1741

Inventio summae cuiusque seriei ex dato termino generali

Leonhard Euler

$+\frac{1}{22}+\frac{1}{132}-\frac{1}{144}-\frac{1}{2000002}-\frac{1}{12000012}+\frac{1}{12000024}+2,928968$ seu $=14,392669$. q. pr.

§. 14. Proposita sit nunc haec series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$ etc., cuius summa in infinitum desideretur. Si primo decem termini initiales addantur habebitur $1,549768$. Pro reliquis vero erit $a=\frac{1}{121}$, $b=\frac{1}{144}$; $x=\frac{1}{(n+10)^2}$ et $y=\frac{1}{(n+11)^2}$. Ex his fit $\int y\,dn=\frac{1}{11}-\frac{1}{n+11}$; atque posito $n=\infty$ erit $\int y\,dn=\frac{1}{11}$. Seriei ergo propositae in infinitum continuatae summa erit $\frac{1}{11}+\frac{7}{12.121}-\frac{1}{12.144}+1,549678$. Quae expressio in partibus decimalibus dat $1,644920$.

INVENTIO SVMMAE CVIVSQVE SERIEI EX DATO TERMINO GENERALI.

AVCTORE
Leonh. Eulero.

§. 1.

Cvm, quae superiore dissertatione de summatione serierum methodo geometrica exposui, diligentius considerassem, eandemque summandi rationem analytice inuestigassem; perspexi, id, quod geometrice elicui, deduci posse ex peculiari quadam summandi methodo, cuius iam ante triennium in Dissertatiotatione de summatione serierum mentionem feceram; postmodum vero de ea non amplius cogitaueram. Vim igitur analyticae methodi penitius perscrutatus, deprehendi non solum formulam geometrice inuentam in ea contineri; sed etiam eius ope adhuc pluribus terminis adiiciendis magis perfici posse, ita vt tandem veram summam absolute exhibeat. Geometrica autem via eosdem terminos inuenire summe difficile videtur.

§. 2. In illa autem dissertatione de summatione serierum, si fuerit terminus generalis cuiuspiam seriei x, eiusque index n, vniuersali modo pro termino summatorio exhibui sequentem formam $\int x\,dn + \frac{x}{2} + \frac{dx}{12\,dn} - \frac{d^3x}{720\,dn^3} +$ etc. ex qua differentialia ipsius x, quia x per n dari ponitur, a differentialis dn, quod constans assumitur, potestatibus, destruentur; ita vt summa algebraica obtineatur, si quidem $x\,dn$ integrationem admittat. In integratione vero ipsius $x\,dn$ tanta adiici debet constans, vt tota expressio euanescat posito $n = 0$.

§. 3. Quia igitur hanc formulam eiusque vsum accuratius in ista dissertatione persequi constitui; ante omnia modum, quo eam formulam sum consecutus exponam: Singularis enim est analysis, qua in hac re sum vsus, et complura satis praeclara in Analytica suppeditat, partim noua partim iam cognita, quae autem nusquam quantum recordor, satis euidenter sunt demonstrata.

§. 4. Ex natura calculi infinitesimalis sequitur, si fuerit y quomodocunque per x et constantes datum, atque loco x ponatur $x + dx$ tum abiturum y in $y + dy$. Si

Si iam porro x elemento dx augeatur, vel x abeat in $x + 2dx$; tum loco y habebitur $y + 2dy + ddy$. Atque si x denuo elemento dx crescat, y transibit in $y + 3dy + 3ddy + d^3y$, vbi coefficientes sunt iidem qui in potestatibus binomii. Ex his sequitur si loco x ponatur $x + mdx$ tum y abire in hanc formam: $y + \frac{m}{1} dy + \frac{m(m-1)}{1.\ 2.} ddy + \frac{m(m-1)(m-2)}{1.\ 2.\ 3.} d^3y +$ etc.

§. 5. Sit iam ad nostrum institutum m numerus infinite magnus, quo mdx quantitatem finitam significare queat; erit valor, quem y, posito $x + mdx$ loco x, habebit, iste: $y + \frac{mdy}{1} + \frac{m^2d^2y}{1.2} + \frac{m^3d^3y}{1.2.3} + \frac{m^4d^4y}{1.2.3.4} +$ etc. Si nunc fiat $mdx = a$ seu $m = \frac{a}{dx}$, induet y, si pro x ponatur $x + a$, hanc formam $y + \frac{ady}{1\,dx} + \frac{a^2ddy}{1.2.dx^2} + \frac{a^3d^3y}{1.2.3.dx^3} +$ etc. in qua omnes termini sunt finitae magnitudinis.

§. 6. Hanc ipsam seriem, quae valorem ipsius y transmutatum exhibet, si loco x ponatur $x + a$, primus produxit *Cl. Taylor* in Methodo Increm. inu. eamque ad multos egregios vsus accommodauit. Sequitur scilicet primum eleuatio binomii ad quamcunque dignitatem. Vt si quaeratur valor ipsius $(x + a)^m$ pono $y = x^m$, eritque $(x + a)^m$ valor ipsius y, si loco x ponatur $x + a$. Cum igitur sit $dy = mx^{m-1}dx$; $d^2y = m(m-1)x^{m-2}dx^2$ et ita porro erit $(x + a)^m = x^m + \frac{max^{m-1}}{1} + \frac{m(m-1)a^2x^{m-2}}{1.2} +$ etc.

 §. 7.

§. 7. Hanc porro feriem *Taylorus* adhibet ad radicem ex quacunque aequatioue proxime inueniendam, id quod hoc pacto perficit. Sit aequatio quaecunque incognitam z inuoluens, nempe $Z = o$, vbi Z eft quantitas ex incognita z et cognitis vtcunque compofita. Deinde fumit x pro valore ipfi z prope aequali, et quantitatem ipfius Z, quae prodit fi loco z ponatur x ponit $= y$, ita vt foret $y = o$, fi x effet verus ipfius z valor.

§. 8. At cum x a vero ipfius z valore aliquantum discrepet, ponit verum ipfius z valorem effe $x + a$. Quare perfpicuum eft fi in y loco x ponatur $x + a$, tum euaniturum y. At loco x fi ponatur $x + a$ tum abibit y in $y + \frac{a\,dy}{1.\,dx} + \frac{a^2\,ddy}{1.2.\,dx^2} + \frac{a^3\,d^3y}{1.2.3.\,dx^3} +$ etc. Hancobrem ergo erit $o = y + \frac{a\,dy}{1.\,dx} + \frac{a^2\,ddy}{1.2.\,dx^2} +$ etc. Ex qua aequatione valor ipfius a erutus dabit complementum a ad x addendum requifitum, quo obtineatur incognita z.

§. 9. Quia autem x ad z prope accedere ponitur, erit a quantitas valde parua, ita vt prae duobus terminis initialibus fequentes omnes euanefcere queant. Hocque pacto oritur $a = -\frac{y\,dx}{dy}$ atque $z = x - \frac{y\,dx}{dy}$, qui eft valor ipfius z multo magis propinquus quam x tantum. Vt pro hac aequatione $z^3 - 3z - 20 = o$ erit $y = x^3 - 3x - 20$ et $dy = 3x^2 - 3$, ideoque $z = x - \frac{x^3 - 3x - 20}{3xx - 3} = \frac{2x^3 + 20}{3xx - 3}$. Sumto nunc primo $x = 3$ erit $z = 3\frac{1}{12}$, hocque valore denuo pro x pofito proxime z inuenietur.

§. 10. Si porro detur conditio quaecunque functionis y, quae certo ipfius x cafu locum habeat, formula fuperior abibit

abibit in aequationem, in qua proprietas ipsius y continebitur. Vt si y huiusmodi fuerit functio ipsius x vt euanescat posito $x = 0$; pono $a = -x$, fiet enim hoc modo $x + a = 0$, atque erit $0 = y - \frac{x dy}{1.dx} + \frac{x^2 ddy}{1.2.dx^2} - \frac{x^3 d^3 y}{1.2.3 dx^3} +$ etc. seu $y = \frac{x dy}{1.dx} - \frac{x^2 ddy}{1.2.dx^2} + \frac{x^3 d^3 y}{1.2.3.dx^3} -$ etc. In qua aequatione omnium earum functionum ipsius x natura continetur, quae euanescunt posito $x = 0$.

§. 11. Si pro y ponatur $\int z dx$; erit $dy = z dx$; $ddy = dz dx$; $d^3 y = d^2 z dx$ etc. quibus valoribus substitutis habebitur $\int z dx = \frac{xz}{1} - \frac{x^2 dz}{1.2.dx} + \frac{x^3 ddz}{1.2.3.dx^2} -$ etc. in qua aequatione integrale ipsius $z dx$ per seriem infinitam exhibetur. Atque haec est generalis quadratura curuarum, quam *Cl. Ioh. Bernoulli* in Act. Lips. tradidit; analysin autem, qua ad hanc seriem peruenit, non adiunxit.

§. 12. Missis autem his, quae ad nostrum institutum minus pertinent, pergo ad series. Sit igitur series quaecunque $A + B + C + D$ - - - - - - - $+ X$; in qua A denotat terminum primum; B secundum; et X eum cuius index est x; ita vt X sit terminus generalis seriei propositae. Ponatur autem summa huius progressionis $A + B + C + D$ - - - - $+ X = S$; erit S terminus summatorius; atque tam S quam X, si series fuerit determinata, ex x et constantibus erunt composita.

§. 13. Quia iam S exhibet summam tot terminorum seriei, quot sunt vnitates in x; si in S loco x scribatur $x - 1$, habebitur prior summa termino vltimo X immi-

X imminuta. Hac igitur substitutione abibit S in S − X. Comparentur ergo haec cum superiore formula; erit S $= y$ et $a = -1$; quamobrem valor ipsius S transmutatus seu S − X erit $= S - \frac{dS}{1\,dx} + \frac{ddS}{1.2.dx^2} - \frac{d^3S}{1.2.3.dx^3} + \text{etc.}$ Ex quo oritur ista aequatio $X = \frac{dS}{1.dx} - \frac{ddS}{1.2.dx^2} + \frac{d^3S}{1.2.3.dx^3} - \frac{d^4S}{1.2.3dx^4} + \text{etc.}$

§. 14. Ope huius ergo aequationis ex dato termino summatorio seriei cuiusque inuenitur terminus generalis. Quod autem, cum alias sit facillimum, superfluum foret hac methodo ad terminum generalem ex summatorio inueniendum vti. Id autem maxime commodi huic aequationi accidit, vt singuli termini sint euoluti, eaque idcirco ad singulares vsus possit accommodari. Methodo enim cognita haec series $X = \frac{dS}{1.dx} - \frac{ddS}{1.2.dx^2} + \frac{d^3S}{1.2.3.dx^3} - \text{etc.}$ potest inuerti vt ex termino generali X determinetur summatorius S; quod ipsum maxime desideratur.

§. 15. Ponamus igitur $\frac{dS}{dx} = \alpha X + \frac{\beta dX}{dx} + \frac{\gamma ddX}{dx^2} + \frac{\delta d^3X}{dx^3} + \frac{\varepsilon d^4X}{dx^4} + \text{etc.}$ ita vt sit $S = \alpha\int X dx + \beta X + \frac{\gamma dX}{dx} + \frac{\delta ddX}{dx^2} + \text{etc.}$ Erit ergo $\frac{ddS}{dx^2} = \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} + \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} + \text{etc.}$ et $\frac{d^3S}{dx^3} = \frac{\alpha ddX}{dx^2} + \frac{\beta d^3X}{dx^3} + \frac{\gamma d^4X}{dx^4} + \text{etc.}$ et $\frac{d^4S}{dx^4} = \frac{\alpha d^3X}{dx^3} + \frac{\beta d^4X}{dx^4} + \text{etc.}$ atque $\frac{d^5S}{dx^5} = \frac{\alpha d^4X}{dx^4} + \text{etc.}$

§. 16. Substituantur ergo istae series loco cuiusque termini superioris seriei; et termini similes inter se comparentur nihiloque aequales ponantur. Quo facto coëfficientes α, β, γ etc. ita determinabuntur, vt sit, vti sequitur: $\alpha = 1$

$$\alpha = 1$$
$$\beta = \frac{\alpha}{2}$$
$$\gamma = \frac{\beta}{2} - \frac{\alpha}{6}$$
$$\delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24}$$
$$\varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120}$$
$$\zeta = \frac{\varepsilon}{2} - \frac{\delta}{6} + \frac{\gamma}{24} - \frac{\beta}{120} + \frac{\alpha}{720} \text{ etc.}$$

§. 17. Coëfficientes ergo $\alpha, \beta, \gamma, \delta$ etc. feriem conftituunt huius indolis, vt quisque terminus ex omnibus antecedentibus determinetur; exiftente termino primo $= 1$. Numeri autem per quos finguli terminorum antecedentium diuidi debent, conftituunt progreffionem a *Wallifio* hypergeometricam dictam; 2, 6, 24, 120, 720, 5040, etc. Ipfa autem feries coëfficientium α, β, γ etc. ita eft comparata, vt vix credam pro ea terminum generalem poffe exhiberi.

§. 18. Pro inftituto ergo noftro contenti effe debemus feriem coëfficientium quousque libuerit continuaffe, id quod ex lege progreffionis facile perfici poteft. Inueni autem hanc feriem, vt fequitur:

$$+ 1 + \frac{1}{1.2} + \frac{1}{1.2.3.2} + 0 - \frac{1}{1.2.3.4.5.6} - 0$$
$$+ \frac{1}{1.2.3.4.5.6.7.6} + 0 - \frac{3}{1.2.3.4.5.6.7.8.9.10}$$
$$- 0 + \frac{5}{1 - - - - - - 11.6} + 0 - \frac{691}{1 - - - - - - 13.210}$$
$$- 0 + \frac{35}{1 - - - - - - - 15.2} + 0 - \frac{3617}{1 - - - - - - 17.30}$$

in qua ferie notabile eft, quod omnes termini pares praeter fecundum euanefcant.

§. 19.

§. 19. Si ergo loco α, β, γ etc. hi termini ſubſtituantur; habebitur terminus ſummatorius $S = \int X dx + \frac{X}{1.2} + \frac{dX}{1.2.3.2dx} - \frac{d^3X}{1.2.3.4.5.6dx^3} + \frac{d^5X}{1 - - - - - 7.6dx^5} - \frac{3d^7X}{1 - - - 9.10.dx^7} + \frac{5d^9X}{1 - - - - - - - - - 11.6dx^9} + \frac{691d^{11}X}{1 - - - 13.210dx^{11}} + \frac{35d^{13}X}{1 - - - - - 15.2dx^{13}} - \frac{3617d^{15}X}{1 - - - - - - - 17.30dx^{15}} +$ etc.

§. 20. Series haec inſignem habet vſum in ſummis progreſſionum algebraicarum inueniendis, quarum in terminis generalibus x nusquam in denominatorem ingreditur. Quia enim hac ratione x vbique habet exponentes affirmatiuos integros, eius differentialia tandem euaneſcent, atque ſeries abrumpetur, vnde ipſe terminus ſummatorius finito terminorum numero reperietur. In quo inueniendo ſtatim omnes termini, qui x non continent reiici poſſunt, quia in $\int X dx$ tanta conſtans addi debet, quae faciat vt fiat $S = o$, poſito $x = o$.

§. 21. Quo vſus huius formulae clarius percipiatur, exempla quaedam afferre conuenit. Sit ergo $X = x$ ſeu ſeries ſummanda haec $1 + 2 + 3 - - - - - + x$; propter $\int X dx = \frac{x^2}{2}$ erit ſumma $S = \frac{x^2 + x}{2}$, eſt enim $\frac{dX}{dx}$ conſtans, et propterea reiicitur, et ſequentia differentialia ſponte euaneſcunt. Sit porro $X = x^2$ ſeu iſta ſeries $1 + 4 + 9 - - - - + x^2$ ſummanda, erit $\int X dx = \frac{x^3}{3}$ et $\frac{dX}{dx} = 2x$ ideoque ſumma ſeriei $S = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}$.

§. 22. Sit nunc haec ſeries generalis poteſtatum numerorum naturalium propoſita $1 + 2^n + 3^n + 4^n + 5^n +$ etc. cuius terminus generalis eſt x^n. Habebitur ergo $X = x^n$ et $\int X dx = \frac{x^{n+1}}{n+1}$. Differentialia autem

ita

ita se habebunt vt sit $\frac{dX}{dx} = n x^{n-1}$; $\frac{d^3 X}{dx^3} = n(n-1)(n-2) x^{n-3}$; $\frac{d^5 X}{dx^5} = n(n-1)(n-2)(n-3)(n-4) x^{n-5}$; etc. His igitur valoribus substitutis erit terminus summatorius seriei propositae

$$S = \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n x^{n-1}}{2.6} - \frac{n(n-1)(n-2) x^{n-3}}{2.3.4.30} + \frac{n(n-1)(n-2)(n-3)(n-4) x^{n-5}}{2.3.4.5.6.42} - \frac{n(n-1)----(n-6) x^{n-7}}{2.3------8.30} + \frac{n(n-1)-----(n-8) 5 x^{n-9}}{2.3------10.66} - \frac{n(n-1)---(n-10) 691 x^{n-11}}{2.3-------13.2730} + \frac{n(n-1)--(n-12) 7 x^{n-13}}{2.3---14.6} - \frac{n(n-1)-----(n-14) 3617 x^{n-15}}{2.3---------16.510} + \text{etc.}$$

Ad quam seriem, quousque opus est continuandam, oportet, vt superior illa series α, β, γ, etc. eousque continuetur.

§. 23. Ex hac igitur generali summatione seriei cuius terminus generalis est x^n confici possunt summae specialium serierum potestatum vt sequitur:

$$\int x^1 = \frac{x^2}{2} + \frac{x}{2}$$
$$\int x^2 = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}$$
$$\int x^3 = \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4}$$
$$\int x^4 = \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}$$
$$\int x^5 = \frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12}$$
$$\int x^6 = \frac{x^7}{7} + \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42}$$

 $\int x^7 =$

$$\int x^7 = \frac{x^8}{8} + \frac{x^7}{2} + \frac{7x^6}{12} - \frac{7x^4}{24} + \frac{x^2}{12}$$
$$\int x^8 = \frac{x^9}{9} + \frac{x^8}{2} + \frac{2x^7}{3} - \frac{7x^5}{15} + \frac{2x^3}{9} - \frac{x}{30}$$
$$\int x^9 = \frac{x^{10}}{10} + \frac{x^9}{2} + \frac{3x^8}{4} - \frac{7x^6}{10} + \frac{x^4}{2} - \frac{3x^2}{20}$$
$$\int x^{10} = \frac{x^{11}}{11} + \frac{x^{10}}{2} + \frac{5x^9}{6} - x^7 + x^5 - \frac{x^3}{2} + \frac{5x}{66}$$
$$\int x^{11} = \frac{x^{12}}{12} + \frac{x^{11}}{2} + \frac{11x^{10}}{12} - \frac{11x^8}{8} + \frac{11x^6}{6} - \frac{11x^4}{8} + \frac{5x^2}{12}$$
$$\int x^{12} = \frac{x^{13}}{13} + \frac{x^{12}}{2} + x^{11} - \frac{11x^9}{6} + \frac{22x^7}{7} - \frac{33x^5}{10} + \frac{5x^3}{3} - \frac{691x}{2730}$$
$$\int x^{13} = \frac{x^{14}}{14} + \frac{x^{13}}{2} + \frac{13x^{12}}{12} - \frac{143x^{10}}{60} + \frac{143x^8}{28} - \frac{143x^6}{20} + \frac{65x^4}{12} - \frac{691x^2}{420}$$
$$\int x^{14} = \frac{x^{15}}{15} + \frac{x^{14}}{2} + \frac{7x^{13}}{6} - \frac{91x^{11}}{30} + \frac{143x^9}{18} - \frac{143x^7}{10} + \frac{91x^5}{6} - \frac{691x^3}{90} + \frac{7x}{6}$$
$$\int x^{15} = \frac{x^{16}}{16} + \frac{x^{15}}{2} + \frac{5x^{14}}{4} - \frac{91x^{12}}{24} + \frac{143x^{10}}{12} - \frac{429x^8}{16} + \frac{455x^6}{12} - \frac{691x^4}{24} + \frac{35x^2}{4}$$
$$\int x^{16} = \frac{x^{17}}{17} + \frac{x^{16}}{2} + \frac{4x^{15}}{3} - \frac{14x^{13}}{3} + \frac{52x^{11}}{3} - \frac{143x^9}{3} + \frac{260x^7}{3} - \frac{1382x^5}{15} + \frac{140x^3}{3} - \frac{3617x}{510}$$

§. 24. Sin autem x non vbique habuerit exponentes affirmatiuos in termino generali feriei, tum quoque expreffio fummae infinitis conftat terminis; quia huiusmodi feries generalem fummationem non admittunt, fed quasque quadraturas inuoluunt. Interim tamen obferuaui ope huius methodi eiusmodi feries facile admodum proxime fummari poffe, quod infignem habet vtilitatem in feriebus, quae parum conuergunt, et alias difficulter fummantur. Quod quomodo efficiendum fit, exemplis docebo.

§. 25. Confiderabo igitur primum feries harmonicas, et prae ceteris quidem hanc $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ etc. cuius terminus generalis eft $\frac{1}{x}$; fummatorius vero, fit S, quaeritur. Eft ergo $X = \frac{1}{x}$ et $\int X\,dx =$ Conft. $+ lx$. Atque porro $\frac{dX}{dx} = \frac{-1}{x^2}$; $\frac{d^3X}{dx^3} = \frac{-1.2.3}{x^4}$; $\frac{d^5X}{dx^5} = -\frac{1.2.3.4.5}{x^6}$ etc. His fubftitutis prodit S = Conft.

+

$+ lx + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \frac{691}{32760x^{12}} - \frac{1}{12x^{14}} +$ etc. Vbi conſtans addenda ita debet eſſe comparata vt poſito $x = o$ fiat $S = o$; Ex hoc vero ob omnes terminos infinite magnos conſtans determinari non poteſt.

§. 26. Ad conſtantem vero determinandam alium caſum aſſumi oportet, quo ſumma ſeriei eſt cognita; qui ergo habebitur, ſi certus terminorum numerus in vnam ſummam colligatur. Addantur ergo 10 termini initiales $1 + \frac{1}{2} + \frac{1}{3} - - - \frac{1}{10}$; reperieturque eorum ſumma $= 2,9289682539682539$; cui aequalis eſſe debet ſumma eorundem terminorum ex formula nempe, Conſt. $+ l10 + \frac{1}{20} - \frac{1}{1200} + \frac{1}{1200000} - \frac{1}{252000000} + \frac{1}{24000000000} - \frac{1}{1320000000000}$ etc. Quo facto reperietur propter $l10 = 2,302585092994045684$ conſtans illa addita $= 0,5772156649015329$; hacque ſemel determinata ſumma quotcunque terminorum huius ſeriei reperietur.

§. 27. Hac igitur ratione inueſtigaui ſummam 100, 1000, 10000 etc. terminorum ſeriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ etc. inuenique vt ſequitur:

$$\int 10 = 2,9289682539682539$$
$$\int 100 = 5,1873775176396203$$
$$\int 1000 = 7,4854708605503449$$
$$\int 10000 = 9,7876060360443823$$
$$\int 100000 = 12,0901461298634280$$
$$\int 1000000 = 14,3927267228657236$$

§. 28. Si primus tantum terminus ſeriei 1 capiatur; erit $S = 1$, et $x = 1$ ideoque $lx = 0$. Habebitur ergo ex aequatione $0,4227843350984670 = \frac{1}{2} - \frac{1}{12} + \frac{1}{120} - \frac{1}{252} + \frac{1}{240} - \frac{1}{132} + \frac{691}{32760} - \frac{1}{12} +$ etc. Huius ergo ſeriei admodum irregularis et ne conuergentis quidem inuenta eſt ſumma quam proxime. Seriei autem in infinitum continuatae ſumma erit $= l\infty + 0,5772156649015329$, quae prodit poſito $x = \infty$.

§. 29. Progrediamur nunc ad hanc ſeriem $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$ etc. conſiderandam, in qua eſt $X = \frac{1}{2x-1}$ et $\int X\,dx =$ Conſt. $+ \frac{1}{2}l(2x-1)$, atque $\frac{dX}{dx} = \frac{-2}{(2x-1)^2}$; $\frac{d^3X}{dx^3} = \frac{-2.4.6}{(2x-1)^4}$; $\frac{d^5X}{dx^5} = \frac{-2.4.6.8.10}{(2x-1)^6}$ etc. His igitur inuentis erit ſeriei propoſitae ſumma $S =$ Conſt. $+ \frac{1}{2}l(2x-1) + \frac{1}{2(2x-1)} - \frac{1}{6(2x-1)^2} + \frac{1}{15(2x-1)^4} - \frac{8}{63(2x-1)^6} + \frac{8}{15(2x-1)^8} - \frac{128}{33(2x-1)^{10}} + \frac{256.691}{4095(2x-1)^{12}} - \frac{2048}{3(2x-1)^{14}} + \frac{1024.3617}{255(2x-1)^{16}} -$ etc.

§. 30. Conſtans autem quantitas in hoc caſu actu addendis aliquot terminis non tam expedite poteſt determinari quam in caſu praecedenti. Hoc vero caſu ſubſidium aliquod vſu venit, quo haec conſtans ex praecedente determinari poteſt. Scilicet ſeriei $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$ etc. in infinitum continuatae ſumma eſt $=$ Conſt. $+ \frac{1}{2}l\infty$. Subtrahatur ab huius ſeriei duplo prior ſeries harmonica; habebitur $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{5}$ etc. cuius ſumma vt conſtat eſt $l2$. Erit ergo $l2 =$ conſt. $+ l\infty - l\infty - 0,577215$ etc. ideoque haec conſtans quaeſita $= 0,6351814227307392$.

§. 31.

§. 31. Pergo ad ſeries magis compoſitas, et conſidero hanc $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc. reciprocam quadratorum, cuius terminus generalis eſt $\frac{1}{x^2} = X$. Erit ergo $\int X\, dx =$ Conſt. $- \frac{1}{x}$, atque $\frac{dX}{dx} = \frac{-2}{x^3}$; $\frac{d^3X}{dx^3} = \frac{-2.3.4}{x^5}$; $\frac{d^5X}{dx^5} = \frac{-2.3.4.5.6}{x^7}$ etc. His ſubſtitutis erit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} - - - - \frac{1}{x^2} = S =$ Conſt. $- \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} + \frac{1}{30x^9} - \frac{5}{66x^{11}} + \frac{691}{2730x^{13}} - \frac{7}{6x^{15}} +$ etc. Vbi conſtantis quantitas ex caſu ſpeciali debet determinari.

§. 32. Ipſo ergo actu addidi decem terminos initiales ſeriei iſtius, quorum ſummam inueni 1,549767731166540. Ad hanc ergo cum ſit hoc caſu $x = 10$, ſi addatur $\frac{1}{10} - \frac{1}{200} + \frac{1}{6000} - \frac{1}{3000000} + \frac{1}{4200000000} - \frac{1}{300000000000} + \frac{1}{13200000000000} - \frac{691}{27300000000000000} + \frac{7}{6000000000000000000}$ — etc. Ex hoc ergo prodit conſtans illa addenda = 1,64493406684822643647. Huicque conſtanti aequalis eſt ſeriei in infinitum continuatae ſumma; poſito enim $x = \infty$ fit $S =$ Conſt. euaneſcentibus omnibus terminis.

§. 33. Simili modo pro ſerie reciproca cuborum $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} +$ etc. ſi addantur decem termini initiales habebitur eorum ſumma haec 1,197531985674193. Vnde inuenitur conſtans, quae in ſummatione huius ſeriei addi debet = 1,202056903159594. Atque huic numero aequalis eſt ſeriei $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64}$ in infinitum continuatae ſumma. Atque pro biquadratis $1 + \frac{1}{16} + \frac{1}{81} +$ etc. et ſumma = 1,0823232337110824.

34 Confideremus nunc hac methodo feriem, qua area circuli, cuius diameter eft $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. vel $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15}$ etc. cuius terminus generalis eft $\frac{2}{(4x-3)(4x-1)}$ vel refoluendo in factores $\frac{1}{4x-3} - \frac{1}{4x-1}$. Ad fummam ergo huius feriei quam proxime inueniendam eft $X = \frac{1}{4x-3} - \frac{1}{4x-1}$ atque $\int X\,dx =$ Conft. $- \frac{1}{4} l \frac{4x-1}{4x-3}$; et $\frac{dX}{dx} = \frac{-4}{(4x-3)^2} + \frac{4}{(4x-1)^2}$; $\frac{d^3X}{dx^3} = \frac{-4.8.12}{(4x-3)^4} + \frac{4.8.12}{(4x-1)^4}$ etc. Ex his erit feriei $\frac{2}{1.3} + \frac{2}{5.7} - - - - + \frac{2}{(4x-3)(4x-1)}$ fumma $S =$ Conft. $- \frac{1}{4} l \frac{4x-1}{4x-3} + \frac{1}{2}\left(\frac{1}{4x-3} - \frac{1}{4x-2}\right) - \frac{1}{3}\left(\frac{1}{(4x-3)^2} - \frac{1}{(4x-1)^2}\right) + \frac{8}{15}\left(\frac{1}{(4x-3)^4} - \frac{1}{(4x-1)^4}\right) - \frac{256}{63}\left(\frac{1}{(4x-3)^6} - \frac{1}{(4x-1)^6}\right) + \frac{1024}{15}\left(\frac{1}{(4x-3)^8} - \frac{1}{(4x-1)^8}\right) - \frac{4^8}{33}\left(\frac{1}{(4x-3)^{10}} - \frac{1}{(4x-1)^{10}}\right) +$ etc. Haec vero feries etiamfi decem termini addantur non fatis conuergit, quo valor conftantis commode poffit exhiberi. Conftans autem quater fumta exhibet peripheriam circuli exiftente diametro $= 1$.

INVE-