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De progressionibus harmonicis observationes

Leonhard Euler

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DE
PROGRESSIONIBVS HARMONICIS
OBSERVATIONES.

AVCTORE
Leonh. Eulero.

§. I.

Progressionum harmonicarum nomine intelliguntur omnes series fractionum, quarum numeratores sunt aequales inter se, denominatores vero progressionem arithmeticam constituunt. Huiusmodi ergo forma generalis est $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2b}$, $\frac{c}{a+3b}$, etc. Quique enim tres termini contigui vt $\frac{c}{a+b}$, $\frac{c}{a+2b}$, $\frac{c}{a+3b}$ hanc habent proprietatem, vt differentiae extremorum a medio sint ipsis extremis proportionales. Scilicet est $\frac{c}{a+b} - \frac{c}{a+2b} : \frac{c}{a+2b} - \frac{c}{a+3b} = \frac{c}{a+b} : \frac{c}{a+3b}$. Cum autem haec sit proprietas proportionis harmonicae; vocatae sunt istiusmodi fractionum series progressionem harmonicae. Vocari etiam possent reciprocae primi ordinis, quia in termino generali $\frac{c}{a+(n-1)b}$ index n vnicam eamque negativam habet dimensionem.

§. 2. Quanquam in his seriebus termini perpetuo decrescunt; tamen summa huiusmodi seriei in infinitum continuatae semper est infinita. Ad hoc demonstrandum non opus est methodo hasce series summam; sed veritas facile ex sequente principio elucebit. Series quae in infinitum continuata summam habet finitam, etiam-
si ea

si ea duplo longius continuetur nullum accipiet augmentum, sed id quod post infinitum adiicitur cogitatione, re vera erit infinite paruum. Nisi enim hoc ita se haberet, summa seriei etfi in infinitum continuatae non esset determinata et propterea non finita. Ex quo consequitur, si id, quod ex continuatione vltra terminum infinitesimum oritur, sit finitae magnitudinis, summam seriei necessario infinitam esse debere. Ex hoc ergo principio iudicare poterimus, vtrum seriei cuiusque propositae summa sit infinita an finita.

§. 3. Sit itaque series $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2b}$ etc. in infinitum continuata, terminusque infinitesimus $\frac{c}{a+(i-1)b}$, denotante i numerum infinitum, qui sit index huius termini. Iam haec series vltcrius continuetur a termino $\frac{c}{a+ib}$ vsque ad terminum $\frac{c}{a+(ni-1)b}$ cuius exponens est ni . Horum terminorum igitur insuper adiectorum numerus est $(n-1)i$; Summa eorum vero minor erit quam $\frac{(n-1)ic}{a+ib}$ maior vero quam $\frac{(n-1)ic}{a+(ni-1)b}$. Sed quia i est infinite magnum, evanesct a in vtroque denominatore. Quare summa maior erit quam $\frac{(n-1)c}{nb}$ at minor quam $\frac{(n-1)c}{b}$, Ex quo perspicitur hanc summam esse finitam, atque consequenter seriei propositae $\frac{c}{a}$, $\frac{c}{a+b}$, etc. in infinitum continuatae summam infinite magnam.

§. 4. Huius autem summae terminorum ab i ad ni limites propiores ex sequentibus proportionis harmonicae proprietatibus eliciuntur. Scilicet omnis proportio harmonica ita est comparata, vt terminus medius minor sit

sit quam pars tertia summae terminorum omnium. Hanc
 ob rem terminus medius inter $\frac{c}{a+i b}$ et $\frac{c}{a+(n-1)b}$, qui est
 $\frac{c}{a+\frac{(n+i-1)}{2}b}$, ductus in terminorum numerum $(n-1)i$,
 seu $\frac{(n-1)ic}{a+\frac{(n+i-1)}{2}b}$ minor erit quam summa terminorum
 Siue terminorum summa hinc maior erit quam
 $\frac{c(n-1)}{(n+i)b}$ ob i infinitum. Praeterea medium arithmeti-
 cum inter terminos extremos maius est parte tertia
 summae terminorum. Ex hoc sequitur fore etiam in
 serie harmonica terminorum summam minorem quam
 $(n-1)i$ in medium arithmeticum terminorum ex-
 tremorum, quod est $\frac{(ca+(n+i-1)b)c}{2(a+ib)(a+(n-1)b)}$ seu $\frac{(n-1)c}{2nib}$, du-
 ctum. Quare summa erit minor quam $\frac{(n^2-1)c}{2nb}$, ita ut
 hi duo limites sint $\frac{c(n-1)}{(n+i)b}$ et $\frac{(n^2-1)c}{2nb}$, adeoque summa
 proxime $=\frac{(n-1)c}{b\sqrt{n}}$ quod est medium proportionale inter
 limites.

§. 5. Ex his colligere licet, quibus casibus haec
 series magis vniuersalis $\frac{c}{a}, \frac{c}{a+b}, \frac{c}{a+2^\alpha b}$ etc. in infi-
 nitum usque ad $\frac{c}{a+i^\alpha b}$ habeat summam finitam vel in-
 finitam. Sequantur enim terminum vltimum termini $(n-1)i$,
 eritque horum summa minor quam $\frac{(n-1)c}{i^{\alpha-1}b}$, at maior
 quam $\frac{(n-1)c}{n^\alpha i^{\alpha-1}c}$. Quare si fuerit α numerus unitate
 maior.

maior, summa horum terminorum sequentium erit ∞ , et propterea summa progressionis finita. At si sit $a < 1$, summa terminorum sequentium erit infinita; quocirca ipsius progressionis summa in infinitum maiore gradu erit infinita. Inter has igitur progressionis sola harmonica, in qua $a = 1$, hanc habet proprietatem, ut summa eius in infinitum continuatae sit infinite magna, terminorum vero sequentium post terminum infinitesimum summa finita.

§. 6. Quanta vero sit summa terminorum a termino indicis i ad terminum indicis ni sequenti modo inuestigo. Ponatur summa seriei $\frac{c}{a}, \frac{c}{a+ib}, \frac{c}{a+2ib}, \dots, \frac{c}{a+(i-1)ib}$ ad terminum indicis i vsque $= s$, quae est quantitas ex a, b, c et i determinanda. Crescat i unitate, habebitque s pro augmento terminum sequentem $\frac{c}{a+ib}$. Quare erit $di : ds = 1 : \frac{c}{a+ib}$ seu $ds = \frac{c di}{a+ib}$. Vnde inuenitur $s = C + \frac{c}{b} l(a+ib)$, denotante C quantitatem quandam constantem. Apparet quoque ex hac forma summam eiusdem seriei ab initio ad terminum indicis ni continuatae fore $= C + \frac{c}{b} l(a+nib)$. Harum igitur summarum differentia $\frac{c}{b} l \frac{a+nb}{a+ib} = \frac{c}{b} ln$ (evanescente a) dabit summam terminorum ab $\frac{c}{a+ib}$ vsque ad $\frac{c}{a+nib}$. Quia autem huius summae limites supra assignauimus erit $\frac{c}{b} ln$ maior quam $\frac{2(n-1)c}{(n+1)b}$ atque minor quam $\frac{(n^2-1)c}{2nb}$, seu $ln > \frac{2(n-1)}{n+1}$ atque $ln < \frac{n^2-1}{2n}$.

§. 7. Infra ostendemus quantitatem illam constantem C esse finitam, eamque definire conabimur. Eua-
 Tom. VII. V nescet

nescet ergo C in summa, fietque progressionis $\frac{c}{a}, \frac{b}{a+b}$ $\frac{c}{a+(i-1)b}$ existente terminorum numero infinito $= i$, summa $= \frac{c}{b} l(a+ib) = \frac{c}{b} li$. Quamobrem summa erit vt logarithmus numeri terminorum, hincque infinities minor quam radix quantumvis magnae potestatis ex numero terminorum; nihilo tamen minus est infinite magna.

§. 8. Ex hac consideratione innumerabiles oriuntur series ad logarithmos quorumvis numerorum designandos. Sumamus primo hanc progressionem harmonicam $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$ etc. pro qua fit $a=1, b=1, c=1$. Differentia igitur inter hanc seriem $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i}$ ad terminum indicis i continuatam, et eandem $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{ni}$ ad terminum indicis ni continuatam, erit $= ln$. Quare illa series ab hac subtracta relinquit ln . Quia autem huius seriei numerus terminorum est n vicibus maior quam illius, ab n terminis seriei $1 + \frac{1}{2} + \dots + \frac{1}{ni}$ subtrahi oportet vnicum alterius seriei $1 + \frac{1}{2} + \dots + \frac{1}{i}$, quo subtractio in infinitum eodem modo possit perfici. Quare erit $ln = 1 + \frac{1}{2} + \dots + \frac{1}{ni}$

$$+ \frac{1}{n+1} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \dots + \frac{1}{2n} + \dots + \frac{1}{3} + \dots$$

Si igitur inferioris seriei singuli termini a supra scriptis terminis superioris seriei actu subtrahantur, et pro n numeri integri scribantur 2, 3, 4 --- etc. successive sequentes logarithmorum series obtinebimus.

$$\begin{aligned}
 l_2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} \text{ etc.} \\
 l_3 &= 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - \frac{1}{5} + \frac{2}{6} - \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \frac{1}{10} - \frac{1}{11} + \frac{2}{12} \text{ etc.} \\
 l_4 &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{2}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{3}{12} \text{ etc.} \\
 l_5 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} \text{ etc.} \\
 l_6 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{5}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} - \frac{5}{12} \text{ etc.} \\
 &\text{etc. etc.}
 \end{aligned}$$

Vnde pro cuiusvis numeri logarithmo facile series convergens inuenitur.

§. 9. Ex his seriebus aliae eiusdem formae, quae summam habeant rationalem, possunt deriuari, Nam, quia seriei l_2 duplum aequale est l_4 , si series $1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4}$ etc. subtrahatur ab hac $2 - \frac{2}{2} + \frac{2}{3} - \frac{2}{4}$ etc. residuum, nempe haec series $1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6}$ etc. erit $= 0$, seu $\frac{1}{2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{3}{10}$ etc. Similiter si series l_6 exhibens subtrahatur a summa serierum l_2 et l_3 exhibentium, residuum, nempe $1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9} - \frac{1}{10}$ etc. erit $= 0$ seu $1 = \frac{1}{2} + \frac{2}{3} + \frac{1}{4} - \frac{1}{5} - \frac{2}{6} - \frac{1}{7} + \frac{1}{8} + \frac{2}{9} + \frac{1}{10}$ etc. Pari modo huiusmodi series innumerabiles poterunt inueniri.

§. 10. Series illae logarithmos exprimentes convergunt quidem, sed admodum tarde, quare, quo earum ope logarithmi commode erui queant, requiritur aliquod subsidium. Ad quod inueniendum notari oportet eas series non aequabiliter progredi, sed certas habere reuolutiones, quae tot terminis absoluuntur, quot n habet unitates, tot igitur terminos simul sumtos vnum seriei membrum vocabo. Ita in seriei pro l_2 duo termini constituent vnum membrum, in serie pro l_3 , tres, in serie pro l_4 quatuor et ita porro. Membra igitur ista

aequabilem constituent seriem, et ad logarithmos inueniendos oportet aliquot membra addi. Ponamus iam m membra esse addita ad logarithmum binarii inueniendum; poteritque loco omnium sequentium addi $\frac{x}{4m}$, id quod eo propius accedet, quo maior fuerit numerus m . Ad l_3 inueniendum ad m membra iam addita loco omnium sequentium addatur $\frac{x}{9m}$. Similiter pro l_4 addi debet $\frac{x}{16m}$ et ita porro. Fluunt haec ex modo summandi (§. 6) adhibito, in quo cum m debeat esse quantitas valde magna, neglexi in differentiali numeros ipsi m adiectos, ne integratio a logarithmis pendeat.

§. 11. Quo autem seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{i}$ summa, etiam si infinita accurate determinetur, singulos terminos sequenti modo exprimo.

$$\text{Est } 1 = l_2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} \text{ etc.}$$

$$\text{atque } \frac{1}{2} = l_{\frac{3}{2}} + \frac{1}{2 \cdot 4} - \frac{1}{3 \cdot 8} + \frac{1}{4 \cdot 16} - \frac{1}{5 \cdot 32} \text{ etc.}$$

$$\frac{1}{3} = l_{\frac{4}{3}} + \frac{1}{2 \cdot 9} - \frac{1}{3 \cdot 27} + \frac{1}{4 \cdot 81} - \frac{1}{5 \cdot 243} \text{ etc.}$$

$$\frac{1}{4} = l_{\frac{5}{4}} + \frac{1}{2 \cdot 16} - \frac{1}{3 \cdot 64} + \frac{1}{4 \cdot 256} - \frac{1}{5 \cdot 1024} \text{ etc.}$$

⋮

⋮

⋮

$$\frac{1}{i} = l_{\frac{i+1}{i}} + \frac{1}{2 \cdot i^2} - \frac{1}{3 \cdot i^3} + \frac{1}{4 \cdot i^4} - \frac{1}{5 \cdot i^5} \text{ etc.}$$

His seriebus additis prodibit

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} = l(i+1) + \frac{1}{2}(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots) \text{ in infin.}$$

$$- \frac{1}{3}(1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots) \text{ etc.}$$

$$+ \frac{1}{4}(1 + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots) \text{ etc.}$$

etc.

Quae

Quae series, cum sint conuergentes, si proxime summentur prodibit $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} = l(i+1) + 0,577218$
 Si summa dicatur s , foret, vt supra fecimus, $ds = \frac{dx}{i+1}$, ideoque $s = l(i+1) + C$. Huius igitur quantitatis constantis c valorem deteximus, quippe est $C = 0,577218$.

§. 12. Si series $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i}$ ulterius in infinitum continuetur, et in membra diuidatur, quorum quodvis vt ipsa series i terminos contineat; erit membrum inter $\frac{1}{i}$ et $\frac{1}{2i}$ contentum $= l2$, sequens $= l\frac{3}{2}$, tertium $= l\frac{4}{3}$, etc. Atque cum ipsius seriei summa sit log. infiniti, poterit ad analogiam poni $l\frac{1}{0}$. Hocque modo sequens schema obtinebimus non parum curiosum

$$\begin{array}{l} \text{Series. } 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} \\ \text{Summae } l\frac{1}{0} \quad \left| \quad l\frac{2}{1} \quad \left| \quad l\frac{3}{2} \quad \left| \quad l\frac{4}{3} \quad \left| \quad l\frac{5}{4} \quad \left| \quad \dots \right. \right. \right. \end{array} \text{ etc.}$$

§. 13. Difficile quidem videatur has easdem proprietates progressionum harmonicarum et logarithmorum expressiones analytice, eoque modo, quem alibi ad series summandas tradidi, inuenire. At rem attentius perpendenti hoc non solum fieri, sed multo generalius etiam fieri posse deprehensum est. Considero enim non simplicem progressionem harmonicam, sed cum geometrica coniunctam, cuiusmodi est $\frac{cx}{a} + \frac{cx^2}{a+b} + \frac{cx^3}{a+2b} + \frac{cx^4}{a+3b} + \dots$
 etc. Huius summam pono s , et vtroque per $bx^{\frac{a-b}{b}}$

$$\text{multiplicato erit } bx^{\frac{a-b}{b}} s = \frac{bcx^{\frac{a}{b}}}{a} + \frac{bcx^{\frac{a+b}{b}}}{a+b} + \frac{bcx^{\frac{a+2b}{b}}}{a+2b} + \dots$$

$$\text{Sumtisque differentialibus habebitur } bD. x^{\frac{a-b}{b}} s = dx (cx^{\frac{a-b}{b}} + \dots)$$

$$+ cx^{\frac{a}{b}} + cx^{\frac{a+b}{b}} + \text{etc.}) = \frac{cx^{\frac{a-b}{b}} dx}{1-x} \text{ Sumtisq; iterum inte-}$$

$$\text{gralibus erit } bx^{\frac{a-b}{b}} s = c \int \frac{x^{\frac{a-b}{b}} dx}{1-x} \text{ atque } s = \frac{cx^{a-b} \int x^{\frac{a-b}{b}} dx}{b}$$

$$\text{Ab hac serie iam subtrahō hanc } \frac{fx^m}{g} + \frac{fx^{2m}}{g+b} + \frac{fx^{3m}}{g+2b}$$

etc. cuius summa fit t . Multiplicetur per $\frac{b}{m} x^{\frac{m(g-b)}{b}}$ erit $\frac{b}{m}$

$$x^{\frac{m(g-b)}{b}} t = \frac{fbx^{\frac{mg}{b}}}{mg} + \frac{fbx^{\frac{m(g+b)}{b}}}{m(g+b)} + \frac{fbx^{\frac{m(g+2b)}{b}}}{m(g+2b)} \text{ etc.}$$

$$\text{Sumtisq; differentialibus fiet } \frac{b}{m} D. x^{\frac{m(g-b)}{b}} t = dx (fx^{\frac{mg-b}{b}} \\ + fx^{\frac{m(g+b)-b}{b}} + fx^{\frac{m(g+2b)-b}{b}} \text{ etc.}) = \frac{fx^{\frac{mg-b}{b}} dx}{1-x^m}$$

$$\text{Quare habebitur } t = \frac{fm}{bx^{\frac{m(g-b)}{b}}} \int \frac{x^{\frac{m(g-b)}{b}} dx}{1-x^m}. \text{ Ideoque}$$

$$s - t = \frac{c}{bx^{\frac{a-b}{b}}} \int \frac{x^{\frac{a-b}{b}} dx}{1-x} - \frac{fm}{bx^{\frac{m(g-b)}{b}}} \int \frac{x^{\frac{m(g-b)}{b}} dx}{1-x^m}. \text{ Subtra-}$$

ctio vero ita debet fieri, vt a termino indicis m seriei s subtrahatur terminus primus seriei t , et a termino indicis $2m$ illius, terminus secundus huius seriei et ita porro.

§. 14. Quo nostras series logarithmicas eruamus, sit $a=b$ et $g=b$. Quo factō erit $s = \frac{c}{b} \int \frac{dx}{1-x} = \frac{c}{b} l \frac{1}{1-x}$ et $t = \frac{f}{b} \int \frac{mx^{m-1} dx}{1-x^m} = \frac{f}{b} l \frac{1}{1-x^m}$. Ergo $s - t =$

$l(1 -$

$l \frac{(1-x^m)^{\frac{f}{b}}}{(1-x)^{\frac{c}{b}}}$. Quo autem haec expressio fiat finita facto $x=1$, debeat esse $\frac{f}{b} = \frac{c}{b}$, hanc ob rem fiant omnes hae litterae $=1$, eritque $s-t = l \frac{1-x^m}{1-x} = l(1+x+x^2+\dots+x^{m-1})$ Quae expressio dat differentiam inter has series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$ etc. et $\frac{x^m}{1} + \frac{x^{2m}}{2} + \frac{x^{3m}}{3}$ etc. Quare si $m=2$ erit $l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ etc. si $m=3$, erit $l(1+x+x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \dots$ similique modo $l(1+x+x^2+x^3) = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \dots$ In his si fiat $x=1$, prodibunt eadem series pro logarithmis numerorum naturalium, quas ante dedimus.

§. 15. Si $b=2g$, erit $t = \frac{fx^{\frac{m}{2}}}{b} \int \frac{mx^{\frac{m}{2}-2} dx}{1-x^m}$. Ponatur $x^m = y$, erit $t = \frac{f\sqrt{y}}{b} \int \frac{dy}{(1-y)\sqrt{y}} = \frac{f\sqrt{y}}{b} l \frac{1+\sqrt{y}}{1-\sqrt{y}} = \frac{fx^{\frac{m}{2}}}{b} l \frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$. Si praeterea sit $a=b$, erit $s = \frac{c}{b} l \frac{1}{1-x}$. At s est summa huius seriei $\frac{cx}{a} + \frac{cx^2}{2a} + \frac{cx^3}{3a}$ etc. atque $tx^{\frac{-m}{2}} = \frac{f}{b} l \frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$ dat hanc feriem $fx^{\frac{m}{2}} + \frac{fx^{\frac{3m}{2}}}{3g} + \frac{fx^{\frac{5m}{2}}}{5g} + \dots$ Sit $a=1$ et $g=1$ erit $s-tx^{\frac{-m}{2}} = cl \frac{1}{1-x}$.

$$\frac{f}{2} \int \frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}} = \int \frac{(1-x^2)^{\frac{m}{2}}}{(1-x)^c (1+x^{\frac{m}{2}})^f}$$
 Quae expressio quo fiat finita si $x=1$ oportet sit $\frac{f}{2} = c$ seu $f = 2c$. Sit igitur $c=1$, et $m=2n$ erit serierum $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ etc. et $\frac{2x^n}{1} + \frac{2x^{3n}}{3} + \frac{2x^{5n}}{5}$ etc. differentia = $\int \frac{1-x^n}{(1-x)(1+x^n)}$. Ponatur $n=2$ erit differentia haec = $\int \frac{1-x}{1+x^2}$ factoque $x=1$, erit ea = 0, quare haec series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$ etc. erit = 0, vt iam supra inuenimus.

§. 16. Huiusmodi series summam rationalem habentes nunc ex hac ipsa forma $\int \frac{1+x}{1+x^2}$, infinitae aliae possunt inueniri, assumendis aliis formis similibus quae facto $x=1$ euanescant. Ex hac enim forma $\int \frac{1+x}{1+x^2}$ si per series exprimatur statim resultat series inuenta. Est nimirum $\int (1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} -$ etc. Atque $\int (1+x^2) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \frac{x^{10}}{10} -$ etc. Haec igitur series a superiore subtracta relinquit hanc $\frac{x}{1} - \frac{3x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{3x^6}{6}$ etc. cuius summa erit $\int \frac{1+x}{1+x^2}$. Similiter $\int \frac{1+x^3}{1+x^3}$ dabit hanc seriem $\frac{x}{1} - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} - \frac{2x^9}{9}$ etc. Ergo posito $x=1$ erit $0 = 1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9}$ quam eandem iam §. 9. inuenimus.

§. 17. Hac ratione omnium huiusmodi irregularium serierum, quae tamen secundum membra regulariter procedunt, summae poterunt inueniri, semper enim vt differentiae duarum serierum sunt aestimandae. Vt sit proposita haec series $1 - \frac{2}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8}$ etc. Haec est differentia harum serierum $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$

$+\frac{x^4}{4} + \frac{x^5}{5}$ etc. et $\frac{3x^2}{2} + \frac{3x^5}{5} + \frac{3x^8}{8}$ etc. facto $x=1$. Il-
 lius autem summa est $\int \frac{1}{1-x^3}$ huius vero summa est $\int \frac{x dx}{1-x^3}$
 seu $\int \frac{x}{1-x^3} + \frac{1}{2} \int (x^2 + x + 1) + \frac{\sqrt{-3}}{2} \int \frac{2x+1-\sqrt{-3}}{2x+1+\sqrt{-3}} - \frac{\sqrt{-3}}{2} \int \frac{1-\sqrt{-3}}{1+\sqrt{-3}}$
 Hac igitur ab illa subtracta factoque $x=1$ prodibit $-\frac{1}{2} \int 3$
 $+\frac{\sqrt{-3}}{2} \int \frac{2+\sqrt{-3}}{3-\sqrt{-3}} - \frac{\sqrt{-3}}{2} \int \frac{1+\sqrt{-3}}{1-\sqrt{-3}}$, pro summa progressionis
 propositae. Est vero $\frac{\sqrt{-3}}{2} \int \frac{2+\sqrt{-3}}{3-\sqrt{-3}}$ peripheria circuli di-
 uisa per $\sqrt{3}$ posito diametro $=1$ et $\frac{\sqrt{-3}}{2} \int \frac{1-\sqrt{-3}}{1+\sqrt{-3}}$ huius
 dimidium. Quare seriei summa quam proxime erit
 $0,3576$.

§. 18. At si etiam ipsa membra non vniformiter
 incedunt difficilius summa assignatur. Sumamus hanc
 seriem $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{3}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$
 $+\frac{1}{13} - \frac{4}{14}$ etc. Haec est differentia inter has series $1 +$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \dots - \frac{1}{i \left(\frac{i+3}{2} \right)} \text{ et } \frac{2}{2} + \frac{3}{5} + \frac{4}{9} + \frac{5}{14} - \dots - \frac{i+1}{i \left(\frac{i+3}{2} \right)} \text{ ita in infinitum continuatas, vt extre-}$$

mi termini eundem habeant denominatorem $i \left(\frac{i+3}{2} \right)$. Ha-
 rum serierum prioris summa est $C + li + l(i+3) - l2$,
 denotante C constantem §. 11. inuentam, nempe $0,577718$.
 Altera series subtrahenda in has duas resoluitur $\frac{2}{3} (1 +$
 $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \dots - \frac{1}{i})$ et $\frac{4}{3} (\frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \dots - \frac{1}{i+3})$. Illius
 summa est $\frac{2}{3} C + \frac{2}{3} li$; huius vero $\frac{4}{3} C - \frac{22}{9} + \frac{4}{3} l(i+3)$.
 Quae ambae ab illa summa $C + li + l(i+3) - l2$
 subtractae relinquunt $-C + \frac{22}{9} - l2$ seu $1,173078$ quam
 proxime pro summa seriei propositae.