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Methodus universalis series summandi ulterius promota

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METHODVS VNIVERSALIS SERIES SVMMANDI VLTERIVS PROMOTA.

AVCTORE

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§. 1.

Methodus vniuerſalis ſeries ſummandi, quam exeunte praeterito anno expoſui, latiſſime quidem patet, cum ex ſolo termino generali ſeriei dato exhibeat formulam ſummae ſeriei aequalem; interim tamen difficulter ad eiusmodi ſeries, quarum termini generales algebraice exprimi non poſſunt, ſed vel exponentiales quantitates inuoluunt vel etiam tranſcendentes, accommodatur. Cum enim poſito termino X, cuius index eſt x, ſit ſumma ſeriei a primo termino vsque ad X aequalis $\int X\,dx + \frac{X}{1.2} + \frac{dX}{1.2.3.2\,dx} - \frac{d^3X}{1.2.3.4.5.6\,dx^3} +$ etc. facile apparet ſi X ſaltem huiumodi quantitates a^x inuoluat, tam expreſſionem $\int X\,dx$ quam differentialia ipſius X ad logarithmos deduci, vnde maxima oritur moleſtia in ſumma quaeſita ſaltem proxime aſſignanda.

§. 2. Praeterea etſi X fuerit quantitas algebraica, tamen ſaepius eius differentialia, quae ad ſummam obtinendam ſumi debent, tam fiunt complicata, vt non ſolum difficulter exhiberi queant, ſed etiam ſeriem non multum conuergentem praebeant, vti id euenit in ſerie $\frac{1}{3} + \frac{1}{35} + \frac{1}{99}$ etc. circuli quadraturam continente. Cuius difficultatis ratio in eo potiſſimum verſatur, quod indices terminorum vnitate crescentes assumsi, quas si alio numero crescentes sumsissem, terminus generalis X forte tractabilior prodiisset. Denique si terminus generalis X ne quidem exhiberi potest, vt id in plurimis seriebus accidit, tum data formula summam exhibens ne vsum quidem habere potest.

§. 3. His difficultatibus quomodo occurrere possem diu sum meditatus, tandemque obseruaui ex eodem principio, cuius ope illam formulam inuenissem, alias quoque formulas elici posse ad quasque series summandas idoneas; quibus exhibitis pro quauis oblata serie, ea formula sit eligenda, quae esset commodissima. Ex quouis autem huiusmodi formularum genere conueniens visum est, vt binae formulae tradantur, quarum altera apta sit ad series a termino primo ad datum vsque terminum summandas, cuiusmodi erat formula iam ante a me communicata, altera vero ad series a dato termino in infinitum vsque summandas. Quamuis enim haec posterior summatio ex priore fluat, tamen expediet pro hoc casu peculiarem formulam praebuisse.

§. 4. Incipiam igitur a seriebus, quarum terminus generalis algebraice potest exhiberi, pro quibus etiam methodus in praecedente schediasmate data inseruit; sed indices in progressione quacunque arithmetica progredientes assumam, quo formula inuenta latius pateat saepiusque commodiorem calculum suppeditet. Sit igitur series ab initio ad datum vsque terminum summanda cum indicibus sequens:

$\overset{a}{A}$

$$\overset{a}{A} + \overset{a+b}{B} + \overset{a+2b}{C} - - - - - - + \overset{x}{X} = S$$

vbi indices quantitate b crefcunt, primique termini A index eft a. Ponatur huius feriei fumma $= S$, in qua expreffione fi loco x fubftituatur $x-b$, perfpicuum eft eam exhibituram fummam eandem demto X feu fore aequalem ipfi $S-X$. At fi in S loco x ponatur $x-b$ tum prodibit $S - \frac{bdS}{1\,dx} + \frac{b^2 ddS}{1.2\,dx^2} - \frac{b^3 d^3 S}{1.2.3\,dx^3} +$ etc. vnde habitur fequens aequatio $X = \frac{bdS}{1.dx} - \frac{b^2 ddS}{1.2\,dx^2} + \frac{b^3 d^3 S}{1.2.3\,dx^3} - \frac{b^4 d^4 S}{1.2.3.4\,dx^4}$ $+$ etc. Ex hac vero aequatione elicitur ifta formula

$$S = \int \frac{Xdx}{b} + \frac{X}{1.2} + \frac{bdX}{1.2.3.2dx} - \frac{b^3 d^3 X}{1.2.3.4.5.6\,dx^3} + \frac{b^5 d^5 X}{1.2.3 - - - 7.6dx^5}$$
$$- \frac{3b^7 d^7 X}{1.2.3 - - - - 9.10dx^7} + \frac{5b^9 d^9 X}{1.2.3 - - - 11.6dx^9} - \frac{691 b^{11} d^{11} X}{1.2.3 - - - 13.210dx^{11}}$$
$$+ \frac{35 b^{13} d^{13} X}{1.2.3 - - - - 15.2dx^{13}} - \frac{3617 b^{15} d^{15} X}{1.2.3 - - - 17.30dx^{15}} + \frac{2423279 b^{17} d^{17} X}{1.2.3 - - - 19.1890dx^{17}}$$

etc. cui expreffioni tanta quantitas conftans eft addenda vt pofito $x=a$ fiat $S=A$, vel pofito $x=a-b$ fiat $S=0$.

§. 5. Si ponatur $X = x^n$, feu fi inuenienda fit fumma huius feriei $a^n + (a+b)^n + (a+2b)^n + - - - + x^n$, erit $\int X dx = \frac{x^{n+1}}{n+1}$; $\frac{dX}{dx} = nx^{n-1}$, $\frac{d^3 X}{dx^3} = n$ $(n-1)(n-2)x^{n-3}$ etc. Hinc ergo erit fumma quaefita

$$S = \frac{x^{n+1}}{(n+1)b} + \frac{x^n}{1.2} + \frac{nbx^{n-1}}{1.2.3.2} - \frac{n(n-1)(n-2)b^3 x^{n-3}}{1.2.3.4.5.6}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)b^5 x^{n-5}}{1.2.3.4.5.6.7.6} - \text{etc.} - \frac{a^{n+1}}{(n+1)b} +$$
$$+ \frac{a^n}{1.2} - \frac{nba^{n-1}}{1.2.3.2} + \frac{n(n-1)(n-2)b^3 a^{n-3}}{1.2.3.4.5.6} - \!\!-\!\!-$$

 n(n−1)

$\frac{n(n-1)(n-2)(n-3)(n-4)b^5 a^{n-6}}{1.2.3.4.5.6.7.6}$ + etc. adiecta debita constante. Summa ergo seriei $a+(a+b)+(a+2b)+ - - - + x$ erit $= \frac{x^2}{2b} + \frac{x}{2} + \frac{b}{12} - \frac{a^2}{2b} + \frac{a}{2} - \frac{b}{12} = \frac{x^2 - a^2 + bx + ab}{2b}$, atque summa seriei $a^2 + (a+b)^2 + (a+2b)^2 + - - - - + x^2 = \frac{x^3}{3b} + \frac{x^2}{2} + \frac{bx}{6} - \frac{a^3}{3b} + \frac{a^2}{2} - \frac{ab}{6} = \frac{2x^3 - 2a^3 + 3bx^2 + 3a^2b + b^2x - ab^2}{6b}$ Quae expressiones similes sunt eis, quas pro summis potestatum numerorum naturalium in superiori dissertatione dedi, atque ex iis quoque facile formantur.

§. 6. Sit nunc ad alteram huius generis formulam inueniendam series a dato termino X, cuius index sit x in infinitum vsque summanda, haec scilicet

$$\overset{x}{X} + \overset{x+b,}{Y} + \overset{x+2b}{Z} + \text{etc. infinitum} = S.$$

In summa ergo S si pro x scribatur $x+b$ prodibit $S-X$, erit adeo $X = \frac{-b\,dS}{1.dx} - \frac{b^2\,ddS}{1.2\,dx^2} - \frac{b^3 d^3 S}{1.2.3\,dx^3}$ — etc. vnde vt supra reperietur $S = -\int \frac{X\,dx}{b} + \frac{X}{1.2} - \frac{b\,dX}{1.2.3.2\,dx} + \frac{b^3 d^3 X}{1.2.3.4.5.6\,dx^3}$ $- \frac{b^5 d^5 X}{1.2.3 - - - 7.6\,dx^5} + \frac{3b^7 d^7 X}{1.2.3. - - - 9.10\,dx^7} - \frac{5b^9 d^9 X}{1.2.3 - 11.6\,dx^9}$ $+ \frac{691\,b^{11} d^{11} X}{1.2.3 - - - 13.210\,dx^{11}} - \frac{35\,b^{13} d^{13} X}{1.2.3 - - - - 15.2\,dx^{13}} + \frac{3617\,b^{15} d^{15} X}{1.2.3 - - - - 17.30\,dx^{15}}$ $- \frac{2423279\,b^{17} d^{17} X}{1.2.3 - - - - 19.1890\,dx^{17}}$ + etc. Cui formulae tanta constans est adiicienda vt fiat $S = 0$, si ponatur $x = \infty$; si enim terminus X iam fuerit infinitesimus seu vltimus in serie, summa debet esse euanescens, si quidem series finitam habeat summam, pro quo casu haec formula est accommodata.

§. 7. Quo vsus huius formulae appareat, sit $X = \frac{1}{x^2}$ seu ista series $\frac{1}{x^2} + \frac{1}{(x+b)^2} + \frac{1}{(x+2b)^2}$ + etc. in infinitum sum-

summanda erit ob $\int X\,dx = -\frac{1}{x}$, $\frac{dX}{dx} = \frac{-2}{x^3}$, $\frac{d^3X}{dx^3} = \frac{-2.3.4}{x^5}$ etc. $S = \frac{1}{bx} + \frac{1}{2x^2} + \frac{b}{6x^3} - \frac{b^3}{30x^5} + \frac{b^5}{42x^7} - \frac{b^7}{30x^9} + \frac{5b^9}{66x^{11}} - \frac{691b^{11}}{2730x^{13}} + \frac{7b^{13}}{6x^{15}} - \frac{3617b^{15}}{510x^{17}} + \frac{2423279b^{17}}{33910x^{19}} -$ etc. quae expressio constantem non requirit, cum per se euanescat posito $x = \infty$. Eo magis autem conuergit haec series, quo maior fuerit x respectu ipsius b. Quare si datae seriei aliquot termini initiales addantur actu, reliquorum hac methodo summa inuenta illi aggregato addita dabit summam propositae seriei in infinitum continuatae.

§. 8. Sed missis his, quae priorem regulam tantum commodiorem reddunt, progredior ad series summandas, ad quas illa formula non sufficit. Sit nimirum series summanda, in qua signa terminorum alternantur, vti

$$\overset{a,}{A} - \overset{a+b,}{B} + \overset{a+2b,}{C} - \overset{a+3b,}{D} - - - - - - + \overset{x,}{X} - \overset{x+b}{Y} = S$$

in qua serie cum sit Y talis functio ipsius $x+b$ qualis X est ipsius x, erit $Y = X + \frac{b\,dX}{1\,dx} + \frac{b^2\,ddX}{1.2.dx^2} + \frac{b^3d^3X}{1.2.3dx^3} +$ etc. Deinde si in S loco x ponatur $x-2b$ prodibit $S - X + Y$; erit ergo $-Y + X = \frac{2b\,dS}{1\,dx} - \frac{4b^2\,ddS}{1.2.dx^2} + \frac{8b^3\,d^3S}{1.2.3dx^3} - \frac{16b^4\,d^4S}{1.2.3.4.dx^4} + \frac{32b^5\,d^5S}{1.2.3.4.5dx^5} -$ etc. $= -\frac{b\,dX}{1\,dx} - \frac{b^2d^2X}{1.2\,dx^2} - \frac{b^3\,d^3X}{1.2.3\,dx^3} - \frac{b^4\,d^4X}{1.2.3.4dx^4} -$ etc. Ponatur $\frac{dS}{dx} = \frac{\alpha\,dX}{dx} + \frac{\beta\,d^2X}{dx^2} + \frac{\gamma\,d^3X}{dx^3} + \frac{\delta\,d^4X}{dx^4}$ etc. et comparandis terminis homologis prodibit $S = C - \frac{X}{2} - \frac{3b\,dX}{4dx} - \frac{b^2\,ddX}{2dx^2} - \frac{3b^3d^3X}{16\,dx^3} - \frac{b^4d^4X}{24dx^4} - \frac{9b^5d^5X}{1440.dx^5} -$ etc. et introducendo Y erit $S = C + Y + \frac{X}{2} + \frac{b\,dX}{1.2.2dx} - \frac{b^3\,d^3X}{1.2.3.4.2dx^3} + \frac{3b^5\,d^5X}{1.2.3.4.5.6.2dx^5}$ etc. Atque ideo erit A − B

+ C

$+ C - D + E - F - - - - - - + X = \text{Conſt.} + \frac{X}{2} + \frac{b\,dX}{1.2.2dx}$ $\frac{b^3 d^3 X}{1.2.3.4.2dx^3} + \frac{3b^5 d^5 X}{1.2.3.4.5.6.2dx^5}$ etc. vbi conſtans vt ante ita debet eſſe comparata, vt fiat haec ſumma $= A$ poſito $x = a$.

§. 9. Vlterius autem huius formulae terminis continuatis prodibit poſita ſumma huius ſeriei

$$\overset{a,}{A} - \overset{a+b,}{B} + \overset{a+2b,}{C} - \overset{a+3b,}{D} - - - - - - + \overset{x}{X} = S$$

$S = \text{Conſt.} + \frac{X}{1.2} + \frac{b\,dX}{1.2.2dx} - \frac{b^3 d^3 X}{1.2.3.4.2dx^3} + \frac{3b^5 d^5 X}{1.2.3.4.5.6.2dx^5} - \frac{17 b^7 d^7 X}{1.2.3.4.5.6.7.8.2dx^7} + \frac{155 b^9 d^9 X}{1.2.3.4.5.6.7.8.9.10.2dx^9} - \frac{2073 b^{11} d^{11} X}{1.2.3 - - - - - 12.2dx^{11}} + \frac{382276 b^{13} d^{13} X}{1.2.3. - - - - - - - 14.2dx^{13}} -$ etc. Si ergo quaerendus ſit valor huius progreſſionis $a^2 - (a+b)^2 + (a+2b)^2 - (a+3b)^2 +$ etc. $- - - - - - + x^2$, qui ſit S: erit ob $\frac{dX}{dx} = 2x$, $S = \text{Conſt.} + \frac{x^2}{2} + \frac{bx}{2}$. Conſtans vero C inuenietur ponendo $x = a$ eritque $S = \frac{a^2 - ab + x^2 + bx}{2}$. Exempli gratia erit $1 - 4 + 9 - 16 + 25 - - - - + 121 = 66$.

§. 10. Conſideremus nunc huiusmodi ſeriem in infinitum productam, ſcilicet

$$S = \overset{x,}{X} - \overset{x+b,}{Y} + \overset{x+2b}{Z} - \text{etc. in infinitum.}$$

Erit ergo $S + \frac{2b\,dS}{1.dx} + \frac{4b^2 ddS}{1.2dx^2} + \frac{8b^3 d^3 S}{1.2.3.dx^3} +$ etcet. $= S - X + Y$, ſeu $X - Y = -\frac{2b\,dS}{1.dx} - \frac{4b^2 ddS}{1.2dx^2}$ etc. vnde cum ſit $Y = X + \frac{b\,dX}{1.dx} + \frac{b^2 ddX}{1.2dx^2} + \frac{b^3 d^3 X}{1.2.3dx^3} +$ etc. inuenietur $S = \frac{X}{1.2} - \frac{b\,dX}{1.2.2dx} + \frac{b^3 d^3 X}{1.2.3.4.2dx^3} - \frac{3b^5 d^5 X}{1.2.3.4.5.6.2dx^5} + \frac{17 b^7 d^7 X}{1.2.3 - - - - 8.2dx^7} - \frac{155 b^9 d^9 X}{1.2.3 - - - 10.2dx^9} + \frac{2073 b^{11} d^{11} X}{1.2.3 - - - 12.2dx^{11}} - \frac{38227 b^{13} d^{13} X}{1.2.3 - 14.2dx^{13}}$ $+$ etc.

$+$ etc. $+$ Conſt. Quae conſtans ita debet eſſe comparata vt fiat $S = 0$ poſito $x = \infty$. Huius igitur formulae ope plurimae ſeries lente conuergentes, in quibus terminorum ſigna alternantur, valde prope ſummari poterunt.

§. 11. Sit $X = \frac{1}{x}$, ita vt inueniri debeat ſumma huius feriei $\frac{1}{x} - \frac{1}{x+b} + \frac{1}{x+2b} - \frac{1}{x+3b} +$ etc. in infinitum. Cum ergo ſit $X = \frac{1}{x}$ erit $\frac{dX}{dx} = \frac{-1}{x^2}$, $\frac{d^3X}{dx^3} = \frac{-2.3}{x^4}$, $\frac{d^5X}{dx^5} = \frac{-2.3.4.5}{x^6}$ hincque fiet $S = \frac{1}{2x} + \frac{b}{2.2x^2} - \frac{b^3}{4.2x^4} + \frac{3b^5}{6.2x^6} - \frac{17b^7}{8.2x^8} + \frac{155b^9}{10.2x^{10}} - \frac{2073b^{11}}{12.2x^{12}} + \frac{38227b^{13}}{14.2x^{14}}$ etc. vbi conſtantis additione non eſt opus. Vt ſit $b = 2$ atque $x = 25$, erit ſeriei $\frac{1}{25} - \frac{1}{27} + \frac{1}{29} - \frac{1}{31} + \frac{1}{33} -$ etc. ſumma $= \frac{2}{100} + \frac{8}{100^2} - \frac{256}{100^4} + \frac{8.4^6}{100^6} - \frac{17.4^8}{100^8} + \frac{31.128.4^{10}}{100^{10}} - \frac{691.256.4^{12}}{100^{12}} + \frac{5461.2048.4^{14}}{100^{14}}$ etc. $= 0,0207974719 18$ q. pr. ad quam ſi praecedentium terminorum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ etc. $- - - - - \frac{1}{23}$ actu inuenta ſumma addatur, prodibit quarta peripheriae pars radio exiſtente $= 1$.

§. 12. Hae iam formulae hactenus traditae, quo facile expediri queant calculo, requirunt, vt X ſit functio algebraica ipſius x, cuius differentialia cuiusque gradus commode exhiberi queant. Vix enim vel ne vix quidem iſtae formulae in vſum vocari poſſent, ſi huiusmodi quantitas exponentialis n^x in termino generali progreſſionis ineſſet. Pro huiusmodi ergo progreſſionibus, hoc termino generali Xn^x, vbi X vt ante denotat functionem algebraicam ipſius x, contentis peculia-

 res

res formulas summatrices erui conueniet. Sit itaque ista Series

$$\overset{\alpha,}{A}n^a + \overset{\alpha+\beta.}{B}n^{a+b} + \overset{\alpha+2b}{C}n^{a+2b} + \cdots \overset{x}{X}n^x$$

ad summandum proposita; ponaturque summa $= Sn^x$. Haec formula autem posito $x-b$ loco x abibit in hanc $n^{x-b}(S - \frac{b\,dS}{1\,dx} + \frac{b^2 d^2 S}{1.2dx^2} - \frac{b^3 d^3 S}{1.2.3dx^3} + \text{etc.})$ quae aequalis esse debet priori summae Sn^x demto vltimo termino Xn^x. Habebitur ergo ista aequatio $Sn^b - Xn^b = S - \frac{b\,dS}{1.dx} + \frac{b^2 d^2 S}{1.2dx^2} - \frac{b^3 d^3 S}{1.2.3dx^3} + \frac{b^4 d^4 S}{1.2.3.4dx^4} - \text{etc.}$ Ex qua aequatione valor ipsius S erui debet.

§. 13. Ponatur igitur $n^b = m$, eritque $S = \frac{mX}{m-1} - \frac{\alpha b\,dX}{a(m-1)^2 dx} + \frac{\beta b^2 ddX}{1.2(m-1)^3 dx^2} - \frac{\gamma b^3 d^3 X}{1.2.3(m-1)^4 dx^3} + \frac{\delta b^4 d^4 X}{1.2.3.4(m-1)^5 dx^4} - \frac{\varepsilon b^5 d^5 X}{1.2.3.4.5(m-1)^6 dx^5} + \text{etc.}$ Hinc terminis homologis comparandis posito breuitatis ergo $m-1=p$ vt sequitur

$\alpha = m$

$\beta = 2\alpha + mp$

$\gamma = 3\beta + 3\alpha p + mp^2$

$\delta = 4\gamma + 6\beta p + 4\alpha p^2 + mp^3$

$\varepsilon = 5\delta + 10\gamma p + 10\beta p^2 + 5\alpha p^3 + mp^4$ etc.

vnde pro litteris $\alpha, \beta, \gamma, \delta$ etc. sequentes obtinentur valores:

$\alpha = m$

$\beta = 2m + mp$

$\gamma = 6m + 6mp + mp^2$

$\delta = 24m + 36mp + 14mp^2 + mp^3$

$\varepsilon = 120m + 240mp + 150mp^2 + 30mp^3 + mp^4$ etc.

hic

hic quilibet numerus est multiplum suprascripti cum praecedente iuncti, vti

$$30 = 2(1 + 14)$$
$$150 = 3(14 + 36)$$
$$240 = 4(36 + 24)$$
$$120 = 5(24 + 0)$$

vel restituto $m - 1$ loco p, vti sequitur:

1. $\alpha = m$
2. $\beta = m + m^2$
3. $\gamma = m + 4m^2 + m^3$
4. $\delta = m + 11\, m^2 + 11\, m^3 + m^4$
5. $\varepsilon = m + 26\, m^2 + 66\, m^3 + 26\, m^4 + m^5$
6. $\zeta = m + 57\, m^2 + 302\, m^3 + 302\, m^4 + 57\, m^5 + m^6$ etc.

qui valores ita progrediuntur, vt sit coefficiens ψ, cuius index est k, $= m + (2^k - \frac{(k+1)}{1}) m^2 + (3^k - \frac{(k+1)}{1} 2^k + \frac{(k+1)k}{1.2}) m^3 + (4^k - \frac{(k+1)}{1} 3^k + \frac{(k+1)k}{1.2} 2^k - \frac{(k+1)k(k-1)}{1.\ 2.\ 3}) m^4 + \text{-------} + m^k$.

§. 14. Ex his ergo colligitur seriei propositae

$$\overset{a,}{A} n^a + \overset{a+b,}{B} n^{a+b} + \overset{a+2b}{C} n^{a+2b} \text{------} + \overset{x}{X} n^x$$

summa

$$= n^x \left(\frac{n^b X}{n^b - 1} - \frac{n^b b\, dX}{1 (n^b - 1)^2 dx} + \frac{(n^{2b} + n^b) b^2 ddX}{1.2 (n^b - 1)^3 dx^2} - \frac{(n^{3b} + 4 n^{2b} + n^b) b^3 d^3 X}{1.\ 2.\ 3 (n^b - 1)^4 dx^3} + \frac{(n^{4b} + 11 n^{3b} + 11 n^{2b} + n^b) b^4 d^4 X}{1.\ 2.\ 3.\ 4 (n^b - 1)^5 dx^4} - \text{etc.} \right)$$
$$+ \text{Const.}$$

$+$ Conft. quae ita debet effe comparata vt pofito $x = a$ fiat fumma $= A n^a$. Si ponatur $n^b = -1$, abibit feries in pure algebraicam, in qua figna terminorum alternantur; hincque eadem formula ponendo -1 loco n^b, refultat, quam pro eodem cafu iam fupra §. 9. inuenimus. Seriei vero in infinitum excurrentis

$$\overset{x}{X} n^x + \overset{x+b}{Y} n^{x+b} + \overset{x+2b}{Z} n^{x+2b} + \text{etc.}$$

in infinitum fumma eft

$$= n^x \left(\frac{-X}{n^b - 1} + \frac{n^b b\, dX}{1(n^b-1)^2 dx} - \frac{(n^{2b}+n^b) b^2\, ddX}{1.2(n^b-1)^3 dx^2} + \frac{(n^{3b}+4n^{2b}+n^b) b^3 d^3 X}{1.2.3(n^b-1)^4 dx^3} - \frac{(n^{4b}+11n^{3b}+11n^{2b}+n^b) b^4 d^4 X}{1.2.3.4(n^b-1)^5 dx^4} + \text{etc.} \right) + \text{Conft.}$$

quae conftans ita debet effe comparata vt pofito $x = \infty$ fumma fiat $= 0$: quod quidem femper accidere folet per fe, ita vt conftante non fit opus, fi quidem feries conuergit, fummamque habet finitam.

§. 15. Ex priore formula pro huiusmodi feriebus ad datum vsque terminum fummandis intelligitur, fi X fuerit functio algebraica talis ipfius x, vt tandem altiora eius differentialia euanefcant, tum terminum fummatorium reuera exhiberi poffe. Quamobrem fi feries cuius terminus generalis eft X fuerit fummabilis, tum quoque feries, cuius terminus generalis eft $X n^x$, erit fummabilis. Sic propofita ferie $a^2 n^a + (a+b)^2 n^{a+b} + (a+2b)^2 n^{a+2b} - - - - - + x^2 n^x$ erit eius fumma $= x^n$

$(n^b x^2$

$\left(\frac{n^b x^2}{n^b - 1} - \frac{2 n^b b x}{(n^b - 1)^2} + \frac{(n^{2b} + n^b) b^2}{(n^b - 1)^3}\right)$ + Const. quae constans facto $x = a$ et summa $= a^2 n^a$ prodibit $= n^a$ $\left(a^2 - \frac{n^b a^2}{n^b - 1} + \frac{2 n^b a b}{(n^b - 1)^2} - \frac{(n^{2b} + n^b) b^2}{(n^b - 1)^3}\right)$.

§. 16. Altera praeterea formula ingentem praestat vtilitatem in seriebus infinitis summandis, quod vt clarius pateat proposita sit series $\frac{n^x}{x} + \frac{n^{x+2}}{x+2} + \frac{n^{x+4}}{x+4} + \frac{n^{x+6}}{x+6} +$ etc. cuius summa sit $= S$. Erit ergo $b = 2$, et $X = \frac{1}{x}$, vnde fiet summa $S = n^x \left(\frac{1}{(n^2 - 1) x} - \frac{2 n^2}{(n^2 - 1)^2 x^2} - \frac{4 (n^4 + n^2)}{(n^2 - 1)^3 x^3} - \frac{8 (n^6 + 4 n^4 + n^2)}{(n^2 - 1)^4 x^4} - \frac{16 (n^8 + 11 n^6 + 11 n^4 + n^2)}{(n^2 - 1)^5 x^5} - \text{etcet.}\right)$ Sit nunc $x = 25$ et $n^2 = -\frac{1}{3}$ seu $n = \frac{1}{\sqrt{-3}}$, erit $\frac{1}{\sqrt{-3}}) \frac{1}{25 \cdot 3^{12}} - \frac{1}{27 \cdot 3^{13}} + \frac{1}{29 \cdot 3^{14}} - \frac{1}{31 \cdot 3^{15}} + \text{etcet.}) = \frac{1}{3^{12} \sqrt{-3}} \left(\frac{3}{4 \cdot 25} + \frac{3}{8 \cdot 25^2} - \frac{3}{8 \cdot 25^3} - \frac{3}{16 \cdot 25^4} + \frac{15}{8 \cdot 25^5} \text{ etc.}\right)$. Cum igitur in circulo radii $= 1$ sit arcus triginta graduum $= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} - \text{etc.}\right)$, si horum terminorum actu 12 addantur, erunt sequentes reliqui $\frac{1}{\sqrt{3}}\left(\frac{1}{25 \cdot 3^{12}} - \frac{1}{27 \cdot 3^{13}} + \frac{1}{29 \cdot 3^{14}} - \text{etc.}\right) = \frac{\sqrt{3}}{3^{12}}\left(\frac{1}{4 \cdot 25} + \frac{1}{8 \cdot 25^2} - \frac{1}{8 \cdot 25^3} - \frac{1}{16 \cdot 25^4} + \frac{5}{8 \cdot 25^5} + \text{etc.}\right)$

§. 17. Si termini seriei summandae ex factoribus fuerint compositi ita vt series huiusmodi habeat formam.

$$\overset{a}{A} + \overset{a+b}{AB} + \overset{a+2b}{ABC} + \cdots + \overset{x}{ABC \ldots VX}$$

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ponatur summa $= S. ABC - - - - - VX$. Abibit ergo haec summa posito $x-b$ loco x in hanc $ABC - - V$ $(S - \frac{b\,dS}{1.dx} + \frac{b^2\,ddS}{1.2.dx^2} - \frac{b^3\,d^3S}{1.2.3.dx^3}$ etc.) quae aequalis esse debet priori summae demto termino vltimo, id est huic quantitati $AB - - - - V(SX - X)$. Hanc ob rem habebitur ista aequatio $SX - X = S - \frac{b\,dS}{1.dx} + \frac{b^2\,ddS}{1.2\,dx^2} - \frac{b^3\,d^3S}{1.2.3\,dx^3}$ $\frac{b^4\,d^4S}{1.2.3.4\,dx^4} -$ etc.

Ex hac aequatione valor ipsius S erutus erit iste $S = \frac{X}{X-1} + E + F + G +$ etc. qui termini ita progrediuntur, vt posito $\frac{X}{X-1} = D$ fit $E = \frac{-b\,dD}{(X-1)\,dx}$; $F = \frac{-b\,dE}{1(X-1)\,dx} + \frac{b^2\,ddD}{1.2(X-1)\,dx^2}$; $G = \frac{-b\,dF}{1(X-1)\,dx} + \frac{b^2\,ddE}{1.2(X-1)\,dx^2} - \frac{b^3\,d^3D}{1.2.3(X-1)\,dX^3}$; atque ita porro. Adeo vt summa seriei propositae fit $= ABC - - - - - VX(D + E + F + G - - - - -$ etc.). Seriei vero in infinitum continuatae

$$\overset{x}{A\,B}\,C - - - VX + A\overset{x+b}{B}CD - - - - VXY + \text{etc.}$$

summa erit $= ABC - - - - - VX(1 - D - E - F - G - - - - -$ etc.) $+$ Const. Vt si quaeratur summa huius seriei $\frac{1}{1.2.3 - - - - - x} + \frac{1}{1.2.3 - - - - x(x+1)} +$ etc. in infinitum erit $b = 1$, $X = \frac{1}{x}$, atque $D = \frac{1}{1-x}$. Tum vero $E = \frac{-x}{(1-x)^3}$; $F = \frac{x(2+x)}{(1-x)^5}$ etc. summa ergo seriei propositae ob constantem euanescentem erit $= \frac{1}{1.2.3 - - x}(1 + \frac{1}{x-1} - \frac{x}{(x-1)^3} + \frac{x(x+2)}{(x-1)^5}$ etc.). Ex his vero traditis facile intelligitur, cuiusmodi summae formam assumere oporteat in quouis casu oblato, quo summa minimo labore inueniatur.

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