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Euler and the Duplication Formula for the Gamma-Function

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Abstract

We show how the formulas in Euler's paper "Variae considerationes circa series hypergeometricas" [4] imply Legendre's duplication formula for the Γ-function. This paper can be seen as an Addendum to [2].

1 Introduction

In [2], we focused on a function defined by Euler in [4] as:

$$\Gamma_E(x) := a \cdot (a+b) \cdot (a+2b) \cdot (a+3b) \cdot \dots \cdot (a+(x-1)b) \quad \text{for} \quad a, b > 0, \tag{1}$$

which we showed to be continueable to non-integer values of x via the expression:

$$\Gamma_E(x) = \frac{b^x}{\Gamma\left(\frac{a}{b}\right)} \cdot \Gamma\left(x + \frac{a}{b}\right). \tag{2}$$

Here, $\Gamma(x)$ means the ordinary Γ-function defined as:

$$\Gamma(x) := \int\limits_0^\infty e^{-t} t^{x-1} dt \quad \text{for} \quad \mathrm{Re}(x) > 0. \tag{3}$$

Equation (1) enabled us to determine the constant A in the asymptotic expansion for the function Γ_E found by Euler via the Euler-Maclaurin summation formula. The asymptotic expansions reads:

$$\Gamma_E(x) \sim A \cdot e^{-x} \cdot (a-b+bx)^{\frac{a}{b}+x-\frac{1}{2}} \quad \text{for} \quad x \to \infty. \tag{4}$$

We found the constant A to be

$$A = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{a}{b}\right)} \cdot e^{1-\frac{a}{b}} \cdot b^{\frac{1}{2}-\frac{a}{b}}. \tag{5}$$

In this paper, we intend to use this result and more of Euler's formulas from the same paper to show that they imply the Legendre duplication formula for the Γ-function, i.e., the relation

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}} \cdot \Gamma\left(\frac{x}{2}\right) \cdot \Gamma\left(\frac{x}{2} + \frac{1}{2}\right). \tag{6}$$

2 Euler's other Functions

2.1 Euler's Definition

Aside from the function Γ_E, in his paper [4], Euler introduced two other related functions:

$$\begin{aligned} \Delta(x) &= a \cdot (a+2b) \cdot (a+4b) \cdot (a+6b) \cdot \dots \cdot (a+(2x-2)b), \\ \Theta(x) &= (a+b) \cdot (a+3b) \cdot (a+5b) \cdot \dots \cdot (a+(2x-1)b). \end{aligned} \tag{7}$$

As it was the case for Γ_E (equation (1)), Euler's definition is only valid for integer values of x, but by using the ideas from [2], we could extend the definition to real numbers.

2.2 Asymptotic Expansions of these Functions

Furthermore, Euler also found asymptotic expansions for his functions Δ and Θ. They are:

$$\begin{aligned} \Delta(x) &\sim B \cdot e^{-x} \cdot (a-2b+2bx)^{\frac{a}{2b}+x-\frac{1}{2}} \\ \Theta(x) &\sim C \cdot e^{-x} \cdot (a-b+2bx)^{\frac{a}{2b}+x}, \end{aligned} \tag{8}$$

where B and C are constants resulting from the application of the Euler-Maclaurin summation formula and the asymptotic expansions are valid for $x \to \infty$.

2.3 Relation among the Constants

Euler was not able to find any of the constants A, B and C. But, using the general relations among his functions Γ_E, Δ and Θ and the respective corresponding asymptotic expansions, he found the following relations:

$$A = \frac{B \cdot C}{\sqrt{e}} \tag{9}$$

and

$$B = C \cdot k \cdot \sqrt{e} \tag{10}$$

with $k = \Delta\left(\frac{1}{2}\right)$. As we will show in the next section, these relations imply the Legendre duplication formula (equation (6)).

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3 Derivation of the Legendre Duplication Formula from Euler's Formulas

As Euler remarked himself in [4], equations (9) and (10) tell us that we only need to find one of the constants A, B and C such that we can calculate the remaining two from the first. Since we discovered the value A (equation (5)), we could do precisely that. But for our task at hand, we need to find the value of k first.

3.1 Evaluation of the Constant k

To evaluate $k = \Delta\left(\frac{1}{2}\right)$, we note that we just have to make the substitution $b \mapsto 2b$ in equation (1) such that the expression for Γ_E goes over into the expression for Δ (equation (8)) in equation (7). Making the same substitution in equation (2), we arrive the the following expression for $\Delta(x)$:

$$\Delta(x) = \frac{(2b)^x}{\Gamma\left(\frac{a}{2b}\right)} \cdot \Gamma\left(x + \frac{a}{2b}\right).$$

Therefore, for $x = \frac{1}{2}$

$$k = \Delta\left(\frac{1}{2}\right) = \frac{(2b)^{\frac{1}{2}}}{\Gamma\left(\frac{a}{2b}\right)} \cdot \Gamma\left(\frac{1}{2} + \frac{a}{2b}\right). \tag{11}$$

3.2 The Legendre Duplication Formula

Having found k, let us use equations (9) and (10) to find the Legendre duplication formula (equation (6)). Substituting the value for C in (10) in for the value of C in (9), we arrive at this equation:

$$A = \frac{B^2}{\Delta\left(\frac{1}{2}\right)} e^{-1}. \tag{12}$$

Next, we note that since $\Delta(x)$ is obtained from $\Gamma_E(x)$ by the substitution $b \mapsto 2b$, the value of the constant B is obtained in the same way from A and reads:

$$B = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{a}{2b}\right)} \cdot (2b)^{\frac{1}{2} - \frac{a}{2b}} \cdot e^{1 - \frac{a}{2b}}. \tag{13}$$

Thus, substituting the respective values for A (equation (5)), B (equation (13)) and k (equation (11)), equation (12) becomes:

$$\frac{\sqrt{2\pi}}{\Gamma\left(\frac{a}{b}\right)} \cdot e^{1-\frac{a}{b}} \cdot b^{\frac{1}{2} - \frac{a}{b}} = \frac{\left(\frac{\sqrt{2\pi}}{\Gamma\left(\frac{a}{2b}\right)} \cdot (2b)^{\frac{1}{2} - \frac{a}{2b}} \cdot e^{1 - \frac{a}{2b}}\right)^2}{\frac{(2b)^{\frac{1}{2}}}{\Gamma\left(\frac{a}{2b}\right)} \cdot \Gamma\left(\frac{1}{2} + \frac{a}{2b}\right)} \cdot e^{-1}.$$

Most terms cancel each other and after this equation simplifies to:

$$\frac{1}{\Gamma\left(\frac{a}{b}\right)} = \frac{\sqrt{2\pi}\cdot 2^{\frac{1}{2}-\frac{a}{b}}}{\Gamma\left(\frac{a}{2b}\right)\cdot\Gamma\left(\frac{1}{2}+\frac{a}{2b}\right)}.$$

Finally, writing x instead of $\frac{a}{b}$ and solving this equation for $\Gamma(x)$, after a little simplification, we arrive at the relation:

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\cdot\Gamma\left(\frac{x}{2}\right)\cdot\Gamma\left(\frac{x+1}{2}\right),$$

which is the Legendre duplication formula for the Γ-function (equation (6)), as we wanted to show.

4 Conclusion

In this note we showed that Legendre's duplication formula, i.e., equation (6) follows from Euler's formulas found in his paper [4]. Indeed, the Legendre duplication formula could also have been shown by Euler himself, if he had set this task for himself, as we argued in more detail in [2]. Furthermore, Euler's ideas that we explained in this and the before-mentioned paper, can be generalized to show the multiplication formula for the Γ-function, i.e, the formula

$$\Gamma(x) = \sqrt{\frac{n}{(2\pi)^{n-1}}}\cdot n^{x-1}\cdot\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdot\cdots\cdot\Gamma\left(\frac{x+n-1}{n}\right).$$

This formula is attributed to Gauss who stated and proved it in [5]. But it was given by Euler (in different form, expressed via Beta functions) in [3], as we demonstrated in [1].

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