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De progressionibus harmonicis observationes

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DE PROGRESSIONIBVS HARMONICIS OBSERVATIONES.

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§. 1.

Progressionum harmonicarum nomine intelliguntur omnes series fractionum, quarum numeratores sunt aequales inter se, denominatores vero progressionem arithmeticam constituunt. Huiusmodi ergo forma generalis est $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2b}$, $\frac{c}{a+3b}$, etc. Quique enim tres termini contigui vt $\frac{c}{a+b}$, $\frac{c}{a+2b}$, $\frac{c}{a+3b}$ hanc habent proprietatem, vt differentiae extremorum a medio sint ipsis extremis proportionales. Scilicet est $\frac{c}{a+b} - \frac{c}{a+2b} : \frac{c}{a+2b} - \frac{c}{a+3b} = \frac{c}{a+b} : \frac{c}{a+3b}$. Cum autem haec sit proprietas proportionis harmonicae; vocatae sunt istiusmodi fractionum series progressiones harmonicae. Vocari etiam possent reciprocae primi ordinis, quia in termino generali $\frac{c}{a+(n-1)b}$ index n vnicam eamque negatiuam habet dimensionem.

§. 2. Quanquam in his seriebus termini perpetuo decrescunt; tamen summa huiusmodi seriei in infinitum continuatae semper est infinita. Ad hoc demonstrandum non opus est methodo hasce series summandi; sed veritas facile ex sequente principio elucebit. Series quae in infinitum continuata summam habet finitam, etiamsi ea ſi ea duplo longius continuetur nullum accipiet augmentum, ſed id quod poſt infinitum adiicitur cogitatione, re vera erit infinite paruum. Niſi enim hoc ita ſe haberet, ſumma ſeriei etſi in infinitum continuatae non eſſet determinata et propterea non finita. Ex quo conſequitur, ſi id, quod ex continuatione vltra terminum infiniteſimum oritur, ſit finitae magnitudinis, ſummam ſeriei neceſſario infinitam eſſe debere. Ex hoc ergo principio iudicare poterimus, vtrum ſeriei cuiusque propoſitae ſumma ſit infinita an finita.

§. 3. Sit itaque ſeries $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2b}$ etc. in infinitum continuata, terminusque infiniteſimus $\frac{c}{a+(i-1)b}$, denotante i numerum infinitum, qui ſit index huius termini. Iam haec ſeries vlterius continuetur a termino $\frac{c}{a+ib}$ vsque ad terminum $\frac{c}{a+(ni-1)b}$ cuius exponens eſt ni. Horum terminorum igitur inſuper adiectorum numerus eſt $(n-1)i$; Summa eorum vero minor erit quam $\frac{(n-1)ic}{a+ib}$ maior vero quam $\frac{(n-1)ic}{a+(ni-1)b}$. Sed quia i eſt infinite magnum, evaneſcet a in vtroque denominatore. Quare ſumma maior erit quam $\frac{(n-1)c}{nb}$ at minor quam $\frac{(n-1)c}{b}$. Ex quo perſpicitur hanc ſummam eſſe finitam, atque conſequenter ſeriei propoſitae $\frac{c}{a}$, $\frac{c}{a+b}$, etc. in infinitum continuatae ſummam infinite magnam.

§. 4. Huius autem ſummae terminorum ab i ad ni limites propiores ex ſequentibus proportionis harmonicae proprietatibus eliciuntur. Scilicet omnis proportio harmonica ita eſt comparata, vt terminus medius minor ſit

ſit quam pars tertia ſummae terminorum omnium. Hanc ob rem terminus medius inter $\frac{c}{a+ib}$ et $\frac{c}{a+(ni-1)b}$, qui eſt $\frac{c}{a+\frac{(ni+i-1)}{2}b}$, ductus in terminorum numerum $(n-1)i$, ſeu $\frac{(n-1)ic}{a+\frac{(ni+i-1)}{2}b}$ minor erit quam ſumma terminorum Siue terminorum ſumma hinc maior erit quam $\frac{2(n-1)c}{(n+1)b}$ ob i infinitum. Praeterea medium arithmeticum inter terminos extremos maius eſt parte tertia ſummae terminorum. Ex hoc ſequitur fore etiam in ſerie harmonica terminorum ſummam minorem quam $(n-1)i$ in medium arithmeticum terminorum extremorum, quod eſt $\frac{(2a+(ni+i-1)b)c}{2(a+ib)(a+(ni-1)b)}$ ſeu $\frac{(n+1)c}{2nib}$, ductum. Quare ſumma erit minor quam $\frac{(n^2-1)c}{2nb}$, ita vt hi duo limites ſint $\frac{2(n-1)c}{(n+1)b}$ et $\frac{(n^2-1)c}{2nb}$, adeoque ſumma proxime $=\frac{(n-1)c}{b\sqrt{n}}$ quod eſt medium proportionale inter limites.

§. 5. Ex his colligere licet, quibus caſibus haec ſeries magis vniuerſalis $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2^{\alpha}b}$ etc. in infinitum vsque ad $\frac{c}{a+i^{\alpha}b}$ habeat ſummam finitam vel infinitam. Sequantur enim terminum vltimum termini $(n-1)i$, eritque horum ſumma minor quam $\frac{(n-1)c}{i^{\alpha-1}b}$, at maior quam $\frac{(n-1)c}{n^{\alpha}i^{\alpha-1}c}$. Quare ſi fuerit α numerus vnitate maior

maior, ſumma horum terminorum ſequentium erit $= 0$, et propterea ſumma progreſſionis finita. At ſi ſit $\alpha < 1$, ſumma terminorum ſequentium erit infinita; quocirca ipſius progreſſiones ſumma in infinities maiore gradu erit infinita. Inter has igitur progreſſiones ſola harmonica, in qua $\alpha = 1$, hanc habet proprietatem, vt ſumma eius in infinitum continuatae ſit infinite magna, terminorum vero ſequentium poſt terminum infiniteſimum ſumma finita.

§. 6. Quanta vero ſit ſumma terminorum a termino indicis i ad terminum indicis ni ſequenti modo inueſtigo. Ponatur ſumma ſeriei $\frac{c}{a}$, $\frac{c}{a+b}$ - - - - - - - - $\frac{c}{a+(i-1)b}$ ad terminum indicis i vsque $= s$, quae eſt quantitas ex a, b, c et i determinanda. Creſcat i vnitate, habebitque s pro augmento terminum ſequentem $\frac{c}{a+ib}$. Quare erit $di : ds = 1 : \frac{c}{a+ib}$ ſeu $ds = \frac{c\,di}{a+ib}$. Vnde inuenitur $s = C + \frac{c}{b} l(a+ib)$, denotante C quantitatem quandam conſtantem. Apparet quoque ex hac forma ſummam eiusdem ſeriei ab initio ad terminum indicis ni continuatae fore $= C + \frac{c}{b} l(a+nib)$. Harum igitur ſummarum differentia $\frac{c}{b} l \frac{a+nb}{a+ib} = \frac{c}{b} ln$ (euaneſcente a) dabit ſummam terminorum ab $\frac{c}{a+ib}$ vsque ad $\frac{c}{a+nib}$. Quia autem huius ſummae limites ſupra aſſignauimus erit $\frac{c}{b} ln$ maior quam $\frac{2(n-1)c}{(n+1)b}$ atque minor quam $\frac{(n^2-1)c}{2nb}$, ſeu $ln > \frac{2(n-1)}{n+1}$ atque $ln < \frac{n^2-1}{2n}$.

§. 7. Infra oſtendemus quantitatem illam conſtantem C eſſe finitam, eamque definire conabimur. Eua-

 neſcet

nescet ergo C in summa, fietque progressionis $\frac{c}{a}, \frac{b}{a+b} - - - - \frac{c}{a+(i-1)b}$ existente terminorum numero infinito $= i$, summa $= \frac{c}{b} l(a+ib) = \frac{c}{b} li$. Quamobrem summa erit vt logarithmus numeri terminorum, hincque infinities minor quam radix quantumuis magnae potestatis ex numero terminorum; nihilo tamen minus est infinite magna.

§. 8. Ex hac consideratione innumerabiles oriuntur series ad logarithmos quorumuis numerorum designandos. Sumamus primo hanc progressionem harmonicam $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$ etc. pro qua fit $a = 1$, $b = 1$, $c = 1$. Differentia igitur inter hanc seriem $1 + \frac{1}{2} + \frac{1}{3} - - - - - \frac{1}{i}$ ad terminum indicis i continuatam, et eandem $1 + \frac{1}{2} + \frac{1}{3} - - - - - \frac{1}{ni}$ ad terminum indicis ni continuatam, erit $= ln$. Quare illa series ab hac subtracta relinquit ln. Quia autem huius seriei numerus terminorum est n vicibus maior quam illius, ab n terminis seriei $1 + \frac{1}{2} - - - - - - \frac{1}{ni}$ subtrahi oportet vnicum alterius seriei $1 + \frac{1}{2} - - - - - \frac{1}{i}$, quo subtractio in infinitum eodem modo possit perfici. Quare erit

$$ln = \begin{array}{l} 1 + \frac{1}{2} - - - - - - \frac{1}{n} + \frac{1}{n+1} - - - - \frac{1}{2n} + \frac{1}{2n+1} - - - - - \frac{1}{3n} + \text{etc.} \\ \phantom{1 + \frac{1}{2} - - - - - -} - 1 \phantom{+ \frac{1}{n+1} - - - -} - \frac{1}{2} \phantom{+ \frac{1}{2n+1} - - - - -} - \frac{1}{3} \end{array}$$

Si igitur inferioris seriei singuli termini a supra scriptis terminis superioris seriei actu subtrahantur, et pro n numeri integri scribantur 2, 3, 4 - - - etc. successiue sequentes logarithmorum series obtinebimus.

$$l2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{1}{12} \text{ etc.}$$
$$l3 = 1 + \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{2}{6} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{2}{9} + \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{2}{12} \text{ etc.}$$
$$l4 = 1 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{3}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{3}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{3}{12} \text{ etc.}$$
$$l5 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{4}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{4}{10} + \tfrac{1}{11} + \tfrac{1}{12} \text{ etc.}$$
$$l6 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{5}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{5}{12} \text{ etc.}$$

etc. etc.

Vnde pro cuiusuis numeri logarithmo facile ſeries conuergens inuenitur.

§. 9. Ex his ſeriebus aliae eiusdem formae, quae ſummam habeant rationalem, poſſunt deriuari, Nam, quia ſeriei $= l2$ duplum aequale eſt $l4$, ſi ſeries $1 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{3}{4}$ etc. ſubtrahatur ab hac $2 - \tfrac{2}{2} + \tfrac{2}{3} - \tfrac{2}{4}$ etc. reſiduum, nempe haec ſeries $1 - \tfrac{3}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{3}{6}$ etc. erit $= 0$, ſeu $\tfrac{1}{2} = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{3}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{3}{10}$ etc. Similiter ſi ſeries $l6$ exhibens ſubtrahatur a ſumma ſerierum $l2$ et $l3$ exhibentium, reſiduum, nempe $1 - \tfrac{1}{2} - \tfrac{2}{3} - \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{2}{6} + \tfrac{1}{7} - \tfrac{1}{8} - \tfrac{2}{9} - \tfrac{1}{10}$ etc. erit $= 0$ ſeu $1 = \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{1}{4} - \tfrac{1}{5} - \tfrac{2}{6} - \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{2}{9} + \tfrac{1}{10}$ etc. Pari modo huiusmodi ſeries innumerabiles poterunt inueniri.

§. 10. Series illae logarithmos exprimentes conuergunt quidem, ſed admodum tarde, quare, quo earum ope logarithmi commode erui queant, requiritur aliquod ſubſidium. Ad quod inueniendum notari oportet eas ſeries non aequabiliter progredi, ſed certas habere reuolutiones, quae tot terminis abſoluuntur, quot n habet vnitates, tot igitur terminos ſimul ſumtos vnum ſeriei membrum vocabo. Ita in ſeriei pro $l2$ duo termini conſtituent vnum membrum, in ſerie pro $l3$, tres, in ſerie pro $l4$ quatuor et ita porro. Membra igitur iſta

 aequa-

aequabilem conſtituent ſeriem, et ad logarithmos inueniendos oportet aliquot membra addi. Ponamus iam m membra eſſe addita ad logarithmum binarii inueniendum; poteritque loco omnium ſequentium addi $\frac{1}{4m}$, id quod eo propius accedet, quo maior fuerit numerus m. Ad $l3$ ininueniendum ad m membra iam addita loco omnium ſequentium addatur $\frac{1}{9m}$. Similiter pro $l4$ addi debet $\frac{1}{16m}$ et ita porro. Fluunt haec ex modo ſummandi (§. 6) adhibito, in quo cum m debeat eſſe quantitas valde magna, neglexi in differentiali numeros ipſi m adiectos, ne integratio a logarithmis pendeat.

§. 11. Quo autem ſeriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}----\frac{1}{i}$ ſumma, etiamſi infinita accurate determinetur, ſingulos terminos ſequenti modo exprimo.

$$\text{Eſt } 1 = l2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} \text{ etc.}$$

$$\text{atque } \frac{1}{2} = l\frac{3}{2} + \frac{1}{2.4} - \frac{1}{3.8} + \frac{1}{4.16} - \frac{1}{5.32} \text{ etc.}$$

$$\frac{1}{3} = l\frac{4}{3} + \frac{1}{2.9} - \frac{1}{3.27} + \frac{1}{4.81} - \frac{1}{5.243} \text{ etc.}$$

$$\frac{1}{4} = l\frac{5}{4} + \frac{1}{2.16} - \frac{1}{3.64} + \frac{1}{4.256} - \frac{1}{5.1024} \text{ etc.}$$

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$$\frac{1}{i} = l\frac{i+1}{i} + \frac{1}{2.i^2} - \frac{1}{3.i^3} + \frac{1}{4.i^4} - \frac{1}{5.i^5} \text{ etc.}$$

His ſeriebus additis prodibit

$$1+\frac{1}{2}+\frac{1}{3}-----\frac{1}{i} = l(i+1) + \frac{1}{2}(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}) \text{ in infin.}$$

$$-\frac{1}{3}(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\text{etc.})$$

$$+\frac{1}{4}(1+\frac{1}{16}+\frac{1}{81}--+\frac{1}{256}+\text{etc.}$$

etc.

Quae

Quae ſeries, cum ſint conuergentes, ſi proxime ſummentur prodibit $1+\frac{1}{2}+\frac{1}{3}---\frac{1}{i}=l(i+1)+0{,}577218$ Si ſumma dicatur s, foret, vt ſupra fecimus, $ds=\frac{di}{i+1}$, ideoque $s=l(i+1)+C$. Huius igitur quantitatis conſtantis c valorem deteximus, quippe eſt $C=0{,}577218$.

§. 12. Si ſeries $1+\frac{1}{2}+\frac{1}{3}-----\frac{1}{i}$ vlterius in infinitum continuetur, et in membra diuidatur, quorum quodvis vt ipſa ſeries i terminos contineat; erit membrum inter $\frac{1}{i}$ et $\frac{1}{2i}$ contentum $=l\,2$, ſequens $=l\frac{3}{2}$, tertium $=l\frac{4}{3}$, etc. Atque cum ipſius ſeriei ſumma ſit log. infiniti, poterit ad analogiam poni $l\frac{1}{0}$. Hocque modo ſequens ſchema obtinebimus non parum curioſum

Series.	$1+\frac{1}{2}---\frac{1}{i}$	$----\frac{1}{2i}$	$---\frac{1}{3i}$	$---\frac{1}{4i}$	$---\frac{1}{5i}$	etc.
Summae	$l\frac{1}{0}$	$l\frac{2}{1}$	$l\frac{3}{2}$	$l\frac{4}{3}$	$l\frac{5}{4}$	

§. 13. Difficile quidem videatur has easdem proprietates progreſſionum harmonicarum et logarithmorum expreſſiones analytice, eoque modo, quem alibi ad ſeries ſummandas tradidi, inuenire. At rem attentius perpendenti hoc non ſolum fieri, ſed multo generalius etiam fieri poſſe deprehenſum eſt. Confidero enim non ſimplicem progreſſionem harmonicam, ſed cum geometrica coniunctam, cuiusmodi eſt $\frac{cx}{a}+\frac{cx^2}{a+b}+\frac{cx^3}{a+2b}+\frac{cx^4}{a+3b}$ etc. Huius ſummam pono s, et vtroque per $bx^{\frac{a-b}{b}}$ multiplicato erit $bx^{\frac{a-b}{b}}s=\frac{bcx^{\frac{a}{b}}}{a}+\frac{bcx^{\frac{a+b}{b}}}{a+b}+\frac{bcx^{\frac{a+2b}{b}}}{a+2b}$ Sumtisque differentialibus habebitur $b\mathrm{D}.\,x^{\frac{a-b}{b}}s=dx(cx^{\frac{a-b}{b}}$

$+cx^{\frac{a}{b}}+cx^{\frac{a+b}{b}}+$etc.$)=\frac{cx^{\frac{a-b}{b}}dx}{1-x}$ Sumtisq; iterum integralibus erit $bx^{\frac{a-b}{b}}s=c\int\frac{x^{\frac{a-b}{b}}dx}{1-x}$ atque $s=\frac{cx^{a-b}}{b}\int\frac{x^{\frac{a-b}{b}}dx}{1-x}$

Ab hac ſerie iam ſubtraho hanc $\frac{fx^m}{g}+\frac{fx^{2m}}{g+b}+\frac{fx^{3m}}{g+2b}$ etc. cuius ſumma ſit t. Multiplicetur per $\frac{b}{m}x^{\frac{m(g-b)}{b}}$ erit $\frac{b}{m}$ $x^{\frac{m(g-b)}{b}}t=\frac{fbx^{\frac{mg}{b}}}{mg}+\frac{fbx^{\frac{m(g+b)}{b}}}{m(g+b)}+\frac{fbx^{\frac{m(g+2b)}{b}}}{m(g+2b)}$ etc.

Sumtisque differentialibus fiet $\frac{b}{m}D.\,x^{\frac{m(g-b)}{b}}t=dx(fx^{\frac{mg-b}{b}}$ $+fx^{\frac{m(g+b)-b}{b}}+fx^{\frac{m(g+2b)-b}{b}}$ etc.$)=\frac{fx^{\frac{mg-b}{b}}dx}{1-x^m}$.

Quare habebitur $t=\frac{fm}{bx^{\frac{m(g-b)}{b}}}\int\frac{x^{\frac{m(g-b)}{b}}dx}{1-x^m}$. Ideoque $s-t=\frac{c}{bx^{\frac{a-b}{b}}}\int\frac{x^{\frac{a-b}{b}}}{1-x}-\frac{fm}{bx^{\frac{m(g-b)}{b}}}\int\frac{x^{\frac{mg-b}{b}}dx}{1-x^m}$. Subtractio vero ita debet fieri, vt a termino indicis m ſeriei s ſubtrahatur terminus primus ſeriei t, et a termino indicis $2m$ illius, terminus ſecundus huius ſeriei et ita porro.

§. 14. Quo noſtras ſeries logarithmicas eruamus, ſit $a=b$ et $g=b$. Quo facto erit $s=\frac{c}{b}\int\frac{dx}{1-x}=\frac{c}{b}l\frac{1}{1-x}$ et $t=\frac{f}{b}\int\frac{mx^{m-1}dx}{1-x^m}=\frac{f}{b}l\frac{1}{1-x^m}$. Ergo $s-t=$

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$l\frac{(1-x^m)^{\frac{f}{b}}}{(1-x)^{\frac{c}{b}}}$. Quo autem haec expressio fiat finita facto $x=1$, debebit esse $\frac{f}{b}=\frac{c}{b}$, hanc ob rem fiant omnes hae litterae $=1$, eritque $s-t=l\frac{1-x^m}{1-x}=l(1+x+x^2 - - - - - x^{m-1})$ Quae expressio dat differentiam inter has series $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}$ etc. et $\frac{x^m}{1}+\frac{x^{2m}}{2}+\frac{x^{3m}}{3}$ etc. Quare si $m=2$ erit $l(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$ etc. si $m=3$, erit $l(1+x+x^2)=x+\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}-\frac{2x^6}{6}+$ etc. similique modo $l(1+x+x^2+x^3)=x+\frac{x^2}{2}+\frac{x^3}{3}-\frac{3x^4}{4}+$ etc. In his si fiat $x=1$, prodibunt eaedem series pro logarithmis numerorum naturalium, quas ante dedimus.

§. 15. Si $b=2g$, erit $t=\frac{fx^{\frac{m}{2}}}{b}\int\frac{mx^{\frac{m-2}{2}}dx}{1-x^m}$. Ponatur $x^m=y$, erit $t=\frac{f\surd y}{b}\int\frac{dy}{(1-y)\surd y}=\frac{f\surd y}{b}l\frac{1+\surd y}{1-\surd y}=\frac{fx^{\frac{m}{2}}}{b}l\frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$. Si praeterea sit $a=b$ erit $s=\frac{c}{b}l\frac{1}{1-x}$. At s est summa huius seriei $\frac{cx}{a}+\frac{cx^2}{2a}+\frac{cx^3}{3a}$ etc. atque $tx^{\frac{-m}{2}}=\frac{f}{b}l\frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$ dat hanc seriem $\frac{fx^{\frac{m}{2}}}{g}+\frac{fx^{\frac{3m}{2}}}{3g}+\frac{fx^{\frac{5m}{2}}}{5g}+$ etc. Sit $a=1$ et $g=1$ erit $s-tx^{\frac{-m}{2}}=cl\frac{1}{1-x}-$

$\frac{f}{2} l \frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}} = l \frac{(1-x^{\frac{m}{2}})^{\frac{f}{2}}}{(1-x)^c (1+x^{\frac{m}{2}})^{\frac{f}{2}}}$. Quae expressio quo fiat finita si $x=1$ oportet sit $\frac{f}{2}=c$ seu $f=2c$. Sit igitur $c=1$, et $m=2n$ erit serierum $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}$ etc. et $\frac{2x^n}{1}+\frac{2x^{3n}}{3}+\frac{2x^{5n}}{5}$ etc. differentia $= l \frac{1-x^n}{(1-x)(1+x^n)}$. Ponatur $n=2$ erit differentia haec $= l \frac{1+x}{1+x^2}$ factoque $x=1$, erit ea $=0$, quare haec series $1-\frac{3}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{3}{6}+\frac{1}{7}$ etc. erit $=0$, vt iam supra inuenimus.

§. 16. Huiusmodi series summam rationalem habentes nunc ex hac ipsa forma $l \frac{1+x}{1+xx}$, infinitae aliae possunt inueniri, assumendis aliis formis similibus quae facto $x=1$ euanescant. Ex hac enim forma $l \frac{1+x}{1+x^2}$ si per series exprimatur statim resultat series inuenta. Est nimirum $l(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-$ etc. Atque $l(1+x^2)=\frac{x^2}{1}-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\frac{x^{10}}{5}-$ etc. Haec igitur series á superiore subtracta relinquit hanc $\frac{x}{1}-\frac{3x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}-\frac{3x^6}{6}$ etc. cuius summa erit $l \frac{1+x}{1+x^2}$. Similiter $l \frac{1+x}{1+x^3}$ dabit hanc seriem $\frac{x}{1}-\frac{x^2}{2}-\frac{2x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\frac{2x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}-\frac{2x^9}{9}$ etc. Ergo posito $x=1$ erit $0=1-\frac{1}{2}-\frac{2}{3}-\frac{1}{4}+\frac{1}{5}+\frac{2}{6}+\frac{1}{7}-\frac{1}{8}-\frac{2}{9}$ quam eandem iam §. 9. inuenimus.

§. 17. Hac ratione omnium huiusmodi irregularium serierum, quae tamen secundum membra regulariter procedunt, summae poterunt inueniri, semper enim vt differentiae duarum serierum sunt aestimandae. Vt sit proposita haec series $1-\frac{2}{2}+\frac{1}{3}+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}+\frac{1}{7}-\frac{2}{8}$ etc. Haec est differentia harum serierum $x+\frac{x^2}{2}+\frac{x^3}{3}$

$+\frac{x^4}{4}$

$+\frac{x^4}{4}+\frac{x^5}{5}$ etc. et $\frac{3x^2}{2}+\frac{3x^5}{5}+\frac{3x^8}{8}$ etc. facto $x=1$. Il-
lius autem summa est $l\frac{1}{1-x}$, huius vero summa est $\int\frac{3xdx}{1-x^3}$
seu $l\frac{1}{1-x}+\frac{1}{2}l(x^2+x+1)+\frac{\sqrt{-3}}{2}l\frac{2x+1-\sqrt{-3}}{2x+1+\sqrt{-3}}-\frac{\sqrt{-3}}{2}l\frac{1-\sqrt{-3}}{1+\sqrt{-3}}$.
Hac igitur ab illa subtracta factoque $x=1$ prodibit $-\frac{1}{2}l3$
$+\frac{\sqrt{-3}}{2}l\frac{3+\sqrt{-3}}{3-\sqrt{-3}}-\frac{\sqrt{-3}}{2}l\frac{1+\sqrt{-3}}{1-\sqrt{-3}}$, pro summa progressionis
propositae. Est vero $\frac{\sqrt{-3}}{2}l\frac{3+\sqrt{-3}}{3-\sqrt{-3}}$ peripheria circuli di-
uisa per $\sqrt{3}$ posito diametro $=1$ et $\frac{\sqrt{-3}}{2}l\frac{1-\sqrt{-3}}{1+\sqrt{-3}}$ huius
dimidium. Quare seriei summa quam proxime erit
0,3576.

§. 18. At si etiam ipsa membra non vniformiter incedunt difficilius summa assignatur. Sumamus hanc seriem $1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{3}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}$
$+\frac{1}{13}-\frac{4}{14}$ etc. Haec est differentia inter has series $1+$
$\frac{1}{2}+\frac{1}{3}++\frac{1}{4}+\frac{1}{5}$ - - - - - - - $\frac{1}{i(\frac{i+3}{2})}$ et $\frac{2}{2}+\frac{3}{5}+\frac{4}{9}+\frac{5}{14}$ -

- - - - - - $\frac{i+1}{i(\frac{i+3}{2})}$ ita in infinitum continuatas, vt extre-

mi termini eundem habeant denominatorem $i(\frac{i+3}{2})$. Harum serierum prioris summa est $C+li+l(i+3)-l2$, denotante C constantem §. 11. inuentam, nempe 0,577218. Altera series subtrahenda in has duas resoluitur $\frac{2}{3}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ - - - - $\frac{1}{i})$ et $\frac{4}{3}(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$ - - - - - - $\frac{1}{i+3})$. Illius summa est $\frac{2}{3}C+\frac{2}{3}li$; huius vero $\frac{4}{3}C-\frac{22}{9}+\frac{4}{3}l(i+3)$. Quae ambae ab illa summa $C+li+l(i+3)-l2$ subtractae relinquunt $-C+\frac{22}{9}-l2$ seu 1, 173078 quam proxime pro summa seriei propositae.