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Analytical Observations (Translation of E326)

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Analytical Observations[a]

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Foreword by Translator

Euler, in this publication with Eneström number E326, provides an induction fallacy which arises from analyzing a particular sequence. Euler wrote this work in 1763, one of only two papers he wrote on sequences and/or series in the 1760's, out of a total of 79 papers on series during his career. His goal in E326 is to investigate the middle terms in the expansion of powers of quadratic trinomial expressions, beginning with the specific simple quadratic $1+x+xx$, before considering the general quadratic $a+bx+cxx$.

The induction fallacy shows up during the analysis of the simple case when Euler first finds an explicit formula for the middle terms, now known as central trinomial coefficients (see the Online Encyclopedia of Integer Sequences, https://oeis.org/A002426). He then investigates a recursive formula which involves pronic and Fibonacci numbers, resulting in two integer sequences which agree for the first nine terms and then disagree from the tenth term onward. [C. Edward Sandifer, *How Euler Did It*, Mathematical Association of America, 2007, p. 143-146.]

Translation

By considering the powers, which arise from the raising of this trinomial form $1+x+xx$, it is seen that the middle terms are revealed to have the greatest numerical coefficients, of which the continuation of the progression, although this is fairly easy to see, yet seems

[a] E326, Original Title *Observationes analyticae.* Published in *Commentarii academiae scientiarum Petropolitanae* 11, 1765, pp. 124-143 and *Opera Omnia*: Series 1, Volume 15, pp. 50 – 69, https://scholarlycommons.pacific.edu/euler-works/324/

[b] The translator wishes to thank the anonymous reviewer for their many helpful suggestions.

worthy of full attention; especially as such speculations generally lead in Analysis to results not to be ignored. First therefore I will explain some simple things to notice about these powers:

Exponent of the power	Powers expanded
0	1
1	$1+x+x^2$
2	$1+2x+3x^2+2x^3+x^4$
3	$1+3x+6x^2+7x^3+6x^4+3x^5+x^6$
4	$1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$
5	$1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$
	etc.

Hence if the middle terms from each of the powers are displayed in order, this progression arises:

$$1, 1x, 3x^2, 7x^3, 19x^4, 51x^5, 141x^6 \text{ etc.}$$

It is not unreasonable to seek out the rules by which these numbers proceed, so that not only the general term or the appropriate coefficient of the indefinite position x^n may be known, but also that the notable qualities of this series[c] may be explored. Towards this goal, I propose the following problems, whose resulting solutions successively lead to other considerations no less interesting.

Problem 1.

To determine the coefficient of the middle term or of the position x^n in $(1+x+xx)^n$ having been expanded by the general power *n*.

Solution.

The proposed power may be represented with the form of the binomial $\left(x(1+x)+1\right)^n$, which, in the usual way, provides the expansion:

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[c] Euler uses the Latin word *series* which could be translated as "series" or "sequence". Although in mathematics today Euler's list would be called a "sequence", "series" will be used in this translation to maintain historical accuracy.

$$x^n(1+x)^n+\frac{n}{1}x^{n-1}(1+x)^{n-1}+\frac{n(n-1)}{1\cdot 2}x^{n-2}(1+x)^{n-2}$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}(1+x)^{n-3}, \text{ etc.}$$

Out of which each member, if expanded further, the term of the form x^n is required to be elicited. The first member produces x^n, while all remaining higher powers of x will emerge from further expansion. From the second member for this power, x^n is derived:

$$\frac{n}{1}x^{n-1}\cdot\frac{n-1}{1}x=\frac{n(n-1)}{1\cdot 1}x^n$$

from the third member in a similar manner we obtain:

$$\frac{n(n-1)}{1\cdot 2}x^{n-2}\cdot\frac{(n-2)(n-3)}{1\cdot 2}x^2=\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 1\cdot 2}x^n.$$

We may combine all these parts into one sum, the desired coefficient of the x^n position to result:

$$1+\frac{n(n-1)}{1\cdot 1}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 1\cdot 2}+\frac{n(n-1)(n-2)(n-3)\cdot(n-4)\cdot(n-5)}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}\text{ , etc.}$$

Corollary 1.

2. Therefore this series, which is broken up so long as *n* is a whole number, produces the coefficient of the x^n position for the given series $1+x+3x^2+7x^3+19x^4+$ etc. and thus with its help, a term, distant from the beginning to as great a degree as you like, is able to be found immediately without the previous ones.

Corollary 2.

3. If we substitute for n the numbers 1, 2, 3 etc. successively, the following values are discovered:[d]

[d] The numbers in this sequence are now known as "trinomial coefficients". [See George E. Andrews, "Euler's 'exemplum memorabile inductionis fallacis' and q-Trinomial Coefficients." J. Amer. Math. Soc. 3, p. 653-669, 1990, and http://mathworld.wolfram.com/TrinomialCoefficient.html.]

n	coefficient of x^n
0	1
1	1
2	1 + 2 = 3
3	1 + 6 = 7
4	1 + 12 + 6 = 19
5	1 + 20 + 30 = 51
6	1 + 30 + 90 + 20 = 141
n	coefficient of x^n
7	1 + 42 + 210 + 140 = 393
8	1 + 56 + 420 + 560 + 70 = 1107
9	1 + 72 + 756 + 1680 + 630 = 3139
10	1 + 90 + 1260 + 4200 + 3150 + 252 = 8953
11	1 + 110 + 1980 + 9240 + 11550 + 2772 = 25653
12	1 + 132 + 2970 + 18480 + 34650 + 16632 + 924 = 73789

Corollary 3.

4. The series of these numbers is so compared, so that each term may be considered to be conveniently matched with three times the preceding term, from which comparison the following differences arise:

1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139
3, 3, 9, 21, 57, 153, 423, 1179, 3321 , etc.

2, 0, 2, 2, 6, 12, 30, 72, 182 etc.

Comment 1.

Printed Marginal Note in Right Margin: A remarkable example of an induction fallacy.

5. If we carefully contemplate these differences, not without reason it seems to happen, that the numbers themselves are pronic[e], or doubled triangular numbers[f]

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[e] Pronic numbers are products of consecutive integers, e.g. $2 \cdot 3 = 6$ or $12 \cdot 13 = 156$. So, in general, pronic numbers are of the form $m(m+1) = m^2 + m$.

[f] Triangular numbers, e.g. 1, 3, 6, 10, …, can be determined by summing up the first *n* positive integers. So, the m^{th} triangular number is $1+2+\cdots+m = \frac{m(m+1)}{2}$ which if doubled, gives a pronic number.

having comprised the form $mm+m$, and also if we consider the roots[g] of these pronic numbers, which make up this series:

1, 0,1, 1, 2, 3, 5, 8, 13. etc.

Clearly it is recurring, where each term is the sum of the preceding two terms. Since the pattern may be recognized from the first ten terms, who would hesitate to assign the same to the entire series? Often, surely, less certain inductions have not failed of success. It will therefore be worthwhile to investigate this aspect carefully, of course while the number 13 may agree with the x^9 term of the series, in general the number for the position x^n will satisfy:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$$ [h]

whose pronic number is:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-4} + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-4} - \frac{2}{5}(-1)^{n-2}.$$ [i]

Hence if two adjacent terms in the proposed series thus may be presented generally:

$$1+x+3x^2+7x^3+19x^4\cdots Px^n+Qx^{n+1}$$ [j]

it will be that $3P-Q=$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-2} + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-2} - \frac{2}{5}(-1)^{n-1}$$ [k]

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[g] While "radix" is usually "square root", it appears Euler took the sequence of pronic numbers $m(m+1)=m^2+m$ from Section 4 and solved for the positive solution, i.e. "root", *m* in each case; e.g. $m(m+1)=m^2+m=2$ has positive solution 1 and $m(m+1)=m^2+m=30$ has positive solution 5.

[h] This formula is sometimes known as the Binet formula for Fibonacci numbers (http://mathworld.wolfram.com/BinetsFibonacciNumberFormula.html) or the Euler-Binet formula (https://proofwiki.org/wiki/Euler-Binet_Formula).

[i] This expression is obtained by forming a pronic number from the previous formula for a Fibonacci number, namely taking the product of itself with the square of itself.

[j] Original has 10 instead of 19 for the coefficient of the x^4 term.

[k] The pronic number formed from the next ($n+1$) Fibonacci number.

from which it is concluded that

$$P=\frac{3^n+(-1)^n}{10}+\frac{1}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n+\frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n+\frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

therefore the proposed series would also be recursive by means of the existing pattern of the relation

$$6,\ -8,\ -8,\ 14,\ 4,\ -3$$ [l]

According to which it will be

$$3139=6\cdot 1107-8\cdot 393-8\cdot 141+14\cdot 51+4\cdot 19-3\cdot 7.$$ [m]

Comment 2.

6. Truly as much as this rule of progression may seem to be supported by a probable induction, since it takes place for the first ten terms; yet a fallacy is discovered, because in the eleventh term 8953 it fails; indeed when this is subtracted from 9417, three times the previous, the remainder is 464, [certainly] not a pronic number, rather there is a pronic root $21=13+8$, that gives $21^2+21=462$, a number which is shy by two from 464, which ought to result from the observed rule. For this reason I will now indeed investigate into the real progression rule of this series, in order that it may be clear in what way each term may actually be determined by a few previous ones.

Problem 2.

7. For the proposed series

$$1,\ x, 3x^2, 7x^3, 19x^4, 51x^5, \text{etc.}$$

to investigate the rule by which each term is determined by means of a few previous ones.

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[l] Original has 13 instead of 14.

[m] Original has 9 instead of 19.

Solution.

In general, several terms of this series themselves may be considered following each other:

$$1,\, x, 3x^2, 7x^3 \ldots\ldots Px^n, Qx^{n+1}, Rx^{n+2}$$

and since in the previous problem, we have seen it to be:

$$P = 1 + \frac{n(n-1)}{1\cdot 1} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3}. \text{ etc.}$$

in a similar manner:

$$Q = 1 + \frac{(n+1)n}{1\cdot 1} + \frac{(n+1)n(n-1)(n-2)}{1\cdot 1\cdot 2\cdot 2} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3}, \text{ etc.}$$

$$R = 1 + \frac{(n+2)(n+1)}{1\cdot 1} + \frac{(n+2)(n+1)n(n-1)}{1\cdot 1\cdot 2\cdot 2} + \frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3}, \text{ etc.}$$

and when we combine whichever from the series have been subtracted:

$$Q - P = \frac{2n}{1} + \frac{2n(n-1)(n-2)}{1\cdot 1\cdot 2} + \frac{2n(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2\cdot 3}, \text{ etc.}$$

$$R - Q = \frac{2(n+1)}{1} + \frac{2(n+1)n(n-1)}{1\cdot 1\cdot 2} + \frac{2(n+1)n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2\cdot 3}, \text{ etc.}$$

hence we may take this form:

$$\frac{n+2}{n+1}(R-Q) = \frac{2(n+2)}{1} + \frac{2(n+2)n(n-1)}{1\cdot 1\cdot 2} + \frac{2(n+2)n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2\cdot 3}, \text{ etc.}$$

from which that [expression] $Q-P$ may be subtracted so that it may result

$$\frac{n+2}{n+1}(R-Q) - (Q-P) = 4 + \frac{4n(n-1)}{1\cdot 1} + \frac{4n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} + \text{ etc.}$$

since this series is $=4P$, we will have

$$R = Q + \frac{(n+1)(Q-P)}{n+2} + \frac{4(n+1)P}{n+2} \text{ or}$$

$$\text{R}=\frac{(2n+3)Q+3(n+1)P}{n+2}.$$

Corollary 1.

8. There is therefore a rule by which each term of the series is determined by the two preceding ones, which therefore is:

$$R = Q + \frac{n+1}{n+2}(Q+3P)$$

from the two next two terms Q and R, the preceding P is defined thus:

$$P = \frac{(n+2)R-(2n+3)Q}{3(n+1)}.$$

Corollary 2.

9. In order that it may be evident, just how this rule may have its place in the given series. Let us illustrate it by several cases:

$$\text{if } n=0, \quad 3 = 1+\frac{1}{2}(1+3\cdot 1)$$

$$\text{if } n=1, \quad 7 = 3+\frac{2}{3}(3+3\cdot 1)$$

$$\text{if } n=2, \quad 19 = 7+\frac{3}{4}(7+3\cdot 3)$$

$$\text{if } n=3, \quad 51 = 19+\frac{4}{5}(19+3\cdot 7)$$

$$\text{if } n=4, \quad 141 = 51+\frac{5}{6}(51+3\cdot 19)$$

etc.

Corollary 3.

10. Therefore since the exponent n enters into a relation that is among the three adjacent terms, it is easily deduced that this series does not belong to a recurring class.

Corollary 4.

11. However among four successive terms *P*, *Q*, *R* and *S* an independent relation may be expressed for the exponent n , while in fact from the previous three terms is

$$n=\frac{2R-3Q-3P}{3P+2Q-R}$$

in a similar way $n+1=\frac{2S-3R-3Q}{3Q+2R-S}$

from which we conclude that

$$S=R+Q+\frac{3P(Q+R)+2QR}{6P+3Q-R}$$

which is a constant relation, by which the following term is defined by the previous three terms.

Comment 1.

12. With the discovered rule, where each term of our progression depends on the 2 preceding [terms], now one may very easily continue this progression for as long as one may want. Therefore since the positions x^{11} and x^{12} have the numbers 25653 and 73789, the coefficient for x^{13} of the sequence will be, from $n=11$:

$$73789+\frac{12}{13}(73789+3\cdot 25653)=212941$$

and of the x^{14} position

$$212941+\frac{13}{14}(212941+3\cdot 73789)=616227$$

from which, continuing our progression up to the twentieth power, therefore:

x

$1x$	$25653x^{11}$
$3x^2$	$73789x^{12}$
$7x^3$	$212941x^{13}$
$19x^4$	$616227x^{14}$
$51x^5$	$1787607x^{15}$
$141x^6$	$5196627x^{16}$
$393x^7$	$15134931x^{17}$
$1107x^8$	$44152809x^{18}$
$3139x^9$	$128996853x^{19}$
$8953x^{10}$	$377379369x^{20}$

about which numbers I observe none to be divisible by 5, but the coefficients of the x^{3a+2} powers however to be divisible by 3, of the x^{7a+3} powers by 7, and certainly from here nothing[n] [else] may be concluded about the character of these numbers. Truly from the law of the progression the main point is found here, accordingly it may be continued to infinity, we will be able to define, to which end the next problem is determined.

Comment 2.

13. If any term of our progression may be subtracted from three times the preceding, the differences of such progression form:

$$1\cdot 2;\, 2\cdot 1;\, 3\cdot 2;\, 4\cdot 3;\, 5\cdot 6;\, 6\cdot 12;\, 7\cdot 26;\, 8\cdot 58;\, 9\cdot 134;$$
$$10\cdot 317;\, 11\cdot 766;\, 12\cdot 1883;\, 13\cdot 4698;\, 14\cdot 11871;$$
$$15\cdot 30330;\, 16\cdot 78249;\, 17\cdot 203622;\, 18\cdot 533955;$$

for which we may set up generally:

$$mp;\, (m+1)\, q;\, (m+2)\, r$$

where the first thing worthy to note occurs, because the previous factors of these terms advance in a series of natural numbers, the following in truth therefore are arranged, so that any one whatever may be constructed from the two preceding in this way:

[n] not anything

$$r=\frac{3mp+2(m+1)q}{m+4}.$$

Problem 3.

14. To investigate the sum, if our series

$$1+x+3x^2+7x^3+19x^4+\text{ etc.}$$

is continued to infinity.

Solution.

Since the relation of each term may be defined from the two preceding ones, let us establish:

$$s=1+x+3x^2+\cdots+Px^n+Qx^{n+1}+Rx^{n+2}+\text{ etc.}$$

whereby it may be noted that $(n+2)R-(2n+3)Q-3(n+1)P=0$

so that we may satisfy this condition, let us take the differential:

$$\frac{ds}{dx}=1+6x+\cdots+nPx^{n-1}+(n+1)Qx^n+(n+2)Rx^{n+1}\text{ etc.}$$

which having been multiplied by $1-2x-3xx$ produces:

$$\begin{array}{l}\frac{ds}{dx}(1-2x-3xx)=\\ 1+6x+21xx\cdots+nPx^{n-1}+(n+1)Qx^n+(n+2)Rx^{n+1}\\ \quad -2\quad -12\qquad\qquad\quad -2nP\qquad\quad -(2n+2)Q\\ \qquad\quad\; -3\qquad\qquad\qquad\qquad\qquad\quad -\qquad 3nP\end{array}$$

which series reduces to:

$$1+4x+6xx\cdots(Q+3P)x^{n+1}+\text{ etc.}$$

But the given series multiplied by $1+3x$ gives

$$s(1+3x)=1+4x+6xx\ldots\ldots(Q+3P)x^{n+1}$$

from which it is clear it will be:

$$\frac{ds}{dx}\left(1-2x-3xx\right)=s\left(1+3x\right) \text{ and therefore}$$

$$\frac{ds}{s}=\frac{dx\left(1+3x\right)}{1-2x-3xx} \text{ , the integration of which produces}$$

$$s=\frac{1}{\sqrt{1-2x-3xx}}=\frac{1}{\sqrt{\left(1+x\right)\left(1-3x\right)}}$$

which is the sum of the given series continued to infinity.

Corollary 1.

15. It is proven, therefore, that the sum of this series is imaginary unless it is assumed that $x<\frac{1}{3}$, however in the case $x=\frac{1}{3}$ it becomes infinite. But by assigning negative values to x, suppose $x=-y$, the sum becomes finite assuming $y<1$, while with the case $y>1$ it turns out to be imaginary. So letting $x=-\frac{1}{2}$,

$$\frac{2}{\sqrt{5}}=1-\frac{1}{2}+\frac{3}{4}-\frac{7}{8}+\frac{19}{16}-\frac{51}{32}+\frac{141}{64}-\text{ etc.}$$

Corollary 2.

16. Now therefore we learned that our series also results if the irrational form $\left(1-2x-3xx\right)^{-\frac{1}{2}}$ is developed in a series in the usual way: which formula may thus be represented as $s=\left(\left(1-x\right)^2-4xx\right)^{-\frac{1}{2}}$; [which] produces

$$s=\frac{1}{1-x}+\frac{2x}{1}\cdot\frac{xx}{\left(1-x\right)^3}+\frac{2\cdot 6}{1\cdot 2}\cdot\frac{x^4}{\left(1-x\right)^5}+\frac{2\cdot 6\cdot 10}{1\cdot 2\cdot 3}\cdot\frac{x^6}{\left(1-x\right)^7}+\text{ etc.}$$

from whose further development emerges:

$$\begin{aligned} s = 1 + x + \ & x^2 + \ x^3 + \ x^4 + \ x^5 \ + \ x^6 \ + \ x^7 \ + \ x^8 + \ x^9 \ + \ x^{10} \\ & + 2\cdot 1 + 2\cdot 3 + 2\cdot 6 + 2\cdot 10 + 2\cdot 15 + 2\cdot 21 + 2\cdot 28 + 2\cdot 36 + 2\cdot 45 \\ & \qquad + 6\cdot 1 + 6\cdot 5 + 6\cdot 15 + 6\cdot 35 + 6\cdot 70 + 6\cdot 126 + 6\cdot 210 \\ & \qquad\qquad + 20\cdot 1 + 20\cdot 7 + 20\cdot 28 + 20\cdot 84 + 20\cdot 210 \\ & \qquad\qquad\qquad + 70\cdot 1 + 70\cdot 9 + 70\cdot 45 \\ & \qquad\qquad\qquad\qquad + 252\cdot 1 \end{aligned}$$

Corollary 3.

17. Hence we deduce in general the numerical coefficient of the power x^n to be expressed as

$$+\frac{2}{1}\cdot\frac{n(n-1)}{1\cdot 2}+\frac{2\cdot 6}{1\cdot 2}\cdot\frac{n(n-1)}{1\cdot 2\cdot}\frac{(n-2)}{3\cdot}\frac{(n-3)}{4} + \text{etc.}$$

which form does not disagree with the first problem we had discovered.

Comment.

18. But if we carefully assess the form of this sum, without difficulty we derive a much more broadly accessible method, with which help the more general power $(a+bx+cxx)^n$ can be thoroughly managed, so that not only the middle terms of each power but also the terms equidistant from the middle on both sides may be assigned. Therefore I will explain this method in the following problem.

Problem 4.

19. If each power of the trinomial $a+bx+cxx$ is expanded, and the middle terms, which are placed equidistantly from each other in the series, to investigate the nature and sum of each of these series.

Solution.

That formula $\dfrac{1}{1-y(a+bx+cxx)}$ may be considered which, developed in a series, produces

$$1+y\left(a+bx+cxx\right)+yy\left(a+bx+cxx\right)^2+y^3\left(a+bx+cxx\right)^3,\ \text{etc.}$$

Whereby where the powers of each proposed trinomial occur, will arise from the expansion:

$$\begin{gathered}1\\ y\left(a+bx+cxx\right)\\ y^2\left(a^2+2abx+2acx^2+2bcx^3+ccx^4\right)+bb\\ y^3\left(a^3+3a^2bx+3a^2cx^2+6abcx^3+3bbcx^4+3bccx^5+c^3x^4\right)+3ab^2+b^3+3acc\\ \text{etc.}\end{gathered}$$

Hence if at first the middle terms, then the terms equidistant on both sides from the middle be taken, the following series will be produced:

$$\begin{aligned}&1+bxy+\left(2ac+bb\right)xxyy+\left(6abc+b^3\right)x^3y^3+\ \text{etc.}\\ &y\left(a+cxx\right)\left(1+2bxy+\left(3ac+3bb\right)xxyy+\text{etc.}\right)\\ &y^2\left(a^2+c^2x^4\right)\left(1+3bxy+\text{etc.}\right)\\ &y^3\left(a^3+c^3x^6\right)\left(1+4bxy+\text{etc.}\right)\\ &y^4\left(a^4+c^4x^8\right)\left(1+5bxy+\text{etc.}\right)\\ &\qquad\qquad\text{etc.}\end{aligned}$$

Therefore by laying aside these multipliers, because the powers of the product xy appear in the series themselves, let us put $xy=z$ and list these series in this manner:

$$\begin{aligned}&1+bz+\left(2ac+bb\right)zz+\left(6abc+b^3\right)z^3 &&=P\\ &1+2bz+\left(3ac+3bb\right)zz+\ \text{etc.} &&=Q\\ &1+3bz+\ \text{etc.} &&=R\\ &1+4bz+\ \text{etc.} &&=S\\ &\qquad\qquad\text{etc.}\end{aligned}$$

thus from $y=\frac{z}{x}$, we have

$$\frac{1}{1-bz-z\left(\frac{a}{x}+cx\right)}=P+z\left(\frac{a}{x}+cx\right)Q+zz\left(\frac{aa}{xx}+ccxx\right)R+z^3\left(\frac{a^3}{x^3}+c^3x^3\right)S+\ \text{etc.}$$

[which] may be multiplied on both sides by $1-bz-z\left(\frac{a}{x}+cx\right)$, and since the quantities *P*, *Q*, *R* etc. depend on *z* alone, all members may be assigned to powers of *x* both positive and negative: having done this we will obtain:

$$\begin{aligned}
1 = P(1-bz) &+ Qz(1-bz)cx + Rzz(1-bz)c^2x^2 + Sz^3(1-bz)c^3x^3 \\
&- Pz\cdot cx \qquad - Qzz\cdot ccx^2 \qquad - Rz^3c^3x^3 \\
-2Qaczz& \\
&- Rz^3\cdot accx \qquad - Sz^4\cdot ac^3x^2 \qquad - Tz^5ac^4x^3 \\
&+ Qz(1-bz)\frac{a}{x} \quad + Rzz(1-bz)\frac{a^2}{x^2} \quad + Sz^3(1-bz)\frac{a^3}{x^3} \\
&- Pz\cdot\frac{a}{x} \qquad - Qzz\cdot\frac{a}{x}\frac{a}{x} \qquad - Rz^3\frac{a^3}{x^3} \\
&- Rz^3ac\cdot\frac{a}{x} \qquad - Sz^4ac\cdot\frac{a}{x}\frac{a}{x} \qquad - Tz^5ac\cdot\frac{a^3}{x^3}
\end{aligned}$$

where it is apparent the negative powers of x are reduced to nothing under the same conditions as the positive ones. Hence we arrive at the following conclusions:

$$Q=\frac{P(1-bz)-1}{2aczz}$$
$$R=\frac{Q(1-bz)-P}{aczz}$$
$$S=\frac{R(1-bz)-Q}{aczz}$$
$$T=\frac{S(1-bz)-R}{aczz}$$
etc.

Therefore we see that the quantities P, Q, R, S etc. progress according to a recurring series, of which the step of the relation is

$$\frac{1-bz}{aczz};-\frac{1}{aczz}$$

hence if indices may be assigned to these quantities:

$$\overset{0}{P},\ \overset{1}{Q},\ \overset{2}{R},\ \overset{3}{S}\ \ldots\ \overset{n}{Z}$$

so that it is *Z* that corresponds with the index *n*, as a result of the nature of the recurrence:

$$Z = A\left(\frac{1-bz-\sqrt{(1-bz)^2-4aczz}}{2aczz}\right)^n + B\left(\frac{1-bz+\sqrt{(1-bz)^2-4aczz}}{2aczz}\right)^n$$

whence it may be established that the quantity *Z* can be expressed in this sort of series, that is:

$$Z = 1+(n+1)bz+..zz+..z^3+..z^4 \text{ etc.}$$

from which it is necessarily evident that $B=0$, because otherwise the terms will arise from the following member containing the negative powers of *z*. Therefore since $B=0$,

$$Z = A\left(\frac{1-bz-\sqrt{1-2bz+(bb-4ac)zz}}{2aczz}\right)^n.$$

Now let $n=0$, and it is necessary to set $A=P$, however putting $n=1$, causes:

$$A\cdot\frac{1-bz-\sqrt{1-2bz+(bb-4ac)ss}}{2aczz}=Q.$$

Consequently since $A=P$ and $2aczzQ+1=P(1-bz)$ it follows that:

$$P(1-bz)-P\sqrt{1-2bz+(bb-4ac)zz}=P(1-bz)-1$$

and therefore $P=\dfrac{1}{\sqrt{1-2bz+(bb-4ac)zz}}$. On account of which the general term of our series $P, Q, R, S \cdots Z$ is:

$$Z=\frac{1}{\sqrt{1-2bz+(bb-4ac)zz}}\left(\frac{1-bz-\sqrt{1-2bz+(bb-4ac)zz}}{2aczz}\right)^n.$$

Therefore putting $y=1$ so that $x=z$, if all powers of the trinomial $a+bz+czz$ are developed, the series of intermediate terms $1+bz+(2ac+bb)zz$ etc. will be $=P$ however from the middle of the terms, for the *n* positions for the preceding [terms] of the further ones the sum is $=a^nZ$, certainly for the same positions in the following [terms] of the further ones the sum is $= c^n z^n Z$. But on the other hand the sum of all these series taken together is $= \dfrac{1}{1-a-bz-czz}$.

Corollary 1.

20. Therefore the quantities *P*, *Q*, *R*, *S* etc. make up a geometric progression, whose first term is $P = \dfrac{1}{\sqrt{1-2bz+(bb-4ac)zz}}$, and the denominator of the progression is:

$$\frac{1-bz-\sqrt{1-2bz+(bb-4ac)zz}}{2aczz}.$$

Corollary 2.

21. If we assume $a=1$, $b=1$, and $c=1$, in the case where we previously considered the powers of the trinomial $1+z+zz$, whose middle terms make up the series, the sum of which $= \dfrac{1}{\sqrt{1-2z-3zz}}$, as we found above.

Problem 5.

22. To convert the formula discovered in the preceding problem:

$$\frac{1}{\sqrt{1-2bz+(bb-4ac)zz}}\left(\frac{1-bz-\sqrt{1-2bz+(bb-4ac)zz}}{2aczz}\right)^n,$$

into a series, the terms of which proceed according to the powers of z.

Solution.

For the sake of brevity let $bb-4ac=e$, and also we may set

$$s = \frac{1}{\sqrt{1-2bz+ezz}}\left(\frac{1-bz-\sqrt{1-2bz+ezz}}{2aczz}\right)^n$$

it is necessary to free that relation between z and s from irrationality through differentiation. To this end, set

$$\frac{1-bz-\sqrt{1-2bz+ezz}}{2aczz}=v \text{ so that } acvvzz-(1-bz)v+1=0$$

from which differentiation gives:

$$dv(2acvzz-1+bz)+vdz(2acvz+b)=0 \text{ or}$$
$$dv\sqrt{1-2bz+ezz}=\frac{vdz}{z}\left(1-\sqrt{1-2bz+ezz}\right)$$

and therefore $\frac{dv}{v}=\frac{dz}{z\sqrt{1-2bz+ezz}}-\frac{dz}{z}$.

Hence this produces a logarithmic differential equation

$$\frac{ds}{s}=\frac{dz(b-ez)}{1-2bz+ezz}-\frac{ndz}{z}+\frac{ndz}{z\sqrt{1-2bz+ezz}}.$$

For the present let us put $\frac{dt}{t}=\frac{ds}{s}+\frac{ndz}{z}-\frac{dz(b-ez)}{1-2bz+ezz}$, so that $\frac{dt}{t}=\frac{ndz}{z\sqrt{1-2bz+ezz}}$, from which we obtain by taking the squares: $zzdt^2(1-2bz+ezz)=nnHdz^2$, which equation, differentiated again by placing the element dz constant, gives:

$$zzddt(1-2bz+ezz)+zdtdz(1-3bz+2ezz)=nntdz^2$$

or

$$\frac{d\,dt}{t}+\frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)}\cdot\frac{dt}{t}-\frac{nndz^2}{zz(1-2bz+ezz)}=0.$$

Now because $\frac{d\,dt}{t}=d.\frac{dt}{t}+\frac{dt^2}{tt}$

$$\frac{ddt}{t}=\frac{dds}{s}-\frac{ds^2}{1s}-\frac{ndz^2}{zz}+\frac{dz^2(e-2bb+2bez-eezz)}{(1-2bz+ezz)^2}+\frac{nndz^2}{zz}-\frac{2ndz^2(b-ez)}{z(1-2bz+ezz)}$$
$$+\frac{ds^2}{ss}+\frac{2ndzds}{sz}-\frac{2dzds(b-ez)}{s(1-2bz+ezz)}+\frac{dz^2(bb-2bez+eezz)}{(1-2bz+ezz)^2}.$$

Then, making the substitution of the above equation into this one, it changes to the form:

$$\frac{dds}{s}+\frac{2ndz}{z}\cdot\frac{ds}{s}-\frac{2dz(b-ez)}{1-2bz+ezz}\cdot\frac{ds}{s}+\frac{n(n-1)dz^2}{zz}-\frac{2ndz^2(b-ez)}{z(1-2bz+ezz)}+\frac{dz^2(e-bb)}{(1-2bz+ezz)^2}$$
$$+\frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)}\cdot\frac{ds}{s}+\frac{ndz^2(1-3bz+2ezz)}{zz(1-2bz+ezz)}-\frac{dz^2\left(b-(e+3bb)z+sbezz-2eez^3\right)}{z(1-2bz+ezz)^2}$$
$$-\frac{nndz^2}{zz(1-2bz+ezz)}=0$$

where if the terms divided by $(1-2bz+ezz)^2$ are combined into one sum, the fraction will be able to be reduced by $1-2bz+ezz$, from which having been reduced we obtain:

$$\frac{dds}{s}+\frac{2ndz}{z}\cdot\frac{ds}{s}+\frac{dz(1-5bz+4ezz)}{z(1-2bz+ezz)}\cdot\frac{ds}{s}-\frac{nndz^2(2b-ez)}{z(1-2bz+ezz)}-\frac{3ndz^2(b-ez)}{z(1-2bz+ezz)}-\frac{dz^2(b-2ez)}{z(1-2bz+ezz)}=0$$

which when arranged simplifies to:

$$zdds(1-2bz+ezz)+dzds\left(2n+1-(4n+5)bz+2(n+2)ezz\right)$$
$$-sdz^2\left((n+1)(2n+1)b-(n+1)(n+2)zz\right)=0.$$

While now it is clear that to set $z=0$ results in $s=1$, we produce this series:

$$s=1+Az+Bzz+Cz^3+Dz^4+Ez^5+\text{ etc.}$$

where after substitution, the following form of the series is going to be reduced to nothing:

$$\begin{array}{r}
2Bz+6Czz\quad +12Dz^3\quad +20Ez^4\\
-4Bb\quad -12Cb\quad -24Db\\
+2Be\quad +6Ce\\
(2n+1)A+2(2n+1)B+3(2n+1)C+4(2n+1)D+5(2n+1)E\\
-(4n+5)A-2(4n+5)Bb-3(4n+5)Cb-4(4n+5)Db\\
+2(n+2)Ae+4(n+2)Be+6(n+2)Ce\\
-(n+1)(2n+1)b-(n+1)(2n+1)Ab-(n+1)(2n+1)Bb-(n+1)(2n+1)Cb-(n+1)(2n+1)De\\
+(n+1)(n+2)e+(n+1)(n+2)Ae+(n+1)(n+2)Be+(n+1)(n+2)Ce
\end{array}$$

from which we deduce these conclusions:

$$A = (n+1)b$$
$$B = \frac{(n+2)((2n+3)Ab - (n+1)e)}{2(2n+2)}$$
$$C = \frac{(n+3)((2n+5)Bb - (n+2)Ae)}{3(2n+3)}$$
$$D = \frac{(n+4)((2n+7)Cb - (n+3)Be)}{4(2n+4)}$$
$$E = \frac{(n+5)((2n+9)Db - (n+4)Ce)}{5(2n+5)}$$
etc

where note $e = bb - 4ac$. And thus every desired term of the series is determined by means of the two preceding terms.

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