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Alexander Aycock

Johannes Gutenberg Universitat, Mainz, aaycock@students.uni-mainz.de

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Euler and the Duplication Formula for the Gamma-Function

Alexander Aycock, Johannes-Gutenberg University Mainz
Staudinger Weg 9, 55128 Mainz
aaycock@students.uni-mainz.de

Abstract

We show how the formulas in Euler’s paper "Variae considerationes circa series hypergeometricas" [4] imply Legendre’s duplication formula for the Γ-function. This paper can be seen as an Addendum to [2].

1 Introduction

In [2], we focused on a function defined by Euler in [4] as:

\[ \Gamma_E(x) := a \cdot (a + b) \cdot (a + 2b) \cdot (a + 3b) \cdots (a + (x - 1)b) \quad \text{for} \quad a, b > 0, \quad (1) \]

which we showed to be continueable to non-integer values of \( x \) via the expression:

\[ \Gamma_E(x) = b^x \frac{\Gamma\left(\frac{a}{b}\right)}{\Gamma\left(x + \frac{a}{b}\right)}. \quad (2) \]

Here, \( \Gamma(x) \) means the ordinary Γ-function defined as:

\[ \Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt \quad \text{for} \quad \text{Re}(x) > 0. \quad (3) \]

Equation (1) enabled us to determine the constant \( A \) in the asymptotic expansion for the function \( \Gamma_E \) found by Euler via the Euler-Maclaurin summation formula. The asymptotic expansions reads:

\[ \Gamma_E(x) \sim A \cdot e^{-x} \cdot (a - b + bx)^{\frac{a}{b} + x - \frac{1}{2}} \quad \text{for} \quad x \to \infty. \quad (4) \]

We found the constant \( A \) to be

\[ A = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{a}{b}\right)} \cdot e^{1 - \frac{a}{b}} \cdot b^{\frac{x}{2} - \frac{a}{b}}. \quad (5) \]
In this paper, we intend to use this result and more of Euler’s formulas from the same paper to show that they imply the Legendre duplication formula for the $\Gamma$-function, i.e., the relation

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}} \cdot \Gamma\left(\frac{x}{2}\right) \cdot \Gamma\left(\frac{x + 1}{2}\right).$$  \(6\)

2 Euler’s other Functions

2.1 Euler’s Definition

Aside from the function $\Gamma_E$, in his paper [4], Euler introduced two other related functions:

$$\Delta(x) = a \cdot (a + 2b) \cdot (a + 4b) \cdot (a + 6b) \cdot \cdots \cdot (a + (2x - 2)b),$$
$$\Theta(x) = (a + b) \cdot (a + 3b) \cdot (a + 5b) \cdot \cdots \cdot (a + (2x - 1)b).$$ \(7\)

As it was the case for $\Gamma_E$ (equation (1)), Euler’s definition is only valid for integer values of $x$, but by using the ideas from [2], we could extend the definition to real numbers.

2.2 Asymptotic Expansions of these Functions

Furthermore, Euler also found asymptotic expansions for his functions $\Delta$ and $\Theta$. They are:

$$\Delta(x) \sim B \cdot e^{-x} \cdot (a - 2b + 2bx)^{\frac{a}{2} + x - \frac{1}{2}}$$
$$\Theta(x) \sim C \cdot e^{-x} \cdot (a - b + 2bx)^{\frac{a}{2} + x},$$ \(8\)

where $B$ and $C$ are constants resulting from the application of the Euler-Maclaurin summation formula and the asymptotic expansions are valid for $x \to \infty$.

2.3 Relation among the Constants

Euler was not able to find any of the constants $A$, $B$, and $C$. But, using the general relations among his functions $\Gamma_E$, $\Delta$ and $\Theta$ and the respective corresponding asymptotic expansions, he found the following relations:

$$A = \frac{B \cdot C}{\sqrt{e}}$$ \(9\)

and

$$B = C \cdot k \cdot \sqrt{e}$$ \(10\)

with $k = \Delta\left(\frac{1}{2}\right)$. As we will show in the next section, these relations imply the Legendre duplication formula (equation (6)).
3 Derivation of the Legendre Duplication Formula from Euler’s Formulas

As Euler remarked himself in [4], equations (9) and (10) tell us that we only need to find one of the constants A, B and C such that we can calculate the remaining two from the first. Since we discovered the value A (equation (5)), we could do precisely that. But for our task at hand, we need to find the value of k first.

3.1 Evaluation of the Constant k

To evaluate $k = \Delta \left( \frac{1}{2} \right)$, we note that we just have to make the substitution $b \rightarrow 2b$ in equation (1) such that the expression for $\Gamma_E$ goes over into the expression for $\Delta$ (equation (8)) in equation (7). Making the same substitution in equation (2), we arrive the the following expression for $\Delta(x)$:

$$\Delta(x) = \frac{(2b)^x}{\Gamma \left( \frac{2b}{b} \right)} \cdot \Gamma \left( x + \frac{a}{2b} \right).$$

Therefore, for $x = \frac{1}{2}$

$$k = \Delta \left( \frac{1}{2} \right) = \frac{(2b)^{\frac{1}{2}}}{\Gamma \left( \frac{2b}{2b} \right)} \cdot \Gamma \left( \frac{1}{2} + \frac{a}{2b} \right). \quad (11)$$

3.2 The Legendre Duplication Formula

Having found k, let us use equations (9) and (10) to find the Legendre duplication formula (equation (6)). Substituting the value for C in (10) in for the value of C in (9), we arrive at this equation:

$$A = \frac{B^2}{\Delta \left( \frac{1}{2} \right)} \cdot e^{-1}. \quad (12)$$

Next, we note that since $\Delta(x)$ is obtained from $\Gamma_E(x)$ by the substitution $b \rightarrow 2b$, the value of the constant $B$ is obtained in the same way from $A$ and reads:

$$B = \frac{\sqrt{2\pi}}{\Gamma \left( \frac{4b}{2b} \right)} \cdot (2b)^{\frac{1}{2}-\frac{a}{2b}} \cdot e^{1-\frac{a}{2b}}. \quad (13)$$

Thus, substituting the respective values for $A$ (equation (5)), $B$ (equation (13)) and $k$ (equation (11)), equation (12) becomes:

$$\frac{\sqrt{2\pi}}{\Gamma \left( \frac{b}{b} \right)} \cdot e^{1-\frac{a}{2b}} \cdot (2b)^{\frac{1}{2}-\frac{a}{2b}} = \left( \frac{\sqrt{2\pi}}{\Gamma \left( \frac{2b}{2b} \right)} \cdot (2b)^{\frac{1}{2}-\frac{a}{2b}} \cdot e^{1-\frac{a}{2b}} \right)^2 \cdot e^{-1}. \quad (13)$$
Most terms cancel each other and after this equation simplifies to:

\[
\frac{1}{\Gamma\left(\frac{a}{b}\right)} = \sqrt{2\pi} \cdot 2^{\frac{1}{2} - \frac{a}{b}} \cdot \frac{\Gamma\left(\frac{a}{b}\right)}{\Gamma\left(\frac{1}{2} + \frac{a}{b}\right)}.
\]

Finally, writing \( x \) instead of \( \frac{a}{b} \) and solving this equation for \( \Gamma(x) \), after a little simplification, we arrive at the relation:

\[
\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}} \cdot \Gamma\left(\frac{x}{2}\right) \cdot \Gamma\left(\frac{x+1}{2}\right),
\]

which is the Legendre duplication formula for the \( \Gamma \)-function (equation (6)), as we wanted to show.

4 Conclusion

In this note we showed that Legendre’s duplication formula, i.e., equation (6) follows from Euler’s formulas found in his paper [4]. Indeed, the Legendre duplication formula could also have been shown by Euler himself, if he had set this task for himself, as we argued in more detail in [2]. Furthermore, Euler’s ideas that we explained in this and the before-mentioned paper, can be generalized to show the multiplication formula for the \( \Gamma \)-function, i.e, the formula

\[
\Gamma(x) = \frac{n}{(2\pi)^{n-1}} \cdot n^{x-1} \cdot \Gamma\left(\frac{x}{n}\right) \cdot \Gamma\left(\frac{x+1}{n}\right) \cdot \ldots \cdot \Gamma\left(\frac{x+n-1}{n}\right).
\]

This formula is attributed to Gauss who stated and proved it in [5]. But it was given by Euler (in different form, expressed via Beta functions) in [3], as we demonstrated in [1].

References


[3] Euler, L. (1772). "Evolutio formulae integralis \( \int x^{f-1} dx (\log(x))^m \) integratione a valore \( x = 0 \) ad \( x = 1 \) extensa" (E421). Novi Commentarii academiae scientiarum Petropolitanae 16 (1772): pp. 91-139. Reprinted in Opera Omnia: Series 1, Volume 34, p. 1050.
17, pp. 316 – 357. Original text available online at https://scholarlycommons.pacific.edu/euler/


[5] Gauss, C. (1813). "Disquisitiones generales circa seriem infinitam 1 + \frac{a \cdot b}{1 \cdot 2} x + \frac{a(a+1) \cdot b(b+1)}{1 \cdot 2 \cdot 3(c+1)} x^2 + \ldots". Commentationes societatis regiae scientiarum Gottengensis recentiores Vol. II. (1812), Göttingen. Reprinted in: Carl Friedrich Gauss, Werke, Volume 3, pp. 123–166.