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Euler's Navigation Variational Problem

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Abstract

In a 1747 publication, *De motu cymbarum remis propulsarum in fluviis* ("On the motion of boats propelled by oars in rivers"), Leonhard Euler (1707-1783) works out various instances of a boat moving at constant speed across a stream flowing in straight streamlines at assigned speeds, in which one of these gives rise to a variational problem consisting of finding the quickest crossing path between two points on opposite side of the river banks, which is generally known as the navigation variational problem. This problem together with the well-known catenary and brachistochrone problems, are considered classical examples in the calculus of variations. Here, we shall present a brief account on Euler's recurrent interests in calculus of variations, mainly laid out in three publications that span between 1738 and 1744. Particular focus will be given to Euler's navigation variational problem. A brief account on Lagrange's contributions to variational calculus is also presented.

Keywords: calculus of variations, variational problems, functionals, Euler-Lagrange equation

1. Introduction

The calculus of variations is a line of research that began with the proposal of the brachistochrone problem, which is to find the shape of a wire joining two given points, so that a bead will slide down the wire under gravity from one point to the other (without friction) in the shortest time. The resulting curve is known as the curve of fast descent or the brachistochrone curve (from Ancient Greek *brákhistos khrónos* 'shortest time'). The problem was first considered by Galileo Galilei (1564–1642) in 1638, but, lacking the necessary mathematical techniques, he concluded erroneously that the solution is the arc of a circle joining the two points. However, it was Johann Bernoulli (1667–1748) who made the problem famous when in June 1696 he challenged readers of the scientific journal *Acta Eruditorum* to solve it, reassuring them that the curve was well known to geometers. He also stated that he would demonstrate the solution at the end of the year, provided that no one else had. In December 1696, Bernoulli extended the time limit to Easter 1697, though by this time he was in possession of the solution by Gottfried Wilhelm Leibniz (1646–1716), sent in a letter dated 16 June 1696—Leibniz having received notification of the problem on 9 June. Isaac Newton (1642–1727) also solved the problem quickly, apparently on the day of receipt, and published his solution anonymously.^a However, Bernoulli immediately recognized the solution as Newton's saying *tanquam ex ungue leonem* (we know the lion by his claw).

Euler's contributions to variational mathematics is concentrated in three publications: E27 *Problematis isoperimetrici in latissimo sensu accepti solutio generalis* (On isoperimetric problems in the widest sense) [1], written in 1732, published in 1738, E56 *Curvarum maximi minive proprietate gaudientium inventio*

^a https://www.open.edu/openlearn/pluginfile.php/1118521/mod_resource/content/3/introduction%20to%20the%20calculus%20of%20variations_ms327.pdf, retrieved on May 15 2022

nova et facilis (New and easy method of finding curves enjoying a maximal or minimal property) [2], written in 1736, published in 1741, and E65 *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti* (A method for finding curved lines enjoying properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense) [3], written in 1743, published in 1744.

Craig G. Fraser has made an extensive examination of these publications [4, 5], with an historical contextualization of other contributions on the subject by Jakob and Johann Bernoulli (1701, 1719), and Taylor (1715), concluding that [5, p. 139-140] "... the organization of the subject adopted by Euler in the *Methodus inveniendi* was taken over by Lagrange when he introduced his δ -algorithm in the 1750s. Lagrange developed his new method exclusively in reference to the examples of Chapters 2 and 3 of Euler's treatise. The result was that in the later eighteenth century the isoperimetric theory became somewhat marginalized within variational mathematics. In his late treatise, the *Leçons sur le calcul des fonctions* of 1806, Lagrange formulated the method of multipliers, a powerful theoretical tool that provided a unified approach to the various problems of the subject. This method forms the basis for the modern perspective on the problems and techniques considered in the present paper..."

Ian Bruce has given a complete English translation of *Methodus inveniendi*,^b in which he provides an overview of the publication as follows: "... This is another project involving one of Euler's favorite topics: the finding of curves satisfying maximum or minimum properties under various circumstances; in the first chapter, a general method of finding such curves is set out and numerous examples of its use presented. In the second and third chapters, the work examines increasingly complicated applications of what was later called the Calculus of Variations. At first determined functions are treated, in which the values of the derivatives of the curve are known at any point; later more abstract constructions are introduced in chapters 2 & 3 where Euler deals with functionals related by a general integral formula; for example, the theory developed can be used to find the shape of the curve a body can fall along in the minimum time subject to various forms of resistance, etc. Chapter 4, section 7 sets out the precepts for the 5 kinds of extrema considered; this is a very useful summary..."

In *Methodus inveniendi*, Euler also develops what later became known as the Euler-Lagrange equation of the variational problem, which is much used in his examples and is fundamental to the whole theory. Most of the applications given by Euler in *Methodus inveniendi* deals with finding curves for which an integral formula shall have a maximum or minimum value. In the present work we review two of Euler's examples of variational problems (the body of minimum resistance and the brachistochrone problem). We will also present an application of the Euler-Lagrange equation to the navigation problem, as laid out by Euler in E94 *De motu cybarum remis propulsarum in fluviis* (On the motion of boats propelled by oars in rivers) [6], written in 1738, published in 1747.

2. The derivation of the Euler-Lagrange equation by Euler

In Chapter II of *Methodus inveniendi*, a curve in the plane joining the points a and z is given (Figure 1). The curve represents geometrically the analytical relation between the abscissa x and the ordinate y . Let M, N, O be three points of the interval AZ infinitely close together. We set $AM = x, AN = x', AO = x''$,

^b <http://www.17centurymaths.com/contents/Euler'smaxmin.htm>, retrieved on May 15 2022

and $Mm = y, Nn = y', Oo = y''$. The differential coefficient or the derivative of p is defined by the relation $dy = p dx$. Euler then writes

$$p = \frac{y' - y}{dx}, p' = \frac{y'' - y'}{dx},$$

which give the values of p at x and x' in terms of dx and the differences of the ordinates y, y' and y'' .

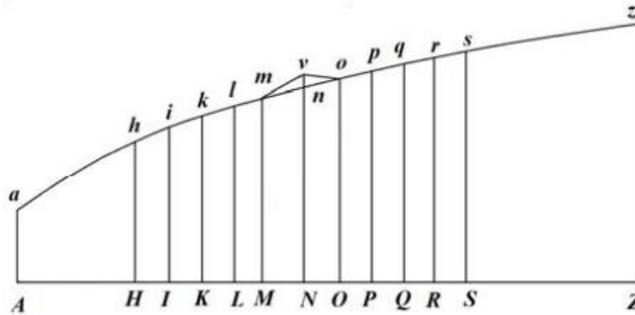


Figure 1: Geometrical elements used by Euler in the development of the Euler-Lagrange equation.

(Source: Figure 4 of *Methodus inveniendi*)

Suppose now that Z is some expression composed of x, y and p . (The symbol Z was also used by Euler to denote the endpoint of the interval AZ .) The definite integral $\int Z dx$ corresponding to the interval from A to Z is

$$\int Z dx \text{ (from } A \text{ to } M) + Z dx + Z' dx + \dots,$$

where Z, Z', \dots are the values of Z at $x, y, p; x', y', p', \dots$. Suppose that the curve az is such that this integral is a maximum or minimum. A comparison curve $amvoz$ is then obtained by increasing the ordinate y' by an infinitely small quantity nv . By hypothesis, the change of the integral calculated along these curves is zero. The only part of the integral that is affected by varying y' is $Z dx + Z' dx$.

Euler then writes

$$dZ = M dx + N dy + P dp,$$

$$dZ' = M' dx + N' dy' + P' dp'.$$

Evidently, where y' is increased by nv , $dx = dy = 0$ and $dy' = nv$, and, then, $dp = \frac{nv}{dx}$ and $dp' = -\frac{nv}{dx}$, resulting in

$$dZ = P \frac{nv}{dx},$$

$$dZ' = N' nv - P' \frac{nv}{dx}.$$

The total change in $\int Z dx (A \text{ to } Z)$ equals to $(dZ + dZ') dx = nv(P + N' nv - P')$, which should be equated to zero. By setting $P' - P = dP$ and replacing N' by N , finally we have that

$$N - \frac{dP}{dx} = 0,$$

from which the final equation of the curve is obtained.

This expression was much used in several examples given by Euler, one of which is to find the shape of the body of minimum resistance (Example V, § 36 of *Methodus inveniendi*). In this case, it can be shown that the body resistance^c is given by

$$\int \frac{ydy^3}{dx^2 + dy^2}.$$

But since $dy = pdx$, this integral transforms into

$$\int \frac{yp^3 dx}{1 + p^2} = \int Z dx$$

which shall be a minimum.

Therefore,

$$Z = \frac{yp^3}{1 + p^2} \text{ and } dZ = \frac{p^3 dy}{1 + p^2} + \frac{y(3p^2 + p^4)dp}{(1 + p^2)^2}.$$

From this expression, it can be seen that $M = 0$, $N = \frac{p^3}{1+p^2}$, and $P = \frac{p^2 y(3+p^2)}{(1+p^2)^2}$.

Recall that $N - \frac{dP}{dx} = 0$, or $Ndy - pdP = 0$, which by integration gives $Ny + a = pP$, where a is a constant of integration. Substituting the expressions for N and P , gives $\frac{yp^3}{1+p^2} + a = \frac{p^3 y(3+p^2)}{(1+p^2)^2}$, or $a(1 + p^2)^2 = 2p^3 y$, which may be rewritten as $y = \frac{a(1+p^2)^2}{2p^3}$.

Recalling that $dx = \frac{dy}{p}$, which upon integration by parts gives $x = \int \frac{dy}{p} = \frac{y}{p} + \int \frac{ydp}{p^2}$.

Substituting the expression for y just found into this expression, results in

$$x = \frac{a(1 + p^2)^2}{2p^4} + a \int \frac{(1 + p^2)^2}{2p^5} dp = \frac{a}{2} \left(\frac{3}{4p^4} + \frac{1}{p^2} + 1 + \log p \right),$$

which together with the expression

$$y = \frac{a(1 + p^2)^2}{2p^3}$$

gives the parametric curve of the body of minimum resistance.

^c A derivation of this result can be found at <https://www.youtube.com/watch?v=gEnMw55dYuQ>, retrieved June 24 2022. This problem is known as Newton's Minimal Resistance Problem, which is to find a solid of revolution which experiences a minimum resistance when it moves through a homogeneous fluid with constant velocity in the direction of the axis of revolution, named after Isaac Newton, who studied the problem in 1685 and published it in 1687 in his *Principia Mathematica*. A historical account of this problem can be found at https://en.wikipedia.org/wiki/Newton%27s_minimal_resistance_problem, retrieved June 24 2022.

To write the Euler-Lagrange equation from $N - \frac{dP}{dx} = 0$, it should be firstly recognized that M, N and P are differential coefficients, which are given by $M = \frac{\partial Z}{\partial x}$, $N = \frac{\partial Z}{\partial y}$, $P = \frac{\partial Z}{\partial p}$. By its turn, $\frac{dP}{dx} = \frac{d}{dx} \left(\frac{\partial Z}{\partial p} \right) = \frac{d}{dx} \left(\frac{\partial Z}{\partial y'} \right)$, and, then $N - \frac{dP}{dx} = \frac{\partial Z}{\partial y} - \frac{d}{dx} \left(\frac{\partial Z}{\partial y'} \right)$, giving the Euler-Lagrange equation as

$$\frac{\partial Z}{\partial y} - \frac{d}{dx} \left(\frac{\partial Z}{\partial y'} \right) = 0.$$

Let us see how this equation can be used to find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time—this is the brachistochrone problem.

The speed v along the curve is given by a simple application of conservation of energy equating kinetic energy to gravitational potential energy, which gives $v = \sqrt{2gy}$, where g is the gravity and y is the vertical coordinate. The element of the curve $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} dx$, where $y' = \frac{dy}{dx}$.

The time t_{12} to travel from point P_1 to another point P_2 is given by the integral $t_{12} = \int_{P_1}^{P_2} \frac{ds}{v} = \int_{P_1}^{P_2} \sqrt{\frac{1+y'^2}{2gy}} dx$. Then, function Z to be minimized is thus $Z = (1 + y'^2)^{1/2} (2gy)^{1/2}$.

However, the function Z does not depend explicitly on x , and, therefore, $\frac{\partial Z}{\partial x} = 0$, and the Euler-Lagrange equation reduces to the so-called Beltrami identity^d

$$Z - y' \frac{\partial Z}{\partial y'} = C.$$

Recognizing that $\frac{\partial Z}{\partial y'} = y'(1 + y'^2)^{-1/2} (2gy)^{-1/2}$, and subtracting $y' \frac{\partial Z}{\partial y'}$ from $Z = (1 + y'^2)^{1/2} (2gy)^{1/2}$, and simplifying then gives

$$\frac{1}{\sqrt{2gy} \sqrt{1 + y'^2}} = C.$$

By squaring both sides and rearranging results in

$$\left[1 + \left(\frac{dy}{dx} \right)^2 \right] y = \frac{1}{2gC^2} = k^2,$$

which after rearranging gives

$$\frac{dy}{dx} = \sqrt{\frac{k^2 - y}{y}}.$$

^d The Beltrami identity, named after Eugenio Beltrami (1835 – 1900), is a special case of the Euler-Lagrange equation in the calculus of variations. More information about this development can be found at https://en.wikipedia.org/wiki/Beltrami_identity, retrieved June 25 2022.

This equation is solved by the parametric equations

$$x = \frac{k^2}{2}(\theta - \sin\theta),$$

$$y = \frac{k^2}{2}(1 - \cos\theta),$$

which are seen to be the equations of a cycloid.

3. Euler navigation problem

In *De motu cymbarum*, Euler considers the following problem: “A boat with a perfect rudder moves at constant speed across a stream flowing in straight streamlines at assigned speeds. Assuming that the downstream velocity of the boat equals that of the river, how should the rudder be set so that the boat traverses a given path? He works out various instances, one of which gives rise to a variational problem, in detail.” (From Clifford Truesdell's *An idiot's fugitive essays on science: methods, criticisms, training, circumstances.*)

The variational problem is the last problem dealt with by Euler in this publication (Problem 5), and it is posed as follows: Knowing the velocity of the river at every location, to find the quickest crossing line AMC , in which the boat leaving from A , reaches C sooner than across any other line joining the points A and C .

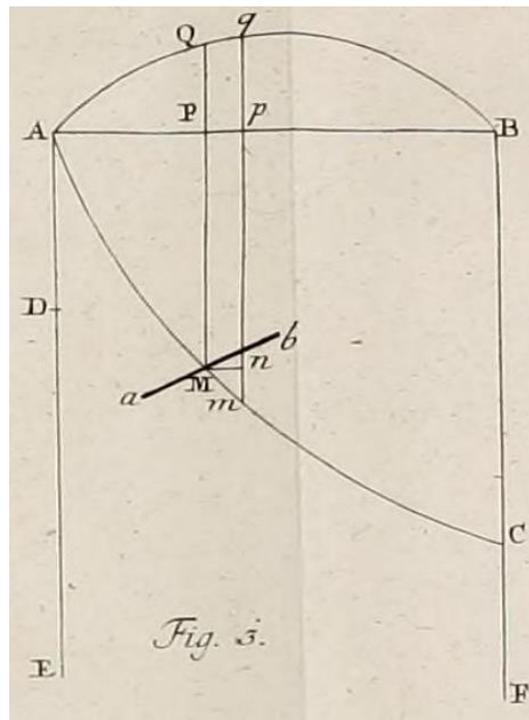


Figure 2: In this figure, aMb represents the direction of the boat spine at this location, AQB indicates the velocity of the river in each location (the river velocity profile), and AMC is the quickest crossing line for a boat leaving from A to reach point C on the opposite bank of the river.

In the previous problem of this publication (Problem 4), Euler shows that the time that the boat takes to traverse the arc AM is given by

$$\int \frac{udy - \sqrt{c^2 ds^2 - u^2 dx^2}}{u^2 - c^2},$$

where u is the velocity of the river, c is the velocity of the boat (supposed constant), y and x are the orthogonal coordinates, and ds is an element of the arc AM , such that $ds = \sqrt{dx^2 + dy^2}$.

Therefore, the integral that should be minimized is

$$\int Z dx = \int \frac{udy - \sqrt{c^2(dx^2 + dy^2) - u^2 dx^2}}{u^2 - c^2} dx = \int \frac{uy' - \sqrt{c^2(1 + y'^2) - u^2}}{u^2 - c^2} dx,$$

and, then

$$Z = \frac{uy' - \sqrt{c^2(1 + y'^2) - u^2}}{u^2 - c^2},$$

where Z as a function of the variables u and p or x and p , since u depends on x only.

Euler gives the solution and only briefly summarizes the method that he used by saying "... Last year I shared a universal method for solving such questions...". We shall now apply the Euler-Lagrange equation to recover Euler's solution to this problem.

Since the function Z does not depend explicitly on y , the Euler-Lagrange equation simplifies to

$$\frac{\partial Z}{\partial y'} = \text{Const.}$$

Evaluating

$$\frac{\partial Z}{\partial y'} = \frac{u}{u^2 - c^2} - \frac{c^2 y'}{(u^2 - c^2) \sqrt{c^2(1 + y'^2) - u^2}},$$

which after substitution gives

$$\frac{u}{u^2 - c^2} - \frac{c^2 y'}{(u^2 - c^2) \sqrt{c^2(1 + y'^2) - u^2}} = \text{Const.} = \frac{1}{g}$$

or, since $y' = p$, this expression can be rewritten as

$$gu\sqrt{c^2 + c^2 p^2 - u^2} - c^2 gp = (u^2 - c^2)\sqrt{c^2 + c^2 p^2 - u^2},$$

which by squaring results in

$$p = \frac{c^2 + gu - u^2}{c\sqrt{(g - u)^2 - c^2}}$$

Since we have that $dy = p dx$, the desired curve will be expressed as $dy = \frac{(c^2 + gu - u^2) dx}{c\sqrt{(g-u)^2 - c^2}}$, from which, because the variables are separated from one another, the curve can be constructed by integration, and, then,

$$y = \int \frac{(c^2 + gu - u^2)}{c\sqrt{(g-u)^2 - c^2}} dx,$$

such that x vanishes for $y = 0$, knowing that, in general, $u = u(x)$.

4. Lagrange contributions to variational calculus^e

Variational calculus is a fundamental instrument for analyzing problems of optimization, and as such, is a fundamental topic in physics, economics, engineering, etc. Its origin traces back to a typical engineering problem—the problem of minimum resistance posed by Newton in 1685—, which was discussed earlier.

Solutions to optimization problems were all essentially geometric in nature, and very *ad hoc*. Even though Euler's treatment was systematic for the first time, his approach as we saw earlier, was somewhat geometric. Lagrange, despite being greatly influenced by Euler's *Methodus inveniendi*, was somewhat dissatisfied, which motivates him (then only at the age of 19 years) to write an article published in 1762 [7] and describing Euler's method as "... an ingenious and fertile as its method is, we must admit that it has not all the simplicity that one might wish in a work of pure analysis. The author [Euler] himself seems to feel this, by his words: it seems desirable to find a method that is independent of geometry ..."

S. Serfaty [8] considers that "... [Lagrange] was able to find anew the results of Euler while freeing himself from geometric intuition (displacing the graph of the function), and replacing it with a "machinery" of operations of calculus. He had seen that Euler's calculus led to defining a new type of differential calculus, in which the objects are no longer functions of real variables, but functions of functions (today called functionals). This crucial conceptual leap (seeing the functions themselves as variables) is truly due to Lagrange, and can be seen as one of his fundamental contributions ..."

Here is what Lagrange says about his method in the *Essai* [7]: "... Now here is a method that requires only a very simple use of the principles of differential and integral calculus, but first of all I must remark that, as this method requires that the same quantities vary in two different ways, so as not to confuse these variations, I introduced in my calculations a new character δ . Thus, δ expresses a difference of Z which is not the same as dZ , but will be however formed by the same rules ...". Later on, in the preface of the *Mécanique Analytique* he says: "... One will find no figures in this work. The method here expounded demands neither constructions nor geometric arguments, but only algebraic operations subject to a regular and uniform procedure ..."

In the brachistochrone problem, one wishes to minimize in the graphs $y(x)$ the quantity

$$F(y) = \int_{P_1}^{P_2} \sqrt{\frac{1 + y'^2}{2gy}} dx,$$

^e This section is a summary of an article by S. Serfaty [Ref. 8].

called the Lagrangian, which depends on a curve.

It is always possible put this expression in the form $L(y, y')$ while trying to minimize^f

$$F(y) = \int_{x_0}^{x_1} L(y, y') dx,$$

Whereas ordinary differential calculus makes it possible to say that if a function f of a variable x reaches its minimum in x , then the derivative of f must be null in x , the question dealt with by the calculus of Lagrange is thus to give a meaning to the equation $F'(y) = 0$, in analogy with ordinary differential calculus.

The fundamental idea of Lagrange is to make, in analogy to the variation $x \rightarrow x + h$ used in differential calculus, variations on the function y itself, by transforming $y(x)$ into $y(x) + hv(x)$, where v is another function and h a small number.

The optimal curve $y = y(x)$ can thus be compared to the curve after the small variation $y = y(x) + hv(x)$. For Lagrange, hv is a δy , and that the derivative of the variation hv is equal to hv' , that is, it is the variation of the derivative, $d\delta = \delta d$.

If y is a minimizing function of F , then

$$\lim_{h \rightarrow 0} \frac{F(y + hv) - F(y)}{h} = 0,$$

if that limit exists.

This is equivalent to

$$\frac{\delta F}{\delta y} = 0.$$

Then, the formal calculation of this limit and from other mathematical considerations leads to

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = 0,$$

which is the Euler–Lagrange equation associated with the problem of minimization.

Lagrange also introduced the method of multipliers—today called Lagrange multipliers—which makes it possible to treat problems with constraints [8]. One example is the geodesic problem with an obstacle: to find the shortest path from A to B that goes around an obstacle. Lagrange's method of variation tells us that the geodesic "takes off" from the obstacle in such a way that it remains tangent to it. This is a particular case of a variational problem, called the "obstacle problem" (Fig. 3).

The general framework [8] is to try to minimize a function of the type

$$F(y) = \int_{x_0}^{x_1} L(y, y') dx,$$

^f We have simplified somewhat the original notation of Ref. [8].

under an integral constraint of the type

$$G(y) = \int_{x_0}^{x_1} K(y, y') dx.$$

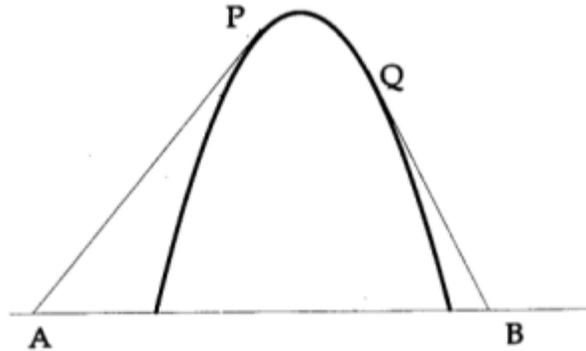


Figure 3: The obstacle problem.

The idea is to allow variations $y = y(x) + hv(x)$ that do not affect (in first order) the constraint, that is, such that $\frac{\delta G}{\delta y}(v) = 0$. By carrying out the calculations as above, one finds that there is a constant λ , such that

$$\frac{\delta F}{\delta y} = \lambda \frac{\delta G}{\delta y'}$$

where λ is called the Lagrange multiplier. The method never gives its value, but this can be found by indirect means [8].

5. Conclusion

Euler's contributions to the calculus of variations are reviewed by presenting Euler's derivation of the Euler-Lagrange equation from geometrical arguments, as found in Euler's publication *Methodus inveniendi*, and by presenting two classical variational problems considered by Euler—the body of minimum resistance and the brachistochrone. Another less known variational problem considered by Euler in another publication *De motu cymbarum* is also reviewed. Finally, a brief account on Lagrange contributions to variational calculus is presented, including his derivation of the Euler-Lagrange equation, and Lagrange's general framework of variational problems with constraints, which leads to the method of Lagrange multipliers.

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