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Abstract

We present a proof given by Euler in his paper *"De serierum determinatione seu nova methodus inveniendi terminos generales serierum"* [4] (E189: "On the determination of series or a new method of finding the general terms of series") for Stirling's formula. Euler's proof uses his theory of difference equations with constant coefficients. This theory outgrew from his earlier considerations on inhomogeneous differential equations with constant coefficients of finite order that he tried to extend to the case of infinite order.

1 Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{for} \quad n \to \infty. \tag{1}$$

was first proven by Stirling. It can be proven by application of the Euler-Maclaurin summation formula or the saddle point approximation. But in his paper *"De serierum determinatione seu nova methodus inveniendi terminos generales serierum"* [4] (E189: "On the determination of series or a new method of finding the general terms of series") Euler gave another proof based on his theory on inhomogeneous linear difference equations with constant coefficients. His theory will be described in section 2. Finally, we will present and discuss Euler's proof in section 3.

2 Euler's Theory of Inhomogeneous Difference Equations ith Constant Coefficients

In this section we will discuss Euler's application of his theory of inhomogeneous difference equations with constant coefficients to the derivation of Stirling's formula (1). Euler reduced them to a differential equation of infinite order. Having treated the finite order case in *"Methodus aequationes differentiales altiorum* *graduum integrandi ulterius promota"* [3] (E188: "The method to integrate differential equations of higher degrees expanded further") before, in [4] he then tried to transfer the results from the before-mentioned paper to the case of infinite order. Unfortunately, this is not possible in the way Euler intended and hence lead Euler to a wrong result when he applied his theory to the case of the logarithm of the factorial. We will explain this in more detail in section 3. But we will briefly state what we need to discuss Euler's solution of inhomogeneous linear differential equations of finite (see section 2.1) and infinite order (see section 2.2) first.

2.1 Inhomogeneous Linear Differential Equations of Finite Order

In his paper [3], Euler considered equations of the form:

$$\left(a_0 + a_1\frac{d}{dx} + a_2\frac{d^2}{dx^2} + \cdots + a_n\frac{d^n}{dx^n}\right) f(x) = g(x), \tag{2}$$

with complex coefficients $a_1, a_2, \cdots, a_n$. Euler did not state any conditions on the function $g(x)$[a]. In §22, Euler described the following procedure: First, find the zeros with their multiplicity of the expression:

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

Assume $z = k$ is a solution of $P(z) = 0$. Then, if k is a simple zero[b] of $P(z)$, a solution of (2) is given by:

$$f(x) = \frac{e^{kx}}{P'(k)} \int e^{-kx} g(x) dx. \tag{3}$$

Note that the indefinite integral introduces a constant of integration.

2.2 Reduction of the Difference Equation to a Differential Equation

2.2.1 General Idea

As we mentioned in section 1, Euler's paper [4] is a paper actually devoted to inhomogeneous difference equations with constant coefficients, i.e., equations of the form:

[a]The conditions on $g(x)$ can be inferred from Euler's solution. But since we will not need this in this paper, we will not elaborate on this subject.

[b]In this note, we will only need the case of simple zeros and hence will only state the corresponding formula. In [3], Euler stated all cases from order 1 to 4 explicitly.

$$a_0 f(x) + a_1 f(x+1) + \cdots + a_n f(x+n) = g(x), \tag{4}$$

with complex coefficients $a_0, a_1, \cdots, a_n$. Euler's idea to solve (4) is as follows: First, rewrite $f(x+1), f(x+2), \cdots, f(x+n)$ in terms of $f(x)$ and its derivatives by applying Taylor's theorem. Next, substitute the corresponding term in equation (4). After some rearrangement, one arrives at an inhomogeneous differential equation of infinite order with constant coefficients, i.e., an equation of the form:

$$\left(A_0 + A_1 \frac{d}{dx} + A_2 \frac{d^2}{dx^2} + \cdots + A_n \frac{d^n}{dx^n} + \cdots \right) f(x) = g(x), \tag{5}$$

where $A_0, A_1, A_2, \cdots$ are complex coefficients.

Having transformed the initial equation (4) into this form, Euler argued that the same procedure outlined in section (2.2) also applies here. More precisely, one has to find all zeros of the expression:

$$A_0 + A_1 z + A_2 z^2 + \cdots + A_n z^n + \cdots \tag{6}$$

and has to construct the solution to (5) from those zeros. In his paper [4] Euler considered various examples; but in this note we are interested in his solution of the simple difference equation.

2.2.2 Example: The Simple Difference Equation

For the sake of explanation and since we will be need the result in section (3), let us consider the simple difference equation, i.e., the equation

$$f(x+1) - f(x) = g(x) \tag{7}$$

and let us describe Euler's solution. First, Euler[c] expanded $f(x+1)$ by using Taylor's theorem:

$$f(x+1) = f(x) + \frac{d}{dx} f(x) + \frac{1}{2!} \frac{d^2}{dx^2} f(x) + \frac{1}{3!} \frac{d^3}{dx^3} f(x) \cdots .$$

Substituting this into equation (7), Euler arrived at the equation

[c]In his paper [4] §55, Euler considered the equation $y(x) - y(x-1) = X(x)$ instead of equation (7). But does not change the final result substantially, of course.

$$\left(\frac{d}{dx} + \frac{1}{2!}\frac{d^2}{dx^2} + \frac{1}{3!}\frac{d^3}{dx^3} + \cdots\right) f(x) = g(x).$$

Thus, according to his theory, Euler needed to find the zeros (and their multiplicity) of the expression

$$P(z) = \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z - 1.$$

The general zero of this expression is $z = \log(1)$. But in his paper *"De la controverse entre Mrs. Leibnitz et Bernoulli sur les logarithmes des nombres négatifs et imaginaires"* [2] (E168: "On the controverse of Leibniz and Bernoulli on the logarithms of negative and imaginary numbers") Euler had demonstrated that the logarithm of a number is a multivalued expression and hence concluded that there are infinitely many zeros, namely:

$$z = 0, \pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \pm 8\pi i, \cdots .$$

Furthermore, all those zeros are simple, since:

$$\lim_{z \to 2k\pi i} \frac{e^z - 1}{z - 2k\pi i} = \lim_{z \to 2k\pi i} \frac{e^z}{1} = e^{2k\pi i} = 1.$$

where L'Hospital's rule was used in the first step. Therefore, Euler used the general solution formula (3). This gave him:

$$f(x) = \int g(x)dx + e^{2\pi ix}\int g(x)e^{-2\pi ix}dx + e^{-2\pi ix}\int g(x)e^{+2\pi ix}dx \qquad (8)$$
$$+e^{4\pi ix}\int g(x)e^{-4\pi ix}dx + e^{-4\pi ix}\int g(x)e^{+4\pi ix}dx + \cdots$$

In [4] § 55, Euler expressed the solutions using sines and cosines instead of the exponentials that we used here.

Thus, we arrived at Euler's general solution of the simple difference equation (7). Unfortunately, as we will see below in section (3.3), there is a mistake in Euler's solution (8).

3 Application to the Factorial

In [4] §56 - §60, Euler applied his general formula (8) to the factorial[d], i.e., the function $y(x)$ satisfying:

[d] More precisely, Euler actually considered the difference equation satisfied by the Γ-function.

$$y(x+1) = xy(x). \tag{9}$$

This equation can be transformed into a simple difference equation by taking logarithms. We have:

$$\log y(x+1) - \log y(x) = \log(x).$$

3.1 Application of the General Formula

Applying (8) with $f(x) = \log y(x)$ and $g(x) = \log(x)$ we get:

$$\begin{aligned} f(x) = x\log x - x + C + e^{2\pi i x}\int \log(x)e^{-2\pi i x}dx + e^{-2\pi i x}\int \log(x)e^{+2\pi i x}dx \\ +e^{4\pi i x}\int \log(x)e^{-4\pi i x}dx + e^{-4\pi i x}\int \log(x)e^{+4\pi i x}dx + \cdots \end{aligned} \tag{10}$$

where $\int \log(x)dx$ was already evaluated and C is a constant of integration[e].

3.2 Derivation of Stirling's Formula

§59 -§60 of [4] contain the derivation of Stirling's formula (1) from (10). Euler first evaluated the general expression:

$$e^{2k\pi i x}\int e^{-2k\pi i x}\log(x)dx.$$

He did so by integrating by parts infinitely many times with $e^{-2k\pi i x}$ as function to be integrated. In modern and compact notation the result is[f]:

$$e^{2k\pi i x}\int e^{-2k\pi i x}\log(x)dx = -\frac{\log(x)}{2k\pi i} + \sum_{n=1}^{\infty}\frac{(-1)^n(n-1)!}{(2k\pi i)^{n+1}x^n} + C_k e^{2k\pi i x}.$$

C_k is a constant of integration. Proceeding in the same way for all other integrals, we have the formal identity:

[e]This is the solution Euler gave in [4] §59. But he represented his solution using sines and cosines.

[f]Since Euler used $\sin(2k\pi x)$ and $\cos(2\pi x)$ instead of $e^{-2k\pi i x}$, his result differs from the one we will find. But the derivation is the same in both cases, of course.

$$\log y(x) = x\log x - x + C + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(C_k e^{2k\pi i x} - \frac{\log(x)}{2k\pi i} + \sum_{n=1}^{\infty} \frac{(-1)^n (n-1)!}{(2k\pi i)^{n+1} x^n} \right)$$

Let us simplify the sum. First, we note that

$$C + \sum_{k\in\mathbb{Z}\setminus\{0\}} C_k e^{2k\pi i x} =: h(x)$$

is a general periodic function, i.e., it satisfies $h(x+1) = h(x)$ for all x. Next,

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\log(x)}{2k\pi i} = 0,$$

since the terms cancel each other. Therefore, we just need to evaluate the double sum. By a formal calculation we have:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{n=1}^{\infty} \frac{(-1)^n (n-1)!}{(2k\pi i)^{n+1} x^n} = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{2}{k^{2n+2}} \cdot \frac{(-1)^n (2n)!}{(2\pi)^{2n+2} \cdot x^{2n+1}}. \tag{11}$$

The sum over k had been evaluated by Euler. The general formula can found, e.g., in [1] and in modern notation reads:

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} (2\pi)^{2n} B_{2n}}{2(2n)!}, \tag{12}$$

where B_n is the n-th Bernoulli number. Inserting this into (11), we find:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{n=1}^{\infty} \frac{(-1)^n (n-1)!}{(2k\pi i)^{n+1} x^n} = \sum_{n=0}^{\infty} 2 \cdot \frac{(-1)^n (2\pi)^{2n+2} B_{2n+2}}{2(2n+2)!} \cdot \frac{(-1)^n (2n)!}{(2\pi)^{2n+2} \cdot x^{2n+1}}.$$

Many terms cancel such that:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{n=1}^{\infty} \frac{(-1)^n (n-1)!}{(2k\pi i)^{n+1} x^n} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1)2n x^{2n-1}}.$$

Therefore, inserting everything we found into (10) we get:

$$\log y(x) = x\log x - x + h(x) + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1)2n x^{2n-1}}, \tag{13}$$

where $h(x)$ satisfies $h(x+1) = h(x)$. This equation is to be understood as an asymptotic series of course and is the formula Euler arrived at in [4] §60, Euler just substituted the explicit numbers for the Bernoulli numbers. Comparing (13) to (1), the term $\log(\sqrt{2\pi})$ is still missing. In [4] Euler argued that it follows from considering a special case, e.g., $x = 1$[g] and the initial condition $y(1) = 1$ to (9) such that one arrives at the final formula:

$$\log y(x) = x\log x - x + \log(\sqrt{2\pi}) + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1)2nx^{2n-1}}, \tag{14}$$

if x is infinitely large. In [4] §60, Euler stated the formula as follows:

$$y(x) = \frac{x^x}{e^x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \cdots\right)\sqrt{2\pi}, \tag{15}$$

which follows by inserting the explicit values for the Bernoulli numbers in (14), taking the exponential and expanding the exponential of the sum.

3.3 Discussion of the Result

As it was remarked by G. Faber in a footnote in the Opera Omnia version of [4], equation (14) and hence (15) is incorrect. The correct formula reads:

$$\log y(x) = x\log x - x + \log(\sqrt{\frac{2\pi}{x}}) + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1)2nx^{2n-1}}, \tag{16}$$

i.e., Euler's formula is off by the term $\log(\sqrt{x})$. Furthermore, the term is not missing due to a calculational error, but due to a conceptional one. More precisely, Euler's idea to construct the solution from the zeros of (6) does not work in general.

We can see how the missing term enters by a formal argument[h]. We are still interested in (7). Writing D for $\frac{d}{dx}$, this equation can also be represented as:

[g]More precisely, Euler argued that $h(x)$ is to be considered as constant in this case and the value of this constant is equal to the sum $1 - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1)2n}$ which Euler claims to be $\frac{1}{2}\log(2\pi)$ without a proof in this paper, although the series does not converge due to the rapid growth of the Bernoulli numbers. But Euler knew that one can ascribe the beforementioned value to the sum, since it corresponds to the constant $\sqrt{2\pi}$ in Stirling's formula (1).

[h]There are also rigorous arguments involving the theory of the Fourier transform. But this would carry us too far away from our actual objective.

$$\left(e^D-1\right)f(x)=g(x).$$

Thus, formally the solution is given as:

$$f(x)=\left(e^D-1\right)^{-1}g(x),$$

such that we have to find out how to express $\left(e^D-1\right)^{-1}$. We only know how to calculate $D^n f(x)$ for $n\in\mathbb{Z}$. Thus, the idea is to expand $\left(e^D-1\right)^{-1}$ into a Laurent series in D around $D=0$ apply it to $g(x)$. There are many possibility to perform this expansion, but for purposes we will only need the direct expansion. This expansion had also been given by Euler, e.g., in *"De seriebus quibusdam considerationes"* [1] (E130: "Considerations on certain series") §27[i]. The expansion reads:

$$\left(e^D-1\right)^{-1}=\sum_{n=0}^{\infty}B_n\frac{D^{n-1}}{n!}=D^{-1}-\frac{1}{2}+\frac{D}{12}-\frac{D^3}{720}+\cdots, \tag{17}$$

where B_n are the Bernoulli numbers again. Interpreting D^{-1} as an integration, we can write:

$$f(x)=\left(e^D-1\right)^{-1}g(x)=\int g(x)dx-\frac{1}{2}g(x)+\frac{1}{12}\frac{d}{dx}g(x)-\cdots, \tag{18}$$

which is nothing but a modern representation of the Euler-Maclaurin summation formula. Thus, Euler's approach, i.e., constructing the solution from the zeros of e^D-1, misses the term $-\frac{1}{2}g(x)$. If we apply (18) to the factorial, i.e., take $g(x)=\log(x)$ we arrive at (16).

4 Conclusion

In this note we briefly mentioned Euler's theory how to solve inhomogeneous ordinary differential equations of infinite order with constant coefficients and Euler's application of his theory to the derivation of Stirling's formula (1). We pointed out the conceptual error in Euler's approach and provided an explanation how to correct it (section 3.3). Nevertheless, there are many intriguing ideas in [4], aside from Euler's derivation of Stirling's formula on which we focused, such that we intend to cover more content from the before-mentioned paper in the future.

[i] Euler considered the function $\frac{z}{1-e^{-z}}$ and did not state the general formula for the coefficients, but explained their origin.

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