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Euler Archive Spotlight

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Euler Archive Spotlight

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Welcome to this issue's Euler Archive spotlight! In what follows, we will explore some English translations of Euler's works that are available on the Euler Archive. More specifically, we are interested in some recent Euler Archive translations that have not appeared as part of *Euleriana*. All translations undergo a review process, though only those in *Euleriana* are accompanied by an article. So, while most translations appear in both *Euleriana* and the Euler Archive, some reside only as part of the Euler Archive.

One notable translation from 2022 was of "Solutio problematis de investigatione trium numerorum, quorum tam summa quam productum nec non summa productorum ex binis sint numeri quadrati" (E270, "The Solution of a Problem of Searching for Three Numbers, of Which the Sum, Product, and the Sum of Their Products Taken Two at a Time, Are Square Numbers") [1], which appeared originally in Volume 8 of the *Novi Commentarii* (pub. 1763). This was the third Euler translation by the team of Mark Snavely and Phil Woodruff.

As E270's lengthy title suggests, Euler wishes to find three integers x, y, z, whose sum $x+y+z$, product xyz, and sum of products taken two at a time (i.e., $xy+xz+yz$) are all squares. Of course, this problem can be reduced to finding a cubic polynomial $z^3 - pz^2 + qz - r$ (where $p = x+y+z$, $q = xy+xz+yz$, and $r = xyz$) with rational roots and square coefficients. After some clever substitutions, Euler manages to classify some solutions according to the value of a rational parameter. This leads to the mind-bogglingly large solution

$$\begin{cases} x = 29{,}158{,}396{,}911{,}435{,}776 \\ y = 55{,}570{,}736{,}301{,}244{,}025 \\ z = 3{,}668{,}981{,}811{,}742{,}224. \end{cases}$$

Euler also derives some slightly more modest solutions (with the smallest comprising 12- and 13-digit numbers) before concluding this short paper. Snavely and Woodruff note that smaller solutions exist, some of which have been found by undergraduate student researchers at Carthage College in Kenosha, WI.

Another recent translation is by Jiahui (Stella) Li, a student at the Emma Willard School in Troy, NY. The article, "Theoremata circa residua ex divisione potestatum relicta" (E262, "Theorems about the residues left by division by powers") [2] contains numerous theorems about power residues modulo a prime number; it appeared originally in Volume 7 of the Novi Commentarii (pub. 1761). What makes Li's translation unique is that it supplements an existing translation by Jordan Bell by providing modern paraphrases and annotations that expand on Euler's work. These additions give context for the modern number-theoretic concepts in the article and provide detailed proofs for all of Euler's results. Li's goal, which she notes in the abstract, is to "reduce the paper's assumptions on readers' mathematical background and make it more easily followed by a wider audience."

One important feature of this paper is that it anticipates the properties of finite groups that would be formulated more explicitly in the 19th century. For instance, consider Bell and Li's respective renderings of Theorem 2:

> If the power a^μ divided by p has the residue r, and the power a^ν the residue s, the power $a^{\mu+\nu}$ will have the residue rs.
>
> If $a^\mu \equiv r \pmod{p}$ and $a^\nu \equiv s \pmod{p}$, then $a^{\mu+\nu} \equiv rs \pmod{p}$.

Of course, both statements express the multiplication operation for the group $U(p)$ of units modulo p. Bell's translation is faithful to Euler's original, while Li uses a modern paraphrase that is familiar to most students of number theory today.

Two other important results from this paper appear as Theorem 13 (Lagrange's theorem for $U(p)$) and Theorem 14 (Fermat's little theorem), with the latter proof relying on a minimality argument. This differs from Euler's original proof in "Theorematum quorundam ad numeros primos spectantium demonstration" (E54, "A proof of certain theorems regarding prime numbers") [3]. In Euler's words (from the Bell translation of E262), "Behold therefore a new demonstration of the extraordinary theorem, found before by Fermat, which differs greatly from that which I gave in the Comment. Acad. Petropol. Volume VIII. For there I called upon the expansion of the binomial $(a+b)^n$ into a series by means of the method of Newton, which reasoning seems quite remote from the proposition; here indeed I have demonstrated the same theorem from the properties of powers alone, by which this demonstration seems much more natural." A combined reading of the Bell and Li translations will indeed make it more easily followed—and appreciated—by a wider audience.

References

[1] Euler, Leonhard (1763). Solutio problematis de investigatione trium numerorum, quorum tam summa quam productum nec non summa productorum ex binis sint numeri quadrati (E270). *Novi Commentarii academiae scientiarum Petropolitanae* 8 (1760-61), pp. 64-73. This article, along with a translation by Mark Snavely and Phil Woodruff, is available online at the Euler Archive.

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[2] — (1761). Theoremata circa residua ex divisione potestatum relicta (E262). *Novi Commentarii academiae scientiarum Petropolitanae*, 7 (1758-59), pp. 49-82. This article, along with translations by Jordan Bell and Stella Li, is available online at the Euler Archive.

[3] — (1741). "Theorematum quorundam ad numeros primos spectantium demonstration" (E54). *Commentarii academiae scientiarum Petropolitanae*, 8 (1736), pp. 141-146. This article, along with a translation by David Zhao, is available online at the Euler Archive.