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Recommended Citation

DOI: https://doi.org/10.56031/2693-9908.1062
Available at: https://scholarlycommons.pacific.edu/euleriana/vol4/iss1/4

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Euler and homogeneous Difference Equations with linear Coefficients

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We present a method outlined by Euler in his paper “De fractionibus continuis observationes” [2] (E123: “Observations on continued fractions”) that can be used to solve homogeneous difference equations with linear coefficients. We will illustrate his ideas by applying it to two familiar examples and explain how it can be understood from a more modern point of view.

1 Introduction

In his papers “De fractionibus continuis observationes” [2] (E123: “Observations on continued fractions”) and “Methodus inveniendi formulas integrales, quae certis casibus datam inter se teneant rationem, ubi sumul methodus traditur fractiones continuas summandi” [4] (E594: “A method of finding integral formulas which in certain cases have a given ratio, where at the same time a method to sum continued fractions is given”) papers Euler (1707 - 1783) described and applied a general method to evaluate continued fractions of certain type. In [1] it was mentioned that Euler’s method can be used to solve arbitrary homogeneous linear difference equations with linear coefficients, i.e., equations of the form:

\[(a_0x + b_0)f(x) + (a_1x + b_1)f(x + 1) + \cdots + (a_nx + b_n)f(x + n) = 0 \quad (1)\]

with complex numbers \(a_0, b_0, a_1, b_1, \ldots, a_n, b_n\). Furthermore, let us assume that at least one of the coefficients \(a_0, a_1, \ldots, a_n\) is different from 0. For, if all of them are equal to zero, (1) reduces to a difference equation with constant coefficients, an equation that Euler actually discussed extensively in his paper “De serierum determinatione seu nova methodus inveniendi terminos generales serierum” [3] (E189: “On the determination of series or a new method of finding the general terms of series”). Furthermore, Euler’s method that we will introduce (see section 2) and apply to certain familiar examples (see section 3) in this note, does not work, if all coefficients vanish. Finally, we will explain why Euler’s method works from a more modern perspective (see section 4).
2 The Method

Euler’s idea is that the solution to (1) is given as an integral of the form

\[ f(x) = \int_{a}^{b} t^x p(t) dt \] (2)

such that he had to find the limits of integration \(a\) and \(b\) and the unknown function \(p(t)\). He achieved this by introducing the equation\(^{[1]}\)

\[ (a_0 x + b_0) \int t^x p(t) dt + \cdots + (a_n x + b_n) \int t^{x+n} p(t) dt + t^x q(t) = 0, \] (3)

where \(q(t)\) is yet another unknown function. Differentiating the previous equation with respect to \(t\) gives:

\[ (a_0 x + b_0) t^x p(t) + \cdots + (a_n x + b_n) t^{x+n} p(t) + xt^{x-1} q(t) + t^x q'(t) = 0. \]

Division by \(t^{x-1} \neq 0\) and rearrangement according to powers of \(x\) then leads to:

\[ (p(t)(a_0 t + \cdots + a_n t^{n+1}) + q(t)) \cdot x + p(t)(b_0 t + \cdots + b_n t^{n+1}) + tq'(t) = 0. \]

Thus, by comparing coefficients of \(x\) Euler obtained a system of coupled differential equations for \(p(t)\) and \(q(t)\)\(^{[2]}\).

Having found \(p(t)\) and \(q(t)\), Euler used them to find the limits of integration. Considering (3), we see that we need to find at least 2 values for \(t\) such that:

\[ t^x q(t) = 0. \] (4)

Having found those solutions, say \(t_1 = a\) and \(t_2 = b\), a solution to (1) is given by:

\[ f(x) = \int_{t_1}^{t_2} t^x p(t) dt. \] (5)

\(^{[1]}\)In his papers [2] and [4] Euler restricted his investigations to the special case of \(n = 2\) such that (1) leads to a continued fraction for \(\frac{f(x)}{f(x+1)}\).

\(^{[2]}\)In his papers [2] and [4] Euler did not consider the general case, but discusses various examples.
3 Examples

3.1 The Gamma-Function

For the sake of explanation, let us start with the most simple special case of (1), namely the equation:

\[ f(x + 1) = xf(x). \]  

(6)

Euler also considered this example in [4], §13. According to his method, we need to consider the auxiliary equation:

\[ \int p(t)x^t \, dt = x \int p(t)x^{t-1} \, dt + t^x q(t). \]

Differentiating this equation gives:

\[ p(t)x^t = xp(t)x^{t-1} + xt^{x-1}q(t) + t^x q'(t) \]

and hence for \( t^x \neq 0 \):

\[ p(t)t = xp(t) + xq(t) + tq'(t). \]

Comparing the respective powers of \( x \), we find the following equations for \( p(t) \) and \( q(t) \):

\[ p(t) = q'(t) \quad \text{and} \quad q(t) = -p(t), \]  

(7)

which are solved by

\[ p(t) = -q(t) = C \cdot e^{-t} \quad \text{with} \quad C \neq 0. \]

Thus, according to Euler’s method, we need to solve

\[ C \cdot t^x e^{-t} = 0, \]  

(8)

which can only be solved by \( t_1 = 0 \) and \( t_2 = \infty \) such that we have the following solution to (6):

\[ f(x) = C \cdot \int_0^\infty t^{x-1}e^{-t} \, dt. \]

(9)

Adding the initial condition \( f(1) = 1 \) [9] reduces to the \( \Gamma \)-function.
3.2 The Legendre Polynomials

The Legendre polynomials occur frequently at various places in physics. They were introduced by Legendre (1752-1833) in his paper “Recherches sur l’attraction des sphéroïdes homogènes” [6] (“Researches on the attraction of homogeneous spheroids”) in his studies of the gravitational potential. Aside from this they show up in the multipole expansion in electrodynamics. Furthermore, they satisfy a difference equation

\[(n + 1)P_{n+1}(t) = (2n + 1)tP_n(t) - nP_{n-1}(t), \quad \text{for} \quad n \in \mathbb{N}_0 \quad (10)\]

with the initial conditions \(P_0(t) = 1\) and \(P_1(t) = t\). Since (10) has the form of (1), Euler’s method can be applied to find the solution. Indeed, Euler also did so himself in his paper “Speculationes super formula integrali \(\int \frac{x^n}{\sqrt{a^2 - 2bx + cx^2}}, ubi simul egregiae observationes circa fractiones continuas occurrant” [5] (E606: “Speculations about the integral formula \(\int \frac{x^n dx}{\sqrt{a^2 - 2bx + cx^2}}, where at the same time extraordinary observations on continued fractions occur”), where in §10 he stated

\[Q_n = \int_{b - \sqrt{b^2 - 4ac}}^{b + \sqrt{b^2 - 4ac}} \frac{x^n}{\sqrt{a^2 - 2bx + cx^2}} dx\]

as the solution to the difference equation

\[na^2Q_{n-1} = (2n + 1)bQ_n - (n + 1)cQ_{n+1},\]

which is easily seen to reduce to (10) for \(a = c = 1\) and \(b = t\). Thus, we can infer that the Legendre polynomial \(P_n(t)\) is given as:

\[P_n(t) = C(t) \cdot \int_{t - \sqrt{1 - 2xt + x^2}}^{t + \sqrt{1 - 2xt + x^2}} \frac{x^n dx}{\sqrt{1 - 2tx + x^2}}. \quad (11)\]

The function \(C(t)\) is found from using the initial conditions to (10) to be \((\log(-1))^{-1}\). Indeed, by a formal calculation, we have:

\[\int_{t - \sqrt{1 - 2tx + x^2}}^{t + \sqrt{1 - 2tx + x^2}} \frac{x^0 dx}{\sqrt{1 - 2tx + x^2}} = \log(-1)\]
In like manner, for the case $n = 1$:

$$
\int_{t-\sqrt{t^2-1}}^{t+\sqrt{t^2-1}} \frac{x^1 dx}{\sqrt{1-2xt+x^2}} = \log(-1) \cdot t.
$$

Hence, it turns out that the function $C(t)$ in (11) is actually a constant. Unfortunately, \( \log(-1) \) is not uniquely defined, the complex logarithm being a multivalued function. If we want to avoid this ambiguity, we can either say that, e.g., the principal part of the complex logarithm is to be taken or we can resort to the indefinite integral

$$
\int \frac{x^n dx}{\sqrt{1-2xt+x^2}} = P_n(t) \text{artanh} \left( \frac{x-t}{\sqrt{1-2xt+x^2}} \right) + H_n(x, t)\sqrt{1-2xt+x^2} + C
$$

where $H_n$ is a polynomial in $x$ and $t$ and $C$ is a constant of integration. This formula can be proved by induction and the difference equation for the Legendre polynomials (10). Thus, if we are just interested in the Legendre polynomials $P_n(t)$, we do not need to know the polynomial $H_n$ explicitly.

Nevertheless, it is a rather curious incident that the function $C(t)$ in (11) is a complex number, although the corresponding difference equation (10) does not involve complex coefficients. Indeed, the same does not happen for the difference equations satisfied by other orthogonal polynomials, like the Hermite polynomials for example. Why the Legendre polynomials have this peculiar property will be the subject of future investigations.

4 Explanation of Euler’s Method

In §50 of [2], Euler described the idea behind his ansatz. Essentially, starting from (2) he assumed that (1) can be solved by integrating by parts the integral containing the highest power of $t$ integrand repeatedly and express it as a linear combination of those integrals with lower powers of $t$. Interestingly, this is also the idea used by physicists in their study of Feynman-Integrals, where they try to find IPB (Integration-by-parts) identities satisfied by them. Confer, e.g., [7].

But Euler’s method can also be understood from another modern point of view. For, Euler’s ansatz (2) is a Mellin transform. The Mellin-Transform, first defined by Mellin (1854 - 1833) in his paper “Über die fundamentale Wichtigkeit...”

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des Satzes von Cauchy für die Theorien der Gamma- und hypergeometrischen Functionen” [8] (“On the fundamental importance of Cauchy's theorem for the theories of the Gamma- and hypergeometric functions”), is given as:

\[ M(p)(s) = \int_0^\infty t^{s-1}p(t)dt =: M(s), \quad (12) \]

if the integral exists, of course. From this definition we see that \( t^ap(t) \) has the Mellin transform \( M(s+a) \) for an arbitrary complex number \( a \) and that \( tp'(t) \) has the Mellin transform \( -sM(s) \) is seen using integration by parts. Thus, we can immediately conclude from the general properties of the Mellin transform that Euler’s ansatz (2) allows to rewrite (1) as a differential equation for \( p(t) \). Euler’s ansatz needs the introduction of the auxiliary function \( q(t) \) in (3) in order to find the limits of integration.

5 Conclusion

In this note we described Euler’s method that he introduced in his papers [2] and [4], papers actually devoted to continued fractions, to solve homogeneous difference equations with linear coefficients [1]. From a modern point of view, Euler’s solution is most conveniently understood as a Mellin transform (12), whereas Euler himself concluded it from integrating by parts. His method then describes how to find the unknown boundaries of integration and the unknown part of the integrand \( p(t) \) in his ansatz (2). For the latter one is led to an ordinary differential equation and the limits of integration are defined from the requirement that the auxiliary equation (3) reduces (1).

His method can be used to derive integral representations of functions satisfying certain difference equations. Euler himself did this for the \( \Gamma \)-function in [4] and the Legendre polynomials in [5], the last of which led to a curious property satisfied by them in contrast to other orthogonal polynomials that will be subject of future investigations.

References


