft maxime irregularis; in his autem termini progrediantur fecundum numeros primos, quorum progreffio non minus eft abftrufa.

Theorema 7.

Factum continuum in infinitum ex his fractionibus $\frac{2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19}{1\cdot2\cdot4\cdot6\cdot10\cdot12\cdot16\cdot18}$ *etc. vbi numeratores funt omnes numeri primi, denominatores vero vnitate deficiunt a numeratoribus. Hoc factum inquam aequale eft fummae huius feriei*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

eftque adeo infinitum.

Demon-

Demonstratio.

Nam sit

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

erit

$$\frac{1}{2}x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \text{etc.}$$

qua serie ab illa demta restat

$$\frac{1}{2}x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{etc.}$$

in qua nulli amplius denominatores pares insunt. Ab hac denuo auferatur ista series

$$\frac{1}{2}\cdot\frac{1}{3}x = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \text{etc.}$$

restabit

$$\frac{1}{2}\cdot\frac{2}{3}x = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

in cuius denominatoribus nec per 2 nec per 3 diuisibiles reperiuntur. Quo autem etiam numeri per 5 diuisibiles egrediantur, subtrahatur ista series

$$\frac{1.2}{2.3}\cdot\frac{1}{5}x = \frac{1}{5} + \frac{1}{25} + \frac{1}{35} + \text{etc.}$$

restabitque

$$\frac{1.2.4}{2.3.5}x = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

Atque simili modo tollendis omnibus terminis tum per 7 tum per 11 etc. omnesque numeros primos diuisibilibus tandem reperietur

$$\frac{1.2.4.6.10.12.16.18.22.\ \text{etc.}}{2.3.5.7.11.13.17.19.23.\ \text{etc.}}\, x = 1.$$

Quare cum sit

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{etc.}$$

erit

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{etc.} = \frac{2.3.5.7.11.13.17.19.23\ \text{etc.}}{1.2.4.6.10.12.16.18.22\ \text{etc.}}$$

 cu-

cuius expressionis numeratores constituunt progressionem numerorum primorum, denominatores vero vnitate ab iis deficiunt. Q. E. D.

Corollarium 1.

Expressiones ergo $\frac{2.3.5.7.11.13.\text{ etc.}}{1.2.4.6.10.12.\text{ etc.}}$ valor est infinitus, et posito absolute infinito $=\infty$, erit istius expressionis valor $=l\infty$, quod infinitum inter omnes infiniti potestates est minimum.

Corollarium 2.

Cum vero haec expressio $\frac{4.9.16.25.36.49\text{ etc.}}{3.8.15.24.35.48\text{ etc.}}$ finitum habeat valorem scilicet 2; sequitur infinities plures esse numeros primos, quam quadratos, in serie omnium omnino numerorum.

Corollarium 3.

Verum etiam hinc intelligitur infinities pauciores extare numeros primos, quam numeros integros; haec enim expressio $\frac{2.3.4.5.6.7\text{ etc.}}{1.2.3.4.5.6\text{ etc.}}$ absolute infinitum habet valorem, cum similis a numeris tantum primis ortae valor sit logarithmus istius infiniti.

Theorema 8.

Si ex serie numerorum primorum sequens formetur expressio

$$\frac{2^n.\ 3^n.\ 5^n.\ 7^n.\ 11^n.\text{ etc.}}{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\text{ etc.}}$$

erit

erit eius valor aequalis summae huius seriei

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+ etc.$$

Demonſtratio.

Sit

$$x=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.}$$

erit

$$\frac{1}{2^n}x=\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\text{etc.}$$

vnde oritur

$$\frac{(2^n-1)}{2^n}x=1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{9^n}+\text{etc.}$$

Porro eſt

$$\frac{(2^n-1)}{2^n}\cdot\frac{1}{3^n}x=\frac{1}{3^n}+\frac{1}{15^n}+\frac{1}{21^n}+\text{etc.}$$

vnde fiet

$$\frac{(2^n-1)(3^n-1)}{2^n\cdot 3^n}x=1+\frac{1}{5^n}+\frac{1}{7^n}+\text{etc.}$$

Similibus ergo operationibus pro ſingulis numeris primis inſtitutis omnes ſeriei termini praeter primum tollentur, reperieturque

$$1=\frac{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\text{ etc.}}{2^n\ 3^n\ 5^n\ 7^n\ 11^n\text{ etc.}}x$$

et loco x ſerie reſtituta fit

$$\frac{2^n\ 3^n\ 5^n\ 7^n\ 11^n\text{ etc.}}{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\text{ etc.}}=1$$

$$= 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

Q. E. D.

Corollarium 1.

Cum posito $n = 2$ fit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = \frac{\pi^2}{6}$ denotante π peripheriam circuli, cuius diameter est 1, erit

$$\frac{4.\ 9.\ 25.\ 49.\ 121.\ 169.\ \text{etc.}}{3.\ 8.\ 24.\ 48.\ 120.\ 168.\ \text{etc.}} = \frac{\pi^2}{6},$$

seu

$$\frac{\pi^2}{6} = \frac{2.\ 2.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 11.\ 11.\ \text{etc.}}{1.\ 3\ \ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ \text{etc.}}$$

Corollarium 2.

Cum praeterea posito $n = 4$ fit

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = \frac{\pi^4}{90}.$$

erit

$$\frac{\pi^4}{90} = \frac{4.\ 4.\ 9.\ 9.\ 25.\ 25.\ 49.\ 49.\ 121.\ 121.\ \text{etc.}}{3.\ 5.\ 8.\ 10.\ 24.\ 26.\ 48.\ 50.\ 120.\ 122.\ \text{etc.}}$$

Hac igitur expressione per illam diuisa prodit

$$\frac{\pi^2}{15} = \frac{4.\ 9.\ 25.\ 49.\ 121.\ 169.\ \text{etc.}}{5.\ 10.\ 26.\ 50.\ 122.\ 170.\ \text{etc.}}$$

Theorema 9.

Si quadrata numerorum primorum imparium omnium resoluantur in duas partes vnitate a se inuicem differentes, harumque partium impares sumantur pro numeratoribus, pares vero pro denominatoribus seriei ex factoribus compositae valor huius expressionis erit $\frac{5.\ 13.\ 25.\ 61.\ 85.\ 145.\ \text{etc.}}{4.\ 12.\ 24.\ 60.\ 84.\ 144.\ \text{etc.}} = \frac{3}{2}$

Demon-

Demonſtratio.

Per theorematis praecedentis Coroll. 1. habemus

$$\frac{\pi^2}{6} = \frac{4.\ 9.\ 25.\ 49.\ 121.\ 169.\ 289.\ \text{etc.}}{3.\ 8.\ 24.\ 48.\ 120.\ 168.\ 288.\ \text{etc.}}$$

At in Coroll. 2. ſequentem elicuimus aequationem

$$\frac{\pi^2}{15} = \frac{4.\ 9.\ 25.\ 49.\ 121.\ 169.\ 289.\ \text{etc.}}{5.\ 10.\ 26.\ 50.\ 122.\ 170.\ 290.\ \text{etc.}}$$

Quarum expreſſionum ſi illa per hanc diuidatur, π ex calculo egredietur, habebiturque

$$\frac{5}{2} = \frac{5.\ 10.\ 26.\ 50.\ 122.\ 170.\ 290.\ \text{etc.}}{3.\ 8.\ 24.\ 48.\ 120.\ 168.\ 288.\ \text{etc.}}$$

cuius expreſſionis numeratores ſunt vnitate maiores quam quadrata numerorum primorum, denominatores vero vnitate minores. Si ergo vtrinque per $\frac{5}{3}$ diuidatur ſingulaeque fractiones per 2 deprimantur, habebitur

$$\frac{3}{2} = \frac{5.\ 13.\ 25.\ 61.\ 85.\ 145.\ \text{etc}}{4.\ 12.\ 24.\ 60.\ 84.\ 144.\ \text{etc.}}$$

vbi numeratores ſunt vnitate maiores quam denominatores respondentes, atque quisque numerator cum ſuo denominatore facit quadratum numeri primi imparis, ob ſublatum diuiſione quadratum numeri primi paris 2. Q. E. D.

Theorema 10.

Si π vt hactenus ſignificet peripheriam circuli, cuius diameter eſt $= 1$, erit

$$\frac{\pi^3}{32} = \frac{80.\ 224.\ 440.\ 624.\ 728.\ \text{etc.}}{81.\ 225.\ 441.\ 625.\ 729.\ \text{etc.}}$$

cuius expreſſionis denominatores ſunt quadrata numerorum imparium non primorum, numeratores vero vnitate minores.

Demonſtratio.

A *Walliſio* habetur ſequens expreſſio pro π, nempe

$$\frac{\pi}{4} = \frac{8.\ 24.\ 48.\ 80.\ 120.\ 168.\ \text{etc.}}{9.\ 25.\ 49.\ 81.\ 121.\ 169.\ \text{etc.}}$$

 quae

quae fractiones ex omnibus omnino quadratis imparibus formantur. Per Coroll. 1. vero Theor. 8 erat

$$\frac{\pi^2}{6} = \frac{4.\ 9.\ 25.\ 49.\ 121.\ 169.\ \text{etc.}}{3.\ 8.\ 24.\ 48.\ 120.\ 168.\ \text{etc.}}$$

siue

$$\frac{\pi^2}{8} = \frac{9.\ 25.\ 49.\ 121.\ 169.\ 289.\ \text{etc.}}{8.\ 24.\ 48.\ 120.\ 168.\ 288.\ \text{etc.}}$$

quae fractiones ex solis numerorum primorum imparium quadratis sunt formatae. Si iam hae duae expressiones in se mutuo ducantur prodibit

$$\frac{\pi^3}{32} = \frac{80.\ 224.\ 440.\ 624.\ 728.\ \text{etc.}}{81.\ 225.\ 441.\ 625.\ 729.\ \text{etc.}}$$

quae ideo fractiones quadrata numerorum imparium non primorum sequuntur. Q. E. D.

Theorema 11.

Sumto π pro peripheria circuli, cuius diameter est 1, erit

$$\frac{\pi}{4} = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ 23.\ \text{etc.}}{4.\ 4.\ 8.\ 12.\ 12.\ 16.\ 20.\ 24.\ \text{etc.}}$$

cuius expressionis numeratores constituunt progressionem numerorum primorum, denominatores vero sunt numeri pariter pares vnitate vel maiores vel minores quam numeratores respondentes.

Demonstratio.

Cum sit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}\ \text{etc.}$$

erit

$$\frac{1}{3}\cdot\frac{\pi}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \text{etc.}$$

quibus seriebus additis fit

$$\frac{4}{3}\cdot\frac{\pi}{4} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13}\ \text{etc.}$$

Deinde

Deinde est

$$\tfrac{1}{5}\cdot\tfrac{4}{3}\cdot\tfrac{\pi}{4}=\tfrac{1}{5}+\tfrac{1}{25}-\tfrac{1}{35}-\tfrac{1}{55}+\text{etc.}$$

qua sublata prodit

$$\tfrac{4}{5}\cdot\tfrac{4}{3}\cdot\tfrac{\pi}{4}=1-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\text{etc.}$$

in qua serie nulli amplius occurrunt denominatores vel per 3 vel per 5 diuisibiles. Simili modo tollentur omnes per 7 diuisibiles addendo

$$\tfrac{1}{7}\cdot\tfrac{4\cdot 4}{5\cdot 3}\cdot\tfrac{\pi}{4}=\tfrac{1}{7}-\tfrac{1}{49}-\tfrac{1}{77}\ \text{etc.}$$

prodibit autem

$$\tfrac{8\cdot 4\cdot 4}{7\cdot 5\cdot 3}\cdot\tfrac{\pi}{4}=1-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}\ \text{etc.}$$

Perspicitur autem denominatores per numerum primum huius formae $4n-1$ diuisibiles tolli additione, vnde iste nouus factor $\frac{4n}{4n-1}$ accedit: denominatores vero per numerum primum formae $4n+1$ diuisibiles tolli subtractione, vnde nouus factor iste $\frac{4n}{4n+1}$ adiicietur. Horum ergo factorum successiue addendorum donominatores erunt numeri primi; numeratores vero numeri pariter pares vnitate vel maiores vel minores quam denominatores. Hoc ergo modo si auferantur omnes termini seriei initio assumtae, prodibit tandem

$$\tfrac{\text{etc.}\ 24\cdot 20\cdot 16\cdot 12\cdot 12\cdot 8\cdot 4\cdot 4}{\text{etc.}\ 23\cdot 19\cdot 17\cdot 13\cdot 11\cdot 7\cdot 5\cdot 3}\cdot\tfrac{\pi}{4}=1.$$

Ex qua oritur

$$\tfrac{\pi}{4}=\tfrac{3\cdot 5\cdot 7\cdot 11\cdot 15\cdot 17\cdot 19\cdot 23\cdot\ \text{etc.}}{4\cdot 4\cdot 8\cdot 12\cdot 12\cdot 16\cdot 20\cdot 24\cdot\ \text{etc.}}\qquad \text{Q. E. D.}$$

Theorema 12.

Si omnes numeri primi impares in duas partes vnitate a se inuicem differentes diuidantur atque partes pares su-

ſumantur pro numeratoribus, impares vero pro denominatoribus, erit factum continuum

$$\frac{2.\ 2.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ \text{etc.}}{1.\ 3.\ 3.\ 5.\ 7.\ 9.\ 9.\ 11.\ \text{etc.}} = 2$$

Demonſtratio.

Cum ſit per Theorema praecedens $\frac{\pi}{4} = \frac{3.\ 5.\ 7.\ \text{etc.}}{4.\ 4.\ 8.\ \text{etc.}}$ erit

$$\frac{16}{\pi^2} = \frac{4.\ 4.\ 4.\ 4.\ 8.\ 8.\ 12.\ 12.\ 12.\ 12.\ 16.\ 16.\ \text{etc.}}{3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 11.\ 11.\ 13.\ 13.\ 17.\ 17\ \text{etc.}}.$$

At ex Coroll. 1. Theor. 8. ſi per $\frac{3}{4}$ multiplicetur habetur

$$\frac{\pi^2}{8} = \frac{3.\ 3.\ 5.\ 5.\ 7.\ 7\ 11.\ 11.\ 13.\ 13.\ \text{etc.}}{2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ 12.\ 14.\ \text{etc.}}$$

quarum expreſſionum vtraque per numeros primos impares formatur. Si ergo hae in ſe inuicem multiplicentur, prioris denominator deſtruet numeratorem poſterioris, atque praeterea tam ex illius numeratore quam huius denominatore medietas terminorum auferetur. Prodibit ſcilicet

$$2 = \frac{4.\ 4\ 8.\ 12.\ 12.\ 16.\ 20.\ 24.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ 22.\ \text{etc.}}$$

vbi numeratores ſunt numeri pariter pares, denominatores vero impariter pares vtrique vnitate vel maiores vel minores quam numeri primi impares. Si ergo ſingulae fractiones per binarium deprimantur, numerator continebit numeros pares, denominator vero impares, atque bini reſpondentes et vnitate a ſe inuicem different, et coniuncti numerum primum conſtituent. Habebitur igitur

$$2 = \frac{2.\ 2.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ \text{etc.}}{1.\ 3.\ 3.\ 5.\ 7.\ 9.\ 9.\ 11.\ \text{etc.}}$$

Q. E. D.

Theo-

Theorema 13.

Si omnes numeri impares non primi in duas partes diuidantur vnitate a se inuicem distantes harumque pares pro numeratoribus impares vero pro denominatoribus accipiantur, erit

$$\frac{\pi}{4} = \frac{4.\ 8.\ 10.\ 12.\ 14.\ 16.\ 18.\ 20.\ 22.\ 24.\ \text{etc.}}{5.\ 7.\ 11.\ 13.\ 13.\ 17.\ 17.\ 19.\ 23.\ 25.\ \text{etc.}}$$

Demonstratio.

Cum per *Wallisii* quadraturam circuli sit

$$\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12.\ 12.\ \text{etc.}}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11.\ 13.\ \text{etc.}}$$

cuius expressionis si singuli numeratores ad suos respondentes denominatores addantur, prodeunt omnes omnino numeri impares. At quoniam similis expressio, si ex numeris imparibus primis tantum formatur, aequatur binario, vti in praecedente Theoremate est demonstratum, quo erat

$$2 = \frac{2.\ 2.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ \text{etc.}}{1.\ 3.\ 3\ \ 5.\ 7.\ 9.\ 9.\ 11.\ \text{etc.}}$$

si illa expressio per hanc diuidatur, proueniet

$$\frac{\pi}{4} = \frac{4.\ 8.\ 10.\ 12.\ 14.\ 16.\ 18.\ 20.\ 22.\ 24.\ \text{etc.}}{5.\ 7.\ 11.\ 13.\ 13.\ 17.\ 17\ \ 19.\ 23.\ 25.\ \text{etc.}}$$

quae similiter ex numeris imparibus non primis formatur. Numeratores scilicet erunt numeri pares, denominatores vero impares vnitate a numeratoribus distantes, atque singuli numeratores ad suos respondentes denominatores additi dabunt omnes numeros impares non primos. Q. E. D.

Theorema 14.

Denotante vt hactenus π circuli peripheriam cuius diameter est 1, dico fore

$$\frac{\pi}{2} = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ 23.\ 29.\ 31.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ 22.\ 30.\ 30.\ \text{etc.}}$$

cuius expressionis numeratores constituunt seriem numerorum primorum imparium, denominatores vero sunt numeri impariter pares vnitate) vel minores vel maiores quam numeratores respondentes.

Demonstratio.

Per Coroll. Theor. 8. si per $\frac{3}{4}$ multiplicetur est

$$\frac{\pi^2}{8} = \frac{3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 11.\ 11.\ 13.\ 13.\ \text{etc.}}{2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ 12.\ 14.\ \text{etc.}}$$

in qua numeratores sunt numeri primi impares bis positi, denominatores vero numeri tam pariter pares quam impariter pares, vnitate vel maiores vel minores quam ipsi numeri primi. Deinde Theoremate 11. demonstrauimus esse

$$\frac{\pi}{4} = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ 23.\ \text{etc.}}{4.\ 4.\ 8.\ 12.\ 12.\ 16.\ 20.\ 24.\ \text{etc.}}$$

in qua expressione numeratores sunt numeri primi impares semel positi, denominatores vero numeri pariter pares vnitate distantes a numeris primis, ita vt haec expressio inpraecedente sit contenta. Quare si illa expressio per hanc diuidatur prodibit

$$\frac{\pi}{2} = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ \text{etc.}}$$

in qua numeri impares primi numeratores constituunt; denominatores vero sunt numeri impariter pares vnitate vel maiores vel minores quam numeratores. Q. E. D.

Theorema 15.

Denotante π *peripheriam circuli, cuius diameter est* 1, *erit*

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \frac{1}{15} - \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23}$$
$$+ \frac{1}{25} + \frac{1}{27} - \frac{1}{29} + \frac{1}{31} + \frac{1}{33} - \frac{1}{35} - \frac{1}{37} \text{ etc.}$$

cuius seriei denominatores sunt numeri impares omnes, ratio

tio signorum autem hoc nititur fundamento. Numeris primus huius formae $4n-1$ datur signum $+$; numeris primis autem huius formae $4n+1$ signum -1. Deinde numeris compositis id tribuitur signum, quod ipsis ratione compositionis ex primis cum suis signis secundum multiplicationis regulam competit.

Demonstratio.

Quemadmodum vsitatis hic operationibus haec series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

conuersa est in hanc expressionem $\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot \text{etc.}}{4 \cdot 4 \cdot 8 \cdot 12 \cdot \text{etc.}}$, ita vicissim methodus potest excogitari, qua hanc expressionem

$$\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \text{etc.}}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot \text{etc.}}$$

in seriem

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \text{ etc.}$$

transmutare liceat. Atque si haec methodus ad expressionem theoremate praecedente inuentam

$$\frac{\pi}{2} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot \text{etc.}}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot \text{etc.}}$$

adhibeatur, ista expressio abibit in istam seriem propositam

$$1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \text{ etc.}$$

cuius propterea summa est $\frac{\pi}{2}$. Ad idem a posteriore colligere licet, ponendo

$$x = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \frac{1}{15} - \frac{1}{17} \text{ etc.}$$

eritque

$$\frac{1}{3}x = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} + \frac{1}{27} + \frac{1}{33} - \text{etc.}$$

atque subtrahendo prodibit

$$\frac{2}{3}x = 1 - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} \text{ etc.}$$

Deinde

Deinde cum ſimili modo ſit

$$\tfrac{1}{3}\cdot\tfrac{2}{3}x = \tfrac{1}{5} - \tfrac{1}{25} + \tfrac{1}{35} + \tfrac{1}{55} \text{ etc.}$$

prodibit addendo

$$\tfrac{6.\ 2}{5.\ 3}x = 1 + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} \text{ etc.}$$

Atque ſimiliter omnibus tollendis terminis praeter primum 1 inuenietur

$$x = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ \text{etc.}} = \frac{\pi}{2}.$$

Atque hinc ſimul ratio ſignorum ſeriei propoſitae colligitur eadem ipſa, quam deſcripſimus. Q. E. D.

Corollarium.

Summa ergo ſeriei propoſitae

$$1 + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} \text{ etc.}$$

duplo maior eſt quam ſumma huius ſeriei

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} \text{ etc.}$$

Quare cum ipſae fractiones ſint vtrinque eaedem, ſolis ſignis effectum eſt, vt altera alterius ſit dupla.

Theorema 16.

Poſito π *vt hactenus pro peripheria circuli cuius diameter eſt* 1, *erit*

$$\frac{\pi}{2} = 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{10} - \tfrac{1}{14} - \tfrac{1}{16} - \tfrac{1}{18} + \tfrac{1}{18} + \tfrac{1}{20} \text{ etc.}$$

Fractionum autem affirmatiuarum denominatores ſunt vnitate minores quam numeri impares non poteſtates; fractionum autem negatiuarum denominatores ſunt vnitate maiores. Cuiusque autem fractionis ſignum congruit cum ſigno numeri imparis vel vnitate maioris vel minoris non poteſtatis in praecedente Theoremate.

Demon-

Demonſtratio.

Haec ipſa ſeries oritur ex conuerſione praecedentis ſecundum modum in Theorematis 1. 2. 3. vſitatis, quo continuo progreſſiones geometricae vel adduntur vel demuntur, quoad ſolus primus terminus ſuperſit. Q.E.D.

Theorema 17.

Si numeris imparibus primis huius formae $4n-1$ *tribuatur ſignum* $+$, *reliquis huius formae* $4n+1$ *ſignum* $-$, *numeris vero compoſitis ea ſigna, quae ipſis per regulas multiplicationis ex primis competunt; erit*

$$\frac{3\pi}{8} = 1 + \frac{1}{5} - \frac{1}{9} - \frac{1}{15} + \frac{1}{21} + \frac{1}{25} + \frac{1}{33} - \frac{1}{35} - \frac{1}{39} + \frac{1}{49} - \frac{1}{51} \text{ etc.}$$

qui denominatores ſunt facta vel ex duobus vel ex 4 vel ex 6 vel etc. numeris primis.

Demonſtratio.

Cum enim per Theorem. 15 ſit

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{7} + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} - \frac{1}{17} + \frac{1}{19} \text{ etc.}$$

vbi denominatores ſunt omnes numeri impares et ſigna eam tenent legem quam praeſcripſimus; atque praeterea ſit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

quarum ſerierum ii termini ſigna habent eadem, quorum denominatores ſunt facta vel ex binis vel quaternis, vel ſenis etc. numeris primis. His ergo ſeriebus additis ſoli iſti termini remanebunt, diuiſioneque facta per 2 erit

$$\frac{3\pi}{8} = 1 + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} + \frac{1}{25} + \frac{1}{33} \text{ etc.}$$

quae eſt ipſa ſeries propoſita, atque ex lege ſignorum ſimul ſequitur, eas fractiones habere ſignum $+$, quarum denominatores in hac forma $4n+1$ continentur; reliquas ſignum $-$. Q. E. D.

Corollarium.

Si ab huius theorematis ferie fubtrahatur ifta:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

prodibit

$$\frac{\pi}{8} = \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} \text{ etc.}$$

cuius denominatoris funt vel ipfi numeri primi vel facta ex ternis vel quinis vel etc. ii vero qui funt formae $4n-1$ fignum habent $+$ reliqui formae $4n+1$ fignum $-$.

Theorema 18.

Si omnibus numeris primis tribuatur fignum $-$; cuique vero numero compofito id fignum quod ipfi fecundum multiplicationis regulas competit, atque ex omnibus numeris fequens formetur feries

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} - \frac{1}{12} - \text{etc.}$$

erit eius fumma in infinitum continuatae $= 0$.

Demonftratio.

Sit enim $x =$ fummae iftius feriei feu

$$x = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} \text{ etc.}$$

erit per operationes in pofterioribus theorematis adhibitas

$$\frac{3}{2}x = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ etc.}$$

atque fimili modo

$$\frac{3}{2} \cdot \frac{4}{3} x = 1 - \frac{1}{5} - \frac{1}{7} - \frac{1}{11} - \frac{1}{13} \text{ etc.}$$

Denique hac operatione infinities repetita erit

$$\frac{3 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdot 14 \cdot \text{etc.}}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \text{etc.}} x = 1.$$

At cum fit per Theor. feptimum

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10} \text{ etc.} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.}\ l\infty.$$

facile intelligitur et noftrum ipfius x coefficientem effe

infinite

infinite magnum. Quare quo factum aequale esse possit 1 erit $x = 0$, et hanc ob rem habebitur

$$0 = 1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{7} - \tfrac{1}{8} \text{ etc.}$$

cuius denominatores, qui sunt vel ipsi primi vel facta ex ternis, quinis, etc. habent signum $-$ reliqui signum $+$. Q.E.D.

Corollarium 1.

Modus ergo apparet, quo in progressione harmonica signa in singulis terminis sint distribuenda, vt summa totius seriei fiat $= 0$.

Corollarium 2.

Cum inuenerimus $x = 0$, erit quoque $\tfrac{1}{2}x = 0$, et hanc ob rem habebimus quoque

$$0 = 1 - \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} - \tfrac{1}{13} + \tfrac{1}{15} \text{ etc.}$$

in qua numeri tantum impares occurrunt; descriptamque ratione signorum tenent legem.

Theorema 19.

Summa seriei reciprocae numerorum primorum

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \tfrac{1}{13} + \text{etc.}$$

est infinite magna; infinities tamen minor, quam summa seriei harmonicae $1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} +$ *etc. Atque illius summa est huius summae quasi logarithmus.*

Demonstratio.

Ponatur $\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \text{etc.} = A$ atque $\tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \text{etc.} = B$ et $\tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{5^3} + \text{etc.} = C$. atque ita porro omnes potestates peculiaribus litteris designando; erit posito e pro numero cuius logarithmus hyperbolicus est 1

 e^{A+}

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\text{etc.}} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\ \text{etc.}$$

Nam est

$$A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D\ \text{etc.} = l\,\frac{2}{1}+l\,\frac{3}{2}+l\,\frac{5}{4}+l\,\frac{7}{6}+\ \text{etc.}$$

ideoque

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D} = \frac{2.\,3.\,5.\,7.\ \text{etc.}}{1.\,2.\,4.\,6.\ \text{etc.}} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ \text{etc.}$$

per Theor. 7. At non solum B, C, D, etc. habebunt valores finitos, sed etiam $\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ etc. valorem habet finitum. Quare quo

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\text{etc.}}\ \text{fit} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \text{etc.} = \infty.$$

oportet vt A sit infinite magnum, cumque ideo eius respectu sequentes termini $\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ etc. euanescant, erit

$$e^{A} = e^{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\text{etc.}} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ \text{etc.}$$

Atque consequenter erit

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\ \text{etc.}$$
$$= l\,(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\ \text{etc.})$$

illius ergo seriei summa erit infinities minor quam huius, atque cum huius summa sit $= l\infty$ erit

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ \text{etc.} = l.\,l\infty.$$

Q. E. D.

DE