Euleriana

Volume 3 | Issue 2 | Article 3

2023

Solution of the Diophantine equation (maa+nbb)=cd(mcc+ndd) using rational numbers

Georg Ehlers
Oak Ridge National Laboratory, ehlersg@ornl.gov

Recommended Citation
Ehlers, Georg (2023) "Solution of the Diophantine equation (maa+nbb)=cd(mcc+ndd) using rational numbers," *Euleriana*: *3*(2), pp.81-123.
DOI: https://doi.org/10.56031/2693-9908.1041
Available at: https://scholarlycommons.pacific.edu/euleriana/vol3/iss2/3

Solution of the Diophantine equation $ab\,(maa+nbb)=cd\,(mcc+ndd)$ using rational numbers [a]

an English translation of

Resolutio formulae Diophanteae $ab\,(maa+nbb)=cd\,(mcc+ndd)$ per numeros rationales

By Leonhard Euler

Nova acta Academiae scientiarum imperialis petropolitanae
Volume 13 (1795/96), pp. 45-63.

Translated by Georg Ehlers, Oak Ridge National Laboratory,
ehlersg@ornl.gov

Abstract

This paper (E716) was published in *Nova acta Academiae scientiarum imperialis petropolitanae*, Volume 13 (1795/96), pp. 45-63. It was also included in *Commentationes Arithmeticae*, Volume II, as Number LXVIII, pp. 281-293 (E791). Euler starts with Fermat's Last Theorem and mentions the proofs for the cases $n=3$ and $n=4$ which he had completed himself earlier. He then moves on to make the sum of powers conjecture, which was later disproved in the second half of the 20th century. In this context he discusses his discovery of $134^4+133^4=158^4+59^4$, which he calls unexpected. Euler derives the title equation from $A^4+B^4=C^4+D^4$, generalizing it to some extent, and derives three methods by which special solutions may be obtained. He gives two different sets of explicit formulas for the case $m=n=1$. Euler shows that each solution of $ab\,(aa+bb)=cd\,(cc+dd)$ leads to a conjugate solution. Euler considers the problem of making two particular binary forms simultaneously square, and shows how its solution is connected to the case $m=n=1$.

All footnotes are comments by the translator.

[a] Original text available online at https://scholarlycommons.pacific.edu/euler-works/716/

§ 1. Of the theorems once demonstrated by Fermat, and lost after his death, this theorem especially deserves the greatest attention: that there are not two powers of any order, whose sum or difference is a power of the same order, where the first and second order are excluded. Thus Fermat denied that two numbers a and b could be shown, so that the formula $a^n \pm b^n$ would be equal to c^n with the same power, and at the same time the exponent n surpasses the number two. In this way, then, all these equations will be impossible:[b]

$$\text{I. } a^3 \pm b^3 = c^3\ ;\ \text{ II. } a^4 \pm b^4 = c^4\ ;\ \text{ III. } a^5 \pm b^5 = c^5\ ;\ \text{ IV. } a^6 \pm b^6 = c^6\ ;$$
$$\text{V. } a^7 \pm b^7 = c^7\ ;\ \text{ VI. } a^8 \pm b^8 = c^8 \text{ etc.}$$

§ 2. The truth has already been shown quite successfully[c] for the first of these formulas, $a^3 \pm b^3 = c^3$, and so the truth follows at the same time for all the formulas in which the exponent n is a multiple of three. Then, however, the same has been demonstrated even more clearly for the second formula, $a^4 \pm b^4 = c^4$, since it was most clearly proved that neither the sum nor the difference of two biquadrates[d] can be a square, much less therefore a biquadrate,[e] and hence all the cases in which the exponent n is a multiple of four have also been eliminated. Meanwhile, however, a demonstration of this kind, which equally extends to all exponents n, is still most wanted, as no mathematician has yet been able to penetrate into this mystery of numbers.

§ 3. It has appeared to several distinguished mathematicians that these theorems can be extended. For in the same way as two cubes cannot be shown, the sum or difference of which is a cube, it is also certain that there can not be found three biquadrates, the sum of which is equally biquadrate, but at least four biquadrates may be required, so that their sum can be a biquadrate, although no one has yet been able to assign such four biquadrates.[f] In the same way it seems that it can also be proven that four fifth powers cannot be found whose sum is a fifth power,[g] and the situation will be the same in the higher

[b]In order to enhance readability, in many instances this transcription uses display math mode rather than inline math mode.

[c]Euler is probably making reference to his proof published in Elements of Algebra, Vol. 2, chapter 15, article 243.

[d]Euler uses the word *biquadrate* to mean a fourth power.

[e]Euler's proof can be found in Elements of Algebra, Vol. 2, chapter 13, article 202.

[f]Euler's conjecture has been proven wrong, Noam D. Elkies, "On $A^4 + B^4 + C^4 = D^4$", Mathematics of Computation, **51**(184) 825-835 (1988).

[g]This conjecture has also been proven wrong, L. J. Lander and T. R. Parkin, "Counterexample to Euler's conjecture on sums of like powers", Bull. Amer. Math. Soc. **72** (6) 1079 (1966).

DOI: 10.56031/2693-9908.1041

powers, so that also all the following equations will be considered impossible:

$$\begin{aligned} \text{I. } & a^3 + b^3 = c^3 \;, \\ \text{II. } & a^4 + b^4 + c^4 = d^4 \;, \\ \text{III. } & a^5 + b^5 + c^5 + d^5 = e^5 \;, \\ \text{IV. } & a^6 + b^6 + c^6 + d^6 + e^6 = f^6 \;, \\ & \text{etc. etc.} \end{aligned}$$

Therefore, the science of numbers should be seen as greatly advanced if it were possible to extend the desired demonstration also to these formulas.

$\S\,4.$ At first glance, it might seem that these last formulas can not only be extended to sums, but also to differences, as in the first $[n = 3]$ case, so that also the equality $a^4 \pm b^4 \pm c^4 = d^4$ would be considered impossible. However, I have observed quite the opposite a few years ago, see in the Commentaries, volume XVII, p. 64,[h] where I have assigned two biquadrates, the sum of which can be resolved in other two biquadrates, that is $a^4 + b^4 = c^4 + d^4$, from which, therefore, the equality $a^4 + b^4 - c^4 = d^4$ is by no means opposed to the truth, but the numbers, which I deduced for the letters, a, b, c, d, through a very tedious calculation, are very immense.

$\S\,5.$ But when I had recently undertaken a treatment of the subject, contrary to every expectation, I fell upon much smaller numbers with this property, to such an extent that these can be considered the smallest numbers: $a = 134$; $b = 133$; $c = 158$; and $d = 59$. It is indeed found that $134^4 + 133^4 = 158^4 + 59^4$, which calculation is not so troublesome to execute, while on the other hand scarcely any one will dare to attempt a proof for those immense numbers.

$\S\,6.$ The method which I used at that time to solve the equation,

$$A^4 + B^4 = C^4 + D^4 \;,$$

proceeded as follows: I considered the equation $A^4 - C^4 = D^4 - B^4$, and on putting

$$A = a + b \;, \quad C = a - b \;, \quad D = c + d \quad \text{and} \quad B = c - d \;,$$

[h] Euler is referencing the paper "Observationes circa bina biquadrata, quorum summam in duo alia biquadrata resolvere liceat", published as E428 in the Eneström index, and also as #33 of E791 (Volume I).

the fairly simple equation $ab\,(aa+bb) = cd\,(cc+dd)$ resulted, and thus the whole task is reduced to the solution of this formula. However, here I am not only going to use the previous method, but also a much more manageable method, and resolve the more general formula presented in the title

$$ab\,(maa+nbb) = cd\,(mcc+ndd)$$

so that whatever numbers are taken for m and n, infinitely many numbers a, b, c, d, satisfying the equation can be found.

§ 7. To resolve this equality I am using first of all the transformation

$$b = cp \quad \text{and} \quad d = aq\ ,$$

and in this way the equation to be resolved will take the form

$$p\,(maa+nccpp) = q\,(mcc+naaqq)\ ,$$

$$\text{from which is deduced} \quad \frac{aa}{cc} = \frac{np^3-mq}{nq^3-mp}\ ,$$

and thus the whole task is to reduce the formula $\frac{np^3-mq}{nq^3-mp}$ to a square. This indeed happens spontaneously when $q = p$, which quickly catches our eye, leading to $\frac{aa}{cc} = 1$. It follows $c = a$ and $b = d$, which most obviously cannot be considered a solution, since both sides of the equation become identical. Meanwhile however, this case may lead to other solutions.

§ 8. Therefore, since our formula actually becomes a square by putting $p = q$, let's set $p = q\,(1+z)$, so that, on taking $z = 0$, this case is clearly recovered, and now our formula will be transformed into the following,[i]

$$\frac{aa}{cc} = \frac{nq^3\,(1+z)^3-m}{nqq-m\,(1+z)}\ , \quad \text{or} \quad \frac{aa}{cc} = \frac{nqq-m+3nqqz+3nqqzz+nqqz^3}{nqq-m-mz}$$

in which fraction the unknown quantity z rises in the numerator to the third power, while in the denominator it does not rise beyond the first power. Such formulas can be treated by known methods which are already sufficiently clear.

§ 9. In order to make this fraction more manageable, let us divide both the numerator and the denominator by $nqq-m$, and let us set for brevity

[i]The numerator of the left equation should be $nqq\,(1+z)^3-m$ instead of $nq^3\,(1+z)^3-m$.

$$\frac{nqq}{nqq-m} = \alpha \quad \text{and} \quad \frac{m}{nqq-m} = \beta\ ,$$

so that $\alpha - \beta = 1$, and therefore $\alpha = 1 + \beta$, in which case the following formula will appear

$$\frac{aa}{cc} = \frac{1 + 3\alpha z + 3\alpha zz + \alpha z^3}{1 - \beta z}\ .$$

Now, by the following method, which I once used, the denominator is rendered a square, by multiplying above and below by $1 - \beta z$, producing

$$\frac{aa}{cc} = \frac{(1-\beta z)\left(1 + 3\alpha z + 3\alpha zz + \alpha z^3\right)}{(1-\beta z)^2}\ .$$

Thus, the task is to reduce the numerator, which turns out to be

$$1 + (3\alpha - \beta)\, z + 3\alpha\,(1-\beta)\, zz + \alpha\,(1 - 3\beta)\, z^3 - \alpha\beta z^4$$

to a square by setting its root to $1 + fz + gzz$, whose square is[j]

$$1 + 2fz + (ff + 2fg)\, zz + 2fgz^3 + ggz^4\ .$$

Now because the first terms mutually cancel each other out, let the two letters f and g be so defined that the second and third terms are also removed. The former will be done by taking $f = \frac{3\alpha-\beta}{2}$, and the latter by setting

$$ff + 2g = 3\alpha\,(1-\beta)\ , \quad \text{hence it becomes} \quad g = \frac{3\alpha\,(1-\beta)}{2} - \frac{1}{2}ff\ .$$

Since, therefore, only the last two terms remain on either side, dividing by z^3 gives this equation: $\alpha\,(1-3\beta) - \alpha\beta z = 2fg + ggz$, from which follows

$$z = \frac{\alpha\,(1-3\beta) - 2fg}{\alpha\beta + gg}\ .$$

§ 10. This is the very solution which I used a long time ago in the place cited. For any value of q chosen, the letters α and β may be known at the same time, from which the appropriate value for z is obtained, which was first found through $p = q\,(1+z)$, and finally it will be

[j] the zz term below should be $(ff+2g)\, zz$ and not $(ff+2fg)\, zz$.

$$\frac{aa}{cc} = \frac{(1+fz+gzz)^2}{(1-\beta z)^2}\,, \qquad \text{hence} \qquad \frac{a}{c} = \frac{1+fz+gzz}{1-\beta z}\,.$$

Then one can take

$$a = 1+fz+gzz \ \text{ and } \ c = 1-\beta z\,;$$

and finally we have $b = cp$ and $d = aq$, and thus the question proposed will be satisfied.

§ 11. Now, in this way, for the case $m = n = 1$ which I treated before, q cannot be taken as 1 because the letters α and β would turn out to be infinite, while the same enormous numbers from earlier are found when either 2 or 3 is used for q. But if not $m = n$, nothing stands in the way of setting $q = 1$, and hence quite convenient solutions can be obtained; but always with $d = a$, which may perhaps be undesired.

§ 12. Therefore, having rejected this method, I am going to go another way, which will lead to much simpler solutions, which are so arranged that one does not arrive at a fourth power of z. I put immediately $\frac{a}{c} = 1+sz$, so that we have

$$\frac{1+3\alpha z+3\alpha zz+\alpha z^3}{1-\beta z} = (1+sz)^2\ ,$$

and developing all terms of this equation, we will arrive at this equation

$$\left.\begin{array}{c} +1+3\alpha z+3\alpha zz+\alpha z^3 \\ -1-2sz-sszz+\beta ssz^3 \\ +\beta z+2\beta szz \end{array}\right\} = 0$$

which is reduced to the form

$$3\alpha-2s+\beta+(3\alpha+2\beta s-ss)\,z+(\alpha+\beta ss)\,zz = 0\,.$$

§ 13. The unknown z can now be easily deduced from this equation, which may be done primarily in two ways, the first of which can be applied, if it will be possible to set

$$\alpha+\beta ss = 0\ , \quad \text{or} \quad ss = -\frac{\alpha}{\beta}\ ,$$

which cannot be unless $-\frac{\alpha}{\beta}=-\frac{nqq}{m}$ was a square. This case may be used only when one of the numbers m or n is negative, and above all their product is a square. Therefore, for this case, let us put $m=\mu\mu$ and $n=-\nu\nu$, so that the equation[k] can be resolved as

$$ab\left(\mu\mu aa-\nu\nu bb\right)=cd\left(\mu\mu cc-\nu\nu dd\right)\ ;$$

then it will be

$$\alpha=\frac{+\nu\nu qq}{\mu\mu+\nu\nu qq}\qquad\text{and}\qquad\beta=\frac{-\mu\mu}{\mu\mu+\nu\nu qq}$$

enabling one to set

$$ss=-\frac{\alpha}{\beta}=\frac{\nu\nu qq}{\mu\mu}\ ,\qquad\text{therefore}\qquad s=\frac{\nu q}{\mu}\ .$$

In this way the last equation will be

$$3\alpha-2s+\beta+\left(3\alpha+2\beta s-ss\right)z=0\ ,\ \text{from which we get}$$

$$z=\frac{2s-3\alpha-\beta}{3\alpha+2\beta s-ss}\ ,$$

and having done the substitutions, it will be

$$z=\frac{\mu\mu\left(\mu\mu-3\nu\nu qq\right)+2\mu\nu q\left(\mu\mu+\nu\nu qq\right)}{\mu\mu\left(3\nu\nu qq-2\mu\nu q\right)-\nu\nu qq\left(\mu\mu+\nu\nu qq\right)}\ .$$

§ 14. But here, without necessity, we have introduced two new letters, μ and ν; for nothing prevents the letters a, b, c, d, from being simply written in place of μa, νb, μc, νd, so that the formula to be resolved becomes $ab\left(aa-bb\right)=cd\left(cc-dd\right)$, just as if we had taken $\mu=1$ and $\nu=1$, from which a much shorter solution will be obtained.

Solution of a particular formula

$$ab\left(aa-bb\right)=cd\left(cc-dd\right)\ .$$

Here, then, with any number taken for q, we will have $\alpha=\frac{qq}{1+qq}$ and $\beta=\frac{-1}{1+qq}$, and then we take $s=q$, from which the last equation will be

[k] Euler means the title equation.

$$3qq - 1 - 2q - 2q^3 + z\left(2qq - 2q - q^4\right) = 0 \ ,$$

hence

$$z = \frac{1 + 2q - 3qq + 2q^3}{q\left(-2 + 2q - q^3\right)} \, .$$

Having found this value, it will be $p = q\,(1+z)$ and $\frac{a}{c} = 1 + qz$, from which the other letters b and d will be easily assigned.

§ 15. Now let us first set $q = 1$, thus $z = -2$, then $\frac{a}{c} = -1$, and therefore $c = -a$, and because of $p = -1$ will be $b = a$ and $d = a$. Therefore in this case all four letters are equal, and both sides of the formula $= 0$. The same would happen if we held $q = -1$, for it will always be helpful to observe that it will be the same if a positive or a negative value is assigned to q. Let us take, therefore, to give a suitable example, $q = 2$, thus $z = -\frac{3}{4}$, and therefore $p = \frac{1}{2}$, and $\frac{a}{c} = -\frac{1}{2}$, therefore $c = -2a$, and then $b = -a$ and $d = 2a$. In this way we face the preceding difficulty. Therefore let $q = 3$, thus $z = -\frac{34}{69}$, hence $p = \frac{35}{23}$, and $\frac{a}{c} = -\frac{11}{23}$, so that $a = 11$ and $c = -23$ can be taken, and $b = -35$ and $d = 3a = 33$. In this way we arrive at the equation:

$$11 \cdot 35\left(35^2 - 11^2\right) = 23 \cdot 33\left(33^2 - 23^2\right) \ ,$$

The reason for this is obvious, since each side is developed into factors as $2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Hence two right triangles[l] can be shown to be of equal area, for the catheti of first one will be $2 \cdot 11 \cdot 35$ and $24 \cdot 56$,[m] and the other triangle will be $2 \cdot 23 \cdot 33$ and $10 \cdot 56$.

Return to the general formula

$$ab\,(maa + nbb) = cd\,(mcc + ndd) \ .$$

§ 16. When it is not possible to reduce the last zz term in the quadratic equation in § 13 to zero,[n] it can always be done with the absolute first term, from which it becomes $3\alpha - 2s + \beta = 0$, from which equation is obtained $2s = 3\alpha + \beta$, or $s = \frac{4\alpha - 1}{2}$. Therefore, after removing this term, the other two divided by z give the equation

[l] Euler means right triangles with legs $(770, 1104)$ and $(1518, 560)$.

[m] This should be $24 \cdot 46$.

[n] It appears that Euler is referring to the last equation in § 12.

$$3\alpha + 2\beta s - ss + (\alpha + \beta ss)\, z = 0\ ,$$

which equation goes into this:

$$\frac{3}{4} + (\alpha + \beta ss)\, z = 0\ , \quad \text{therefore} \quad z = \frac{-3}{4\alpha + 4\beta ss}\ .$$

Therefore, because

$$2s = 4\alpha - 1\ , \quad \text{it will be} \quad z = \frac{-3}{4\alpha + \beta\,(4\alpha - 1)^2}\ ,$$

from which the following solution of our problem is deduced, by which it is required that

$$ab\,(maa + nbb) = cd\,(mcc + ndd)\ .$$

§ 17. By first putting $b = cp$ and $d = aq$, we made $p = q\,(1 + z)$; and then, having taken the letter q as we please, we set for brevity

$$\alpha = \frac{nqq}{nqq - m} \quad \text{and} \quad \beta = \frac{m}{nqq - m},$$

so that $\alpha - \beta = 1$. We find

$$z = \frac{-3}{4\alpha + \beta\,(4\alpha - 1)^2} \quad \text{from which is defined} \quad p = q\,(1 + z).$$

Then having put

$$s = \frac{4\alpha - 1}{2} = \frac{3\alpha + \beta}{2} \quad \text{we found that} \quad \frac{a}{c} = 1 + sz.$$

Since $1 + sz$ is generally a fraction, both letters a and c are easily assigned whole numbers; by which findings, it will $b = cp$ and $d = aq$. We will illustrate the solution by some examples.

Example I.

by which this formula is proposed to be resolved:
$ab\,(aa + 2bb) = cd\,(cc + 2dd)$.

§ 18. Here, therefore, $m = 1$ and $n = 2$, and taking $q = 1$ will result in $\alpha = 2$ and $\beta = 1$. Hence $s = \frac{7}{2}$ and $z = -\frac{1}{19}$, and therefore $p = \frac{18}{19}$ and $\frac{a}{c} = \frac{31}{38}$. Therefore let us take $a = 31$ and $c = 38$, which finally makes $b = 36$ and $d = 31$, so that the solution is going to be

$$31 \cdot 36 \left(31^2 + 2 \cdot 36^2\right) = 38 \cdot 31 \left(38^2 + 2 \cdot 31^2\right) \ ,$$

which will soon become clear by making the calculation. Indeed it is

$$31^2 + 2 \cdot 36^2 = 3553 = 11 \cdot 17 \cdot 19 \text{ and } 38^2 + 2 \cdot 31^2 = 3366 = 2 \cdot 3^2 \cdot 11 \cdot 17,$$

substituting factors on both sides produces the same product $2^2 \cdot 3^2 \cdot 11 \cdot 17 \cdot 19 \cdot 31$.

Another solution of the same example.

§ 19. Let $q = \frac{1}{2}$, be taken here, and it will be $\alpha = -1$ and $\beta = -2$. Then $s = -\frac{5}{2}$ and $z = \frac{1}{18}$, and we conclude that $p = \frac{19}{36}$ and $\frac{a}{c} = \frac{31}{36}$. Let $a = 31$ and $c = 36$, be taken, from which finally $b = 19$ and $d = \frac{31}{2}$; hence, by doubling these numbers, we will have $a = 62$, $b = 38$, $c = 72$ and $d = 31$, so that

$$62 \cdot 38 \left(62^2 + 2 \cdot 38^2\right) = 72 \cdot 31 \left(72^2 + 2 \cdot 31^2\right) \ ,$$

where, by means of factors, it is

$$62^2 + 2 \cdot 38^2 = 6732 = 2^2 \cdot 3^2 \cdot 11 \cdot 17 \text{ and}$$
$$72^2 + 2 \cdot 31^2 = 7106 = 2 \cdot 11 \cdot 17 \cdot 19.$$

This produces the same product on both sides, $2^4 \cdot 3^2 \cdot 11 \cdot 17 \cdot 19 \cdot 31$.

Example II.

by which this formula is proposed to be resolved:

$$ab\,(aa + 3bb) = cd\,(cc + 3dd) \ .$$

§ 20. Here is then $m = 1$ and $n = 3$, hence $\alpha = \frac{3qq}{3qq-1}$ and $\beta = \frac{1}{3qq-1}$. Let's now assume $q = 1$ first, so that $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, and hence

$$s = \frac{5}{2} \text{ and } z = -\frac{6}{37} \ , \text{ so that } p = \frac{31}{37} \text{ and } \frac{a}{c} = \frac{22}{37}.$$

Let $a = 22$ and $c = 37$ be taken, and $b = 31$ and $d = 22$, and consequently we will have

$$22 \cdot 31 \left(22^2 + 3 \cdot 31^2\right) = 37 \cdot 22 \left(37^2 + 3 \cdot 22^2\right) .$$

This will be clearly

$$22^2 + 3 \cdot 31^2 = 3367 = 7 \cdot 13 \cdot 37 \text{ and } 37^2 + 3 \cdot 22^2 = 2821 = 7 \cdot 13 \cdot 31;$$

and thus the same product arises on both sides, namely $2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 37$.

Another solution of the same example.

§ 21. Taking $q = \frac{1}{2}$, one has $\alpha = -3$ and $\beta = -4$, and hence again $s = -\frac{13}{2}$ and $z = \frac{3}{688}$, so we will have $p = \frac{691}{1376}$ and $\frac{a}{c} = \frac{1337}{1376}$. Let us then take $a = 1337$ and $c = 1376$, from which it becomes $b = 691$ and $d = \frac{1337}{2}$. By doubling the values we then have $a = 2674$; $b = 1382$; $c = 2752$ and $d = 1337$.

Yet another solution of the same example.

§ 22. Assuming here $q = 2$, and $\alpha = \frac{12}{11}$ and $\beta = \frac{1}{11}$,[o] hence comes $s = \frac{37}{22}$ and $z = -\frac{3993}{6977}$; which makes $p = \frac{5968}{6977}$, and further $sz = -\frac{13431}{13954}$, and therefore $\frac{a}{c} = \frac{523}{13954}$. Thus may $a = 523$ and $c = 13954$ be taken, and finally $b = 11936$ and $d = 1046$.

§ 23. We reserve the principal case, in which $m = 1$ and $n = 1$, for special discussion, since the discovery of two biquadrates, the sum of which can be resolved into two other biquadrates, depends on it. This will include the further treatment of the formula $ab\,(aa + bb) = cd\,(cc + dd)$.

§ 24. Therefore, since here $m = 1$ and $n = 1$, take $\alpha = \frac{qq}{qq-1}$ and $\beta = \frac{1}{qq-1}$ from any arbitrarily chosen number q. Then we may take $s = \frac{3\alpha+\beta}{2}$ and $z = \frac{-3}{4\alpha+\beta(3\alpha+\beta)^2}$, and the values $p = q\,(1 + z)$ and $\frac{a}{c} = 1 + sz$ will be found. Finally, it will be $b = cp$ and $d = a = aq$,[p] from which premises we shall proceed to the following examples.

[o] The correct values are $s = \frac{37}{22}$, $z = -\frac{3993}{7177}$, $a = 923$, $b = 12736$, $c = 14354$ and $d = 1846$.
[p] This should simply be $d = aq$.

Example I.

where $q = 3$.

§ 24.[q] It is evident that in this case $q = 1$ cannot be taken, because the number α and β would be infinite. Next, it is easy to foresee that setting $q = 3$ provides a simpler solution than $q = 2$. Assuming therefore $q = 3$, one has $\alpha = \frac{9}{8}$ and $\beta = \frac{1}{8}$, hence $s = \frac{7}{4}$ and $z = -\frac{96}{193}$. Hence we get $p = \frac{291}{193}$, and $sz = -\frac{168}{193}$, therefore $\frac{a}{c} = 1 + sz = \frac{25}{193}$. Then $a = 25$ and $c = 193$ should be taken, and it will be $b = 291$ and $d = 75$. And thus we reach this solution

$$25 \cdot 291 \left(25^2 + 291^2\right) = 193 \cdot 75 \left(193^2 + 75^2\right)$$

where it is noted that[r]

$$\begin{aligned} 25^2 + 291^2 &= 85306 = 2 \cdot 13 \cdot 17 \cdot 193 \quad \text{and} \\ 193^2 + 75^2 &= 42879 = 2 \cdot 13 \cdot 17 \cdot 97 \end{aligned}$$

Now it is easily seen that the same product appears on both sides by the same factors

$$2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 17 \cdot 97 \cdot 193.$$

§ 25. Let us transfer this solution to biquadrates, and in general if one has $ab\,(aa + bb) = cd\,(cc + dd)$, let $a = p + q$ and $b = p - q$, and in like manner $c = r + s$ and $d = r - s$, and this equation is produced: $2\left(p^4 - q^4\right) = 2\left(r^4 - s^4\right)$. Then in this way it will be $p^4 + s^4 = q^4 + r^4$, where it is noted that

$$p = \frac{a+b}{2}\,,\; q = \frac{a-b}{2}\,,\; r = \frac{c+d}{2}\,,\; \text{ and } \; s = \frac{c-d}{2}\,,$$

or by duplicating the sums one gets

$$p = a + b\,,\; q = a - b\,,\; r = c + d\,,\; \text{ and } \; s = c - d\,.$$

[q] The original publication repeats the paragraph number 24 and continues the following paragraphs with their numbering off by one. This has not been corrected in order to keep internal references intact.

[r] There is a typo here, $193^2 + 75^2 = 42874 = 2 \cdot 13 \cdot 17 \cdot 97$.

Therefore we collect, using the method just found $p = 158$; $q = 133$; $r = 134$ and $s = 59$.

§ 26. These numbers are in complete agreement with those we mentioned at the beginning, and therefore without a doubt provide a very simple solution to the problem, which I had once discussed, since it is

$$158^4 + 59^4 = 133^4 + 134^4 \,.$$

Here, contrary to expectation, another solution to our formula $ab\,(aa+bb) = cd\,(cc+dd)$, offers itself. It follows from here $158^4 - 134^4 = 133^4 - 59^4$, which can be resolved into factors as

$$24 \cdot 292 \left(158^2 + 134^2\right) = 74 \cdot 192 \left(133^2 + 59^2\right) \;.$$

It may be noted that, since $pp + qq = \frac{1}{2}\left(p+q\right)^2 + \frac{1}{2}\left(p-q\right)^2$, one has $158^2 + 134^2 = \frac{1}{2}\left(292\right)^2 + \frac{1}{2}\left(24\right)^2$, and in the same way $133^2 + 59^2 = \frac{1}{2}\left(192\right)^2 + \frac{1}{2}\left(74\right)^2$, and substituting these, a similar pattern comes out,[s]

$$24 \cdot 292 \left(292^2 + 24^2\right) = 74 \cdot 192^2 \left(74^2 + 192^2\right) \;.$$

Dividing each factor by 2 results in

$$12 \cdot 146 \left(146^2 + 12^2\right) = 37 \cdot 96 \left(37^2 + 96^2\right) \;.$$

§ 27. Thus we have been led to an even simpler solution of our formula $ab\,(aa+bb) = cd\,(cc+dd)$, which is $a = 12$, $b = 146$, $c = 37$, and $d = 96$. If indeed these numbers are noticeably smaller than those found above; then a general method follows to derive another solution from each solution of our formula. For when we assume $a+b = p$, $a-b = q$, $c+d = r$ and $c-d = s$, then the equality $p^4 + s^4 = q^4 + r^4$ is produced. In turn it will be $p^4 - r^4 = q^4 - s^4$, which, when the factors are taken, will be

$$(p+r)\,(p-r)\,(pp+rr) = (q+s)\,(q-s)\,(qq+ss) \;, \quad \text{or}$$

$$(p+r)\,(p-r)\left[(p+r)^2 + (p-r)^2\right] = (q+s)\,(q-s)\left[(q+s)^2 + (q-s)^2\right] \,.$$

Therefore if we set $p+r = a'$; $p-r = b'$; $q+s = c'$ and $q-s = d'$, we will have

[s] The equation below should be $24 \cdot 292\left(292^2 + 24^2\right) = 74 \cdot 192\left(74^2 + 192^2\right)$.

$$a'b'\left(a'a'+b'b'\right)=c'd'\left(c'c'+d'd'\right),$$

which will therefore hold if it is taken $a'=(a+b)+(c+d)$; $b'=(a+b)-(c+d)$; $c'=(a-b)+(c-d)$ and $d'=(a-b)-(c-d)$.

§ 28. Since we have found

$$a=291\,,\ b=25\,,\ c=193\,,\quad \text{et}\ \ d=75\,,\quad \text{it will be}$$
$$a+b=316\,,\ a-b=266\,,\ c+d=268\,,\quad \text{and}\ \ c-d=118\,.$$

Hence we can infer that $a'=584$, $b'=48$, $c'=384$ and $d'=148$, those numbers divided by 4 provide the simplest solution, namely $a'=146$, $b'=12$, $c'=96$ and $d'=37$, so that

$$146\cdot 12\left(146^2+12^2\right)=96\cdot 37\left(96^2+37^2\right)\,,$$

where actually is

$$146^2+12^2=21460=2^2\cdot 5\cdot 29\cdot 37\ \ \text{and}\ \ 96^2+37^2=10585=5\cdot 29\cdot 73,$$

and now both sides produce the same product $2^5\cdot 3\cdot 5\cdot 29\cdot 37\cdot 73$. Therefore, on account of the remarkable use of this rule, let us include it in the following theorem.

§ 29. **Theorem.** *If the four numbers a, b, c, d have been arranged such that*

$$ab\left(aa+bb\right)=cd\left(cc+dd\right),$$

then, if these four numbers are formed $A=(a+b)+(c+d)$; $B=(a+b)-(c+d)$; $C=(a-b)+(c-d)$ and $D=(a-b)-(c-d)$, one will also have

$$AB\left(A^2+B^2\right)=CD\left(C^2+D^2\right).$$

It is therefore clear that for each solution there is always a conjugate solution to be found with the help of this theorem. Two conjugate solutions can always be shown, which are so connected with each other, that by the help of the theorem one may be defined from the other.

§ 30. In order to more easily find other solutions, let us prepare formulas so that the resulting integer values for the numbers a, b, c, d can be immediately obtained. To this end, since it is also possible to take fractions for q, let us immediately put $q=\frac{f}{g}$, and it will be

$$\alpha = \frac{ff}{ff-gg} \ \text{ and } \ \beta = \frac{gg}{ff-gg}\ , \ \text{ so that } \ s = \frac{3ff+gg}{2\,(ff-gg)}\ .$$

From here, let

$$z = \frac{-3}{4\alpha+\beta\,(3\alpha+\beta)^2}\ , \ \text{ then it will be } \ z = \frac{-3\,(ff-gg)^3}{4f^6+f^4gg+10ffg^4+g^6}\ ,$$

so that

$$1+z = \frac{f^6+10f^4gg+ffg^4+4g^6}{4f^6+f^4gg+10ffg^4+g^6}\ .$$

Then it will be

$$sz = -\frac{3\,(ff+gg)\,(ff-gg)^2}{2\,(4f^6+f^4gg+10ffg^4+g^6)}\ ,$$

and therefore

$$\begin{aligned} 1+sz &= -\frac{-f^6+17f^4gg+17ffg^4-g^6}{2\,(4f^6+f^4gg+10ffg^4+g^6)} \\ &= \frac{(ff+gg)\left(-f^4+18ffgg-g^4\right)}{2\,(4f^6+f^4gg+10ffg^4+g^6)}\ . \end{aligned}$$

§ 31. Since we had put

$$p = q\,(1+z) = \frac{f}{g}\,(1+z)\ , \ \text{ it will be } \ p = \frac{f\left(f^6+10f^4gg+ffg^4+4g^6\right)}{g\,(4f^6+f^4gg+10ffg^4+g^6)}\ .$$

Finally we had $\frac{a}{c} = 1+sz$, so that if we take

$$\begin{aligned} a &= (ff+gg)\left(-f^4+18ffgg-g^4\right) \ \text{ and} \\ c &= 2\left(4f^6+f^4gg+10ffg^4+g^6\right)\ , \end{aligned}$$

since it is

$$b = pc \ \text{ and } \ d = aq\ , \text{ it will be } b = \frac{2f\left(f^6+10f^4gg+ffg^4+4g^6\right)}{g}$$

and finally

$$d = \frac{f\,(ff + gg)\left(-f^4 + 18ffgg - g^4\right)}{g}\,.$$

Let us then multiply all these values by g, and we will have as follows

$$\begin{aligned} a &= g\,(ff + gg)\left(-f^4 + 18ffgg - g^4\right) \\ b &= 2f\left(f^6 + 10f^4gg + ffg^4 + 4g^6\right) \\ c &= 2g\left(4f^6 + f^4gg + 10ffg^4 + g^6\right) \\ d &= f\,(ff + gg)\left(-f^4 + 18ffgg - g^4\right). \end{aligned}$$

§ 32. These formulas produce indeed extremely large numbers, but generally they can be reduced to much smaller numbers by a common divisor. If, as we did above, take $f = 3$ and $g = 1$, our formula will give $a = 800$; $b = 9312$; $c = 6176$ and $d = 2400$. All these numbers, however, allow a division by 32, in which way they are reduced to the numbers found above.

Example II.

where $f = 2$ and $g = 1$.

§ 33. Substituting these values, we will find $a = 5 \cdot 55 = 275$; $b = 928$; $c = 626$ and $d = 550$, which cannot be further reduced to smaller numbers. Let us also show its conjugate solution according to the theorem given above; it will be $A = 2379$; $B = 27$; $C = 729$ and $D = 577$. Thus we have obtained two new solutions.

Another solution

of the equation $ab\,(aa + bb) = cd\,(cc + dd)$.

§ 34. Assuming as before $b = cp$ and $d = aq$ we have reached the equation $\frac{aa}{cc} = \frac{p^3-q}{q^3-p}$, giving a fraction that ought to be reduced to a square. Here we may use another method, which could not be applied to the general form. Let us set

$$p = 1 + \alpha z \quad \text{and} \quad q = 1 + \beta z\,, \quad \text{and it will be found}$$

$$\frac{aa}{cc} = \frac{3\alpha - \beta + 3\alpha\alpha z + \alpha^3 zz}{3\beta - \alpha + 3\beta\beta z + \beta^3 zz}\,,$$

and thus we will arrive at a fraction in which the unknown z rises in both the numerator and the denominator to no more than a square. Such formulas can be treated fairly easily, if either the first absolute terms or the final terms are squares.

Treatment of the former case

in which $3\alpha - \beta$ and $3\beta - \alpha$ are squares.

§ 35. Therefore let us set $3\alpha - \beta = ff$ and $3\beta - \alpha = gg$, and it will be

$$\alpha = \frac{3ff + gg}{8} \text{ and } \beta = \frac{ff + 3gg}{8} .$$

In this way we will have

$$\frac{aa}{cc} = \frac{ff + 3\alpha\alpha z + \alpha^3 zz}{gg + 3\beta\beta z + \beta^3 zz} ,$$

and thus let us set

$$aa = ff + 3\alpha\alpha z + \alpha^3 zz \text{ and } cc = gg + 3\beta\beta z + \beta^3 zz .$$

Now to make these forms into squares, let us form this equality:

$$aagg - ccff = 3\left(\alpha\alpha gg - \beta\beta ff\right) z + \left(\alpha^3 gg - \beta^3 ff\right) zz ,$$

and because the left side has two factors

$$ag + cf \text{ and } ag - cf ,$$

the right side must also be resolved into two factors, one of which is

$$ag + cf = \frac{z}{\lambda} ,$$

and the other will be

$$ag - cf = 3\lambda\left(\alpha\alpha gg - \beta\beta ff\right) + \lambda\left(\alpha^3 gg - \beta^3 ff\right) z .$$

To find the value of λ, let first $z = 0$ and add the squares of these two factors. Their sum will be

$$2\,(aagg+ccff)=9\lambda\lambda\,(\alpha\alpha gg-\beta\beta ff)^2\ ;$$

and from these formulas, with $z=0$, since we have $aa=ff$ and $cc=gg$, it will be

$$2\,(aagg+ccff)=4ffgg=9\lambda\lambda\,(\alpha\alpha gg-\beta\beta ff)^2\ .$$

Hence it is found $\lambda=\frac{2fg}{3(\alpha\alpha gg-\beta\beta ff)}$, which form, since $ff=3\alpha-\beta$ and $gg=3\beta-\alpha$, becomes

$$\lambda=\frac{2fg}{3\,(\beta-\alpha)^2}\,.$$

§ 36. Having found this value for λ, the two factors will be

$$ag+cf=\frac{3\,(\beta-\alpha)\,3z}{2fg}\ \text{and}\ ag-cf=2fg+\frac{2fg\left(\alpha^3gg-\beta^3ff\right)z}{3\,(\alpha\alpha gg-\beta\beta ff)}\ ,\ \text{or}$$

$$ag-cf=2fg+\frac{2fg\,(\beta+\alpha)\,(\beta\beta-3\alpha\beta+\alpha\alpha)\,z}{3\,(\beta-\alpha)^2}\,.$$

Let us set for brevity

$$\frac{(\alpha+\beta)\,(\beta\beta-3\alpha\beta+\alpha\alpha)}{3\,(\beta-\alpha)^2}=\Delta\,,\quad \text{thus}\ \ ag-cf=2fg+2fg\Delta z\,.$$

Now let the squares of these forms be added, and one will get

$$2\,(aagg+ccff)=4ffgg+8ffgg\Delta z+4ffgg\Delta^2zz+\frac{9\,(\beta-\alpha)^6\,zz}{4ffgg}\,.$$

From these formulas we infer

$$2\,(aagg-ccff)=4ffgg+6\,(\alpha\alpha gg+\beta\beta ff)\,z+2\left(\alpha^3gg+\beta^3ff\right)zz\,.$$

Canceling the first terms, the other terms divided by z will give the following equation:

$$\begin{aligned}&6\,(\alpha\alpha gg+\beta\beta ff)+2\left(\alpha^3gg+\beta^3ff\right)z\\&=8ffgg\Delta+4ffgg\Delta^2z+\frac{9\,(\beta-\alpha)^6\,z}{4ffgg}\ ,\end{aligned}$$

DOI: 10.56031/2693-9908.1041

from which the value of z can be easily deduced. We will not develop these formulas further, because their treatment will be more easily done for whatever particular case you wish.

Treatment of the case

in which $f = 3$ and $g = 1$.

§ 37. Here it will be $\alpha = \frac{7}{2}$ and $\beta = \frac{3}{2}$, it will thus become

$$aa = 9 + \frac{147}{4}z + \frac{343}{8}zz \text{ and } cc = 1 + \frac{27}{4}z + \frac{27}{8}zz; \text{ furthermore } \Delta = -\frac{25}{48}.$$

Hence, to find the quantity z we will have the equation

$$195 + \frac{293}{2}z = -\frac{75}{2} + \frac{625}{64}z + 16z = -\frac{75}{2} + \frac{1649}{64}z, \text{ or}$$

$$\frac{465}{2} = -\frac{7727}{64}z, \text{ hence } z = -\frac{14880}{7727}.$$

In this way, we are getting very large numbers, and it is not worth the effort to develop them any further.

§ 38. In the meantime, however, it will not be difficult to establish generally suitable values for the letters a, b, c, d. Take any two squares ff and gg as desired, and let the numbers α and β be taken in such a way that $\frac{\alpha}{\beta} = \frac{3ff+gg}{3gg+ff}$. After reducing the fraction $\frac{3ff+gg}{3gg+ff}$ to the smallest values, let the numerator be taken for α, and the denominator for β. The required numbers will then be

$$\begin{aligned} a &= f(\alpha+\beta)(\alpha\alpha - 3\alpha\beta + \beta\beta) \\ b &= g\left(\beta^3 - 5\alpha\beta\beta + 4\alpha\alpha\beta - 2\alpha^3\right) \\ c &= g(\alpha+\beta)(\alpha\alpha - 3\alpha\beta + \beta\beta) \\ d &= f\left(\alpha^3 - 5\alpha\alpha\beta + 4\alpha\beta\beta - 2\beta^3\right). \end{aligned}$$

Or, setting

$$(\alpha+\beta)(\alpha\alpha - 3\alpha\beta + \beta\beta) = \Delta, \text{ it will be}$$

$$a = f\Delta\ ,\ b = g\left(\Delta - 3\alpha\,(\alpha - \beta)^2\right)$$
$$c = g\Delta\ ,\ d = f\left(\Delta - 3\beta\,(\alpha - \beta)^2\right)\ .$$

With the help of these formulas, examples can be developed much more easily.

§ 39. So if we take $f = 3$ and $g = 1$, we will have $\frac{\alpha}{\beta} = \frac{28}{12} = \frac{7}{3}$, therefore $\alpha = 7$ and $\beta = 3$. Hence $\Delta = -50$, by which means the numbers themselves are found to be $a = 150$; $b = 386$; $c = 50$ and $d = 582$, which become $a = 75$; $b = 193$; $c = 25$ and $d = 291$ when dividing by 2.

§ 40. Let us now also consider the case where $f = 2$ and $g = 1$, and $\frac{\alpha}{\beta} = \frac{13}{7}$. Let $\alpha = 13$ and $\beta = 7$ be taken, hence $\Delta = -1100$, and from here, the desired numbers themselves come forth $a = 2200$, $b = 2504$, $c = 1100$ and $d = 3712$. When divided by 4, $a = 550$, $b = 626$, $c = 275$ and $d = 928$, which agrees with the solution found above in § 33.

§ 41. The foundation of this analysis is based on the following problem:

Problem. *Given the two formulas: $xx + 2fxy + hyy$ and $xx + 2gxy + kyy$, find a ratio between the numbers x and y, so that both of these formulas become squares.*

§ 42. **Solution.** Put $xx + 2fxy + hyy = P^2$ and $xx + 2gxy + kyy = Q^2$, giving the difference $PP - QQ = 2\,(f - g)\,xy + (h - k)\,yy$. Let one factor be $P - Q = (f - g)\,y$, and the other will be $P + Q = 2x + \frac{(h-k)y}{f-g}$. Now the squares of these two factors are added, and it will produce

$$2P^2 + 2Q^2 = 4xx + \frac{4\,(h-k)\,xy}{f-g} + \frac{(h-k)^2\,yy}{(f-g)^2} + (f-g)^2\,yy\ ;$$

while from the formulas proposed for P and Q it will be

$$2P^2 + 2Q^2 = 4xx + 4\,(f+g)\,xy + 2\,(h+k)\,yy\ ,$$

hence, since the first terms $4xx$ cancel themselves, and the rest, divided by y, give this equation

$$4\,(f+g)\,x + 2\,(h+k)\,y = \frac{4\,(h-k)\,x}{f-g} + \frac{(h-k)^2\,y}{(f-g)^2} + (f-g)^2\,y\ , \quad \text{or}$$

$$4\,(f-g)\,[ff-gg-h+k]\,x = \Big((h-k)^2+(f-g)^2\Big[(f-g)^2-2\,(h+k)\Big]\Big)\,y\ ;$$

therefore it will be[t]

$$\frac{x}{y}=\frac{(f-g)^4-2\,(h+k)\,(f-g)^2+(h-k)^2}{4\,(f-g)\,(ff-gg-h+k)}\ .$$

§ 43. We have already mentioned another case in which both formulas for aa and cc can be reduced to square (§ 34), where, in order for the final parts to become square, it is necessary that both numbers α and β be square. Let therefore $\alpha = mm$ and $\beta = nn$, and then it will be

$$aa = m^6zz+3m^4z+3mm-nn \ \text{ and } \ cc = n^6zz+3n^4z+3nn-mm\ .$$

§ 44. Let us represent these equalities in this way, so that we recall the form of a previous problem:

$$\begin{aligned} aan^6 &= m^6n^6zz+3m^4n^6z+3mmn^6-n^8 \quad \text{and} \\ ccm^6 &= m^6n^6zz+3n^4m^6z+3nnm^6-m^8 \end{aligned}$$

and by comparison it will be $x = m^3n^3z$, and taking $y = 1$, one will have $2fm^3n^3z = 3m^4n^6z$, therefore $f = \frac{3}{2}mn^3$ and $g = \frac{3}{2}nm^3$, finally $h = 3nnm^6 - m^8$ and $k = 3mmn^6 - n^8$, from which we can infer

$$\begin{aligned} f-g &= \tfrac{3}{2}mn\,(nn-mm) \quad \text{and} \\ h-k &= 3nnm^6+n^8-3mmn^6-m^8 = \left(n^4-m^4\right)\left(n^4+m^4-3mmnn\right)\ . \end{aligned}$$

The solution [above] of our problem already offers the value[u]

$$\frac{x}{y}=m^3n^3z=\frac{(f-g)^4-2\,(h+k)\,(f-g)^2+(h-k)^2}{4\,(f-g)\,(ff-gg-h+k)}\ .$$

§ 45. In lieu of a colophon, I will add yet another analysis for the resolution of the formula $ab\,(aa+bb) = cd\,(cc+dd)$. Since the matter has been brought to this equation:

[t] The original has erroneous brackets in the numerator which have been removed.

[u] The original has erroneous brackets in the numerator which have been removed.

$$\frac{aa}{cc} = \frac{p^3 - q}{q^3 - p}\ ,\quad \text{taking}\ \ p = nn\,(q+1) - 1\ , \text{will make}$$

$$\frac{aa}{cc} = \frac{n^6\,(q+1)^3 - 3n^4\,(q+1)^2 + 3nn\,(q+1) - 1 - q}{q^3 - nn\,(q+1) + 1}\ ,$$

reducing the fraction by a factor $q+1$ will give

$$\frac{aa}{cc} = \frac{n^6\,(q+1)^2 - 3n^4\,(q+1) + 3nn - 1}{qq - q + 1 - nn}\ ,\quad \text{or}$$

$$aa = n^6 qq + n^4\,(2nn-3)\,q + (nn-1)^3$$

then

$$cc = qq - q + 1 - nn\ ,\quad \text{or}\ \ n^6 cc = n^6 qq - n^6 q - n^6\,(nn-1)\ .$$

For the problem, then, $x = n^3 q$ and $y = 1$; the rest is obvious.

Resolutio formulae Diophanteae $ab\,(maa+nbb)=cd\,(mcc+ndd)$ per numeros rationales

Auctore L. Eulero

Nova acta Academiae scientiarum imperialis petropolitanae
Tomus 13 (1795/96), pp. 45-63.

§ 1. Inter theoremata olim a Fermatio demonstrata, et post ejus obitum deperdita, imprimis maximam attentionem meretur hoc theorema: quod non dentur duae potestates cujusque ordinis, quarum summa vel differentia sit potestas ejusdem ordinis, siquidem ordo primus et secundus excipiatur. Ita negavit Fermatius exhiberi posse binos numeros a et b, ut[a] haec formula $a^n \pm b^n$ aequetur simili potestati c^n, simul atque exponens n binarium superaverit. Hoc ergo modo omnes istae positiones erunt impossibiles:

$$\text{I. } a^3 \pm b^3 = c^3\ ;\ \text{II. } a^4 \pm b^4 = c^4\ ;\ \text{III. } a^5 \pm b^5 = c^5\ ;\ \text{IV. } a^6 \pm b^6 = c^6\ ;$$
$$\text{V. } a^7 \pm b^7 = c^7\ ;\ \text{VI. } a^8 \pm b^8 = c^8 \text{ etc.}$$

§ 2. De prima harum formularum $a^3 \pm b^3 = c^3$, veritas jam satis feliciter est ostensa, unde simul quoque veritas sequitur pro omnibus formulis, pro quibus exponens n, est multiplum ternarii. Tum vero multo clarius adhuc demonstrata est formula secunda $a^4 \pm b^4 = c^4$, cum evidentissime comprobatum sit, neque summam neque differentiam duorum biquadratorum posse esse quadratum, multo minus ergo biquadratum, hincque etiam evicti sunt omnes casus, quibus exponens n est multiplum quaternarii. Interim tamen ejusmodi demonstratio, quae se pariter ad omnes exponentes n extendat, maxime adhuc desideratur, neque cuiquam Geometrarum in hoc numerorum mysterium penetrare contigit.

§ 3. Pluribus autem insignibus Geometris visum est haec theoremata latius extendi posse. Quemadmodum enim duo cubi exhiberi nequeunt, quorum summa vel differentia sit cubus, ita etiam certum est, nequidem exhiberi posse tria biquadrata, quorum summa sit pariter biquadratum, sed ad minimum quatuor

[a] While the original text generally has *vt*, *vbi*, etc., this transcription uses *ut*, *ubi*, etc., instead throughout the text.

biquadrata requiri, ut eorum summa prodire queat biquadratum, quamquam nemo adhuc talia quatuor biquadrata assignare potuerit. Eodem modo etiam affirmari posse videtur, non exhiberi posse quatuor potestates quintas, quarum summa etiam esset potestas quinta; similique modo res se habebit in altioribus potestatibus; unde sequentes quoque positiones omnes pro impossibilibus erunt habendae:

I. $a^3 + b^3 = c^3$,

II. $a^4 + b^4 + c^4 = d^4$,

III. $a^5 + b^5 + c^5 + d^5 = e^5$,

IV. $a^6 + b^6 + c^6 + d^6 + e^6 = f^6$,

etc. etc.

Plurimum igitur scientia numerorum promoveri esset censenda, si demonstrationem desideratam etiam ad has formulas extendere liceret.

§ 4. Primo quidem intuitu videri posset has postremas formulas non solum ad summas, sed etiam ad differentias, prouti in prima usu venit, extendi posse, ita ut etiam haec aequalitas $a^4 \pm b^4 \pm c^4 = d^4$ pro impossibili esset habenda; verum hoc longe secus se habere ante aliquot annos observavi, in tomo Commentariorum XVII pag. 64, ubi bina biquadrata assignavi, quorum summa in alia duo biquadrata resolvi queat, ita ut sit $a^4+b^4 = c^4+d^4$, unde ergo haec aequalitas $a^4+b^4-c^4 = d^4$ veritati neutiquam adversatur; verum numeri, quos pro his litteris a, b, c, d, per calculum valde taediosum erui, valde immanes prodierunt.

§ 5. Cum autem nuper idem argumentum tractandum suscepissem, praeter omnem expectationem incidi in numeros multo minores hac indole praeditos, atque adeo minimi numeri hoc praestantes statui possunt isti: $a = 134$; $b = 133$; $c = 158$; et $d = 59$, quandoquidem revera deprehendetur esse $134^4 + 133^4 = 158^4+59^4$, quem calculum exsequi haud adeo molestum est, dum contra comprobationem illorum immensorum numerorum vix quisquam tentare audebit.

§ 6. Methodus autem, qua tum temporis sum usus, ut resolverem hanc aequalitatem:

$$A^4 + B^4 = C^4 + D^4 ,$$

ita procedebat: consideravi scilicet hanc aequationem $A^4 - C^4 = D^4 - B^4$, ac posito

$$A = a + b , \quad C = a - b , \quad D = c + d \quad \text{et} \quad B = c - d ,$$

prodiit ista aequatio satis simplex: $ab\,(aa+bb)=cd\,(cc+dd)$; sicque totum negotium ad resolutionem hujus formulae reducitur. Hic autem non solum methodum ante usitatam, multo tractabiliorem sum redditurus, sed etiam ad resolutionem formulae multo generalioris in titulo exhibitae

$$ab\,(maa+nbb)=cd\,(mcc+ndd)$$

sum accommodaturus, ita ut, quicunque numeri pro m et n accipiantur, semper infinitis modis numeri satisfacientes pro a, b, c, d inveniri queant.

§ 7. Ad aequalitatem autem hanc resolvendam utor ante omnia hac transformatione:

$$b=cp \quad \text{et} \quad d=aq\ ,$$

hocque modo aequatio resolvenda hanc induet formam:

$$p\,(maa+nccpp)=q\,(mcc+naaqq)\ ,$$

$$\text{unde elicitur} \quad \frac{aa}{cc}=\frac{np^3-mq}{nq^3-mp}\ ,$$

sicque totum negotium huc redit, ut ista formula $\frac{np^3-mq}{nq^3-mp}$ ad quadratum reducatur, quod quidem sponte evenire casu $q=p$ mox in oculos incurrit, cum evadat $\frac{aa}{cc}=1$, ideoque $c=a$, tum quoque fiet $b=d$,[b] qui autem casus maxime obvius pro solutione neutiquam haberi potest, quandoquidem ambo membra aequationis prodeunt identica. Interim tamen hic ipse casus ad alias solutiones manuducere poterit.

§ 8. Cum igitur nostra formula revera quadratum evadat posito $p=q$, statuamus $p=q\,(1+z)$, ita ut sumpto $z=0$ ipse casus obvius prodeat; nunc autem nostra formula in sequentem transmutabitur:[c]

$$\frac{aa}{cc}=\frac{nq^3\,(1+z)^3-m}{nqq-m\,(1+z)}\ , \quad \text{sive} \quad \frac{aa}{cc}=\frac{nqq-m+3nqqz+3nqqzz+nqqz^3}{nqq-m-mz}$$

in qua fractione quantitas incognita z in numeratore ad tertiam potestatem, in denominatore autem non ultra primam assurgit, cujusmodi formulas per methodos cognitas tractari posse jam satis liquet.

[b] The original has ∂, which can be understood as d but was likely not intended by Euler.

[c] The numerator of the left equation should be $nqq\,(1+z)^3-m$ instead of $nq^3\,(1+z)^3-m$.

§ 9. Quo hanc fractionem tractabiliorem reddamus, tam numeratorem quam denominatorem dividamus per $nqq - m$, statuamusque brevitatis gratia

$$\frac{nqq}{nqq - m} = \alpha \quad \text{et} \quad \frac{m}{nqq - m} = \beta\ ,$$

ita ut sit $\alpha - \beta = 1$, ideoque $\alpha = 1 + \beta$, quo facto formula sequens prodibit

$$\frac{aa}{cc} = \frac{1 + 3\alpha z + 3\alpha zz + \alpha z^3}{1 - \beta z}\ .$$

Jam per secundam methodum, qua olim sum usus, denominator reddatur quadratum, multiplicando supra et infra per $1 - \beta z$, unde prodit

$$\frac{aa}{cc} = \frac{(1 - \beta z)\left(1 + 3\alpha z + 3\alpha zz + \alpha z^3\right)}{(1 - \beta z)^2}\ .$$

Sicque tantum opus est ut numerator, qui evolutus fit

$$1 + (3\alpha - \beta)\, z + 3\alpha\,(1 - \beta)\, zz + \alpha\,(1 - 3\beta)\, z^3 - \alpha\beta z^4$$

ad quadratum reducatur, quod praestabitur ejus radicem ponendo $1 + fz + gzz$, cujus quadratum est[d]

$$1 + 2fz + (ff + 2fg)\, zz + 2fgz^3 + ggz^4\ .$$

Nunc quia primi termini se mutuo sponte destruunt, binae litterae f et g ita definiantur, ut etiam secundi ac tertii termini tollantur; prius fiet sumendo $f = \frac{3\alpha - \beta}{2}$, posterius vero statuendo

$$ff + 2g = 3\alpha\,(1 - \beta)\ , \quad \text{unde fit} \quad g = \frac{3\alpha\,(1 - \beta)}{2} - \frac{1}{2}ff\ .$$

Quoniam igitur utrinque tantum bini termini postremi remanent, qui per z^3 divisi praebent hanc aequationem: $\alpha\,(1 - 3\beta) - \alpha\beta z = 2fg + ggz$, inde elicitur

$$z = \frac{\alpha\,(1 - 3\beta) - 2fg}{\alpha\beta + gg}\ .$$

[d]the zz term below should be $(ff + 2g)\, zz$ and not $(ff + 2fg)\, zz$.

§ 10. Haec est ea ipsa solutio, qua jam dudum loco citato sum usus, cujus ope pro quovis valore ipsius q ad arbitrium assumpto, simul innotescunt litterae α et β, unde valor idoneus pro z obtinetur, quo invento primo erit $p = q\,(1+z)$, ac denique erit

$$\frac{aa}{cc} = \frac{(1+fz+gzz)^2}{(1-\beta z)^2}\,, \qquad \text{ideoque} \qquad \frac{a}{c} = \frac{1+fz+gzz}{1-\beta z}\,,$$

unde ergo sumi poterit

$$a = 1+fz+gzz \qquad \text{et} \qquad c = 1-\beta z\,;$$

tandem vero habebitur $b = cp$ et $d = aq$, sicque quaestioni propositae erit satisfactum.

§ 11. Hoc autem modo pro casu quem olim tractavi, $m = n = 1$ pro q unitas accipi nequit, quia litterae α et β evaderent infinitae; iidem vero enormes numeri, quo tum exhibui, reperiuntur, dum pro q vel 2 vel 3 assumitur. Quodsi autem non fuerit $m = n$, nihil obstat, quo minus statuatur $q = 1$, hincque solutiones satis commodae obtineri poterunt; semper autem tum erit $d = a$, quod fortasse displicere potest.

§ 12. Hac igitur methodo repudiata aliam viam sum ingressurus, quae ad solutiones multo simpliciores perducet, quaeque ita est comparata, ut non perveniatur ad quartam potestatem ipsius z. Hunc in finem statim pono $\frac{a}{c} = 1+sz$, ita ut habeamus:

$$\frac{1+3\alpha z+3\alpha zz+\alpha z^3}{1-\beta z} = (1+sz)^2\,,$$

qua aequatione evoluta et omnibus terminis ad eandem partem translatis pervenietur ad hanc aequationem:

$$\left.\begin{array}{c} +1+3\alpha z+3\alpha zz+\alpha z^3 \\ -1-2sz-sszz+\beta ssz^3 \\ +\beta z+2\beta szz \end{array}\right\} = 0$$

quae redigitur ad hanc formam:

$$3\alpha-2s+\beta+(3\alpha+2\beta s-ss)\,z+(\alpha+\beta ss)\,zz = 0\,.$$

§ 13. Ut nunc ex hac aequatione incognita z commode deduci queat, hoc imprimis duobus modis fieri poterit, quorum primus adhiberi poterit, si statuere licebit

$$\alpha + \beta ss = 0\ , \quad \text{sive} \quad ss = -\frac{\alpha}{\beta}\ ,$$

id quod locum habere nequit, nisi $-\frac{\alpha}{\beta} = -\frac{nqq}{m}$ fuerit quadratum, quamobrem iste casus tum tantum adhiberi poterit, quando numerorum m et n alter fuerit negativus, insuperque eorum productum quadratum. Pro hoc igitur casu ponamus $m = \mu\mu$ et $n = -\nu\nu$, ut aequalitas resolvenda sit

$$ab\,(\mu\mu aa - \nu\nu bb) = cd\,(\mu\mu cc - \nu\nu dd)\ ;$$

tum igitur erit

$$\alpha = \frac{+\nu\nu qq}{\mu\mu + \nu\nu qq} \quad \text{et} \quad \beta = \frac{-\mu\mu}{\mu\mu + \nu\nu qq}$$

statuique poterit

$$ss = -\frac{\alpha}{\beta} = \frac{\nu\nu qq}{\mu\mu}\ , \quad \text{ideoque} \quad s = \frac{\nu q}{\mu}\ .$$

Hoc modo aequatio postrema erit

$$3\alpha - 2s + \beta + (3\alpha + 2\beta s - ss)\,z = 0\ , \text{ unde colligitur } z = \frac{2s - 3\alpha - \beta}{3\alpha + 2\beta s - ss}\ ,$$

factisque substitutionibus erit

$$z = \frac{\mu\mu\,(\mu\mu - 3\nu\nu qq) + 2\mu\nu q\,(\mu\mu + \nu\nu qq)}{\mu\mu\,(3\nu\nu qq - 2\mu\nu q) - \nu\nu qq\,(\mu\mu + \nu\nu qq)}\ .$$

§ 14. Hic autem praeter omnem necessitatem introduximus binas novas litteras μ et ν; nihil enim impedit, quo minus loco μa, νb, μc, νd, simpliciter scribantur litterae a, b, c, d, ita ut formula resolvenda evadat $ab\,(aa - bb) = cd\,(cc - dd)$, perinde ac si sumpsissemus $\mu = 1$ et $\nu = 1$, unde solutio multo brevior obtinebitur.

Resolutio formulae particularis

$$ab\,(aa - bb) = cd\,(cc - dd)\ .$$

Hic igitur sumpto pro q numero quocumque erit $\alpha = \frac{qq}{1+qq}$ et $\beta = \frac{-1}{1+qq}$, tum vero sumi oportet $s = q$, unde postrema aequatio erit

$$3qq - 1 - 2q - 2q^3 + z\left(2qq - 2q - q^4\right) = 0\ ,$$

ideoque

$$z = \frac{1 + 2q - 3qq + 2q^3}{q\left(-2 + 2q - q^3\right)}\ .$$

Hoc valore invento erit $p = q\,(1 + z)$ et $\frac{a}{c} = 1 + qz$, unde porro reliquae litterae b et d facile assignabuntur.

§ 15. Statuamus nunc primo $q = 1$, eritque $z = -2$, deinde $\frac{a}{c} = -1$, ideoque $c = -a$; porro ob $p = -1$ erit $b = a$ et $d = a$. Hoc ergo casu omnes quatuor litterae essent aequales, et ambo formulae membra $= 0$. Idem prodiret, si poneremus $q = -1$: perpetuo enim observasse juvabit, perinde esse sive ipsi q tribuatur valor positivus, sive negativus. Sumamus igitur, ut idoneum exemplum proferamus, $q = 2$, eritque $z = -\frac{3}{4}$, ideoque $p = \frac{1}{2}$ et $\frac{a}{c} = -\frac{1}{2}$, ergo $c = -2a$, porro $b = -a$ et $d = 2a$. Hoc autem modo in praecedens incommodum incidimus. Sit ergo $q = 3$, eritque $z = -\frac{34}{69}$, hinc $p = \frac{35}{23}$ et $\frac{a}{c} = -\frac{11}{23}$, ita sumi poterit $a = 11$ et $c = -23$, eritque $b = -35$ et $d = 3a = 33$. Hoc modo pervenimus ad hanc aequalitatem:

$$11 \cdot 35\left(35^2 - 11^2\right) = 23 \cdot 33\left(33^2 - 23^2\right)\ ,$$

cujus ratio est manifesta, cum utrumque membrum in factores evolutum praebeat $2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Hinc duo triangula rectangula exhiberi possunt, quorum areae inter se sunt aequales; prioris enim catheti erunt $2 \cdot 11 \cdot 35$ et $24 \cdot 56$,[e] alterius vero trianguli $2 \cdot 23 \cdot 33$ et $10 \cdot 56$.

Reversio ad formulam generalem

$$ab\,(maa + nbb) = cd\,(mcc + ndd)\ .$$

§ 16. Quando autem in aequatione quadrata § 13,[f] ultimum terminum zz ad nihilum redigere non licet, semper hoc fieri potest in primo termino absoluto, unde fit $3\alpha - 2s + \beta = 0$, ex qua aequatione elicitur $2s = 3\alpha + \beta$, sive $s = \frac{4\alpha-1}{2}$. Hoc igitur primo termino sublato duo reliqui per z divisi dant hanc aequationem:

[e] This should be $24 \cdot 46$.

[f] It appears that Euler is referring to the last equation in § 12.

$$3\alpha + 2\beta s - ss + (\alpha + \beta ss)\,z = 0\ ,$$

quae aequatio abit in hanc:

$$\frac{3}{4} + (\alpha + \beta ss)\,z = 0\ , \quad \text{unde fit} \quad z = \frac{-3}{4\alpha + 4\beta ss}\ .$$

Cum igitur sit

$$2s = 4\alpha - 1\ , \quad \text{erit} \quad z = \frac{-3}{4\alpha + \beta\,(4\alpha - 1)^2}\ ,$$

unde deducitur sequens solutio nostri problematis, quo requiritur ut fiat

$$ab\,(maa + nbb) = cd\,(mcc + ndd)\ .$$

§ 17. Primo posito $b = cp$ et $d = aq$ fecimus $p = q\,(1 + z)$; tum vero, sumpta littera q pro lubitu, posuimus brevitatis gratia

$$\alpha = \frac{nqq}{nqq - m} \quad \text{et} \quad \beta = \frac{m}{nqq - m}\ ,$$

ita ut sit $\alpha - \beta = 1$. Quo facto invenimus

$$z = \frac{-3}{4\alpha + \beta\,(4\alpha - 1)^2}\ , \quad \text{unde definitur} \quad p = q\,(1 + z)\ .$$

Deinde posito

$$s = \frac{4\alpha - 1}{2} = \frac{3\alpha + \beta}{2} \quad \text{invenimus esse} \quad \frac{a}{c} = 1 + sz\ ,$$

unde cum $1 + sz$ sit plerumque fractio, hinc ambae litterae a et c facile per numeros integros assignantur; quibus inventis erit $b = cp$ et $d = aq$, quam solutionem aliquot exemplis illustremus.

Exemplum I.

quo haec formula resolvenda proponitur:

$$ab\,(aa + 2bb) = cd\,(cc + 2dd)\ .$$

DOI: 10.56031/2693-9908.1041

§ 18. Hic igitur est $m = 1$ et $n = 2$, unde sumpto numero $q = 1$ erit $\alpha = 2$ et $\beta = 1$, hincque $s = \frac{7}{2}$ et $z = -\frac{1}{19}$, ideoque $p = \frac{18}{19}$; tum vero $\frac{a}{c} = \frac{31}{38}$, quocirca sumamus $a = 31$ et $c = 38$, unde denique fit $b = 36$ et $d = 31$, ita ut solutio futura sit

$$31 \cdot 36 \left(31^2 + 2 \cdot 36^2\right) = 38 \cdot 31 \left(38^2 + 2 \cdot 31^2\right) ,$$

id quod calculum instituenti mox patebit. Est enim

$$31^2 + 2 \cdot 36^2 = 3553 = 11 \cdot 17 \cdot 19 \quad \text{et} \quad 38^2 + 2 \cdot 31^2 = 3366 = 2 \cdot 3^2 \cdot 11 \cdot 17 ,$$

quibus factoribus substitutis utrinque prodit idem productum $2^2 \cdot 3^2 \cdot 11 \cdot 17 \cdot 19 \cdot 31$.

Alia solutio ejusdem exempli.

§ 19. Sumatur hic $q = \frac{1}{2}$, eritque $\alpha = -1$ et $\beta = -2$. Deinde vero erit $s = -\frac{5}{2}$ et $z = \frac{1}{18}$, hinc ergo colligimus $p = \frac{19}{36}$ et $\frac{a}{c} = \frac{31}{36}$. Capiatur ergo $a = 31$ et $c = 36$, ex quibus denique fit $b = 19$ et $d = \frac{31}{2}$; unde, hos numeros duplicando, habebimus $a = 62$, $b = 38$, $c = 72$ et $d = 31$, ita ut sit

$$62 \cdot 38 \left(62^2 + 2 \cdot 38^2\right) = 72 \cdot 31 \left(72^2 + 2 \cdot 31^2\right) ,$$

ubi per factores est

$$62^2 + 2 \cdot 38^2 = 6732 = 2^2 \cdot 3^2 \cdot 11 \cdot 17 \text{ et}$$
$$72^2 + 2 \cdot 31^2 = 7106 = 2 \cdot 11 \cdot 17 \cdot 19 .$$

Sicque utrinque prodit idem productum $2^4 \cdot 3^2 \cdot 11 \cdot 17 \cdot 19 \cdot 31$.

Exemplum II.

quo haec formula resolvenda proponitur:
$$ab\,(aa + 3bb) = cd\,(cc + 3dd) .$$

§ 20. Hic ergo est $m = 1$ et $n = 3$, hincque $\alpha = \frac{3qq}{3qq-1}$ et $\beta = \frac{1}{3qq-1}$. Sumamus nunc primo $q = 1$, ut sit $\alpha = \frac{3}{2}$ et $\beta = \frac{1}{2}$, hinc autem fiet

$$s = \frac{5}{2} \quad \text{et} \quad z = -\frac{6}{37} , \quad \text{quamobrem erit} \quad p = \frac{31}{37} \quad \text{et} \quad \frac{a}{c} = \frac{22}{37} .$$

Capiatur ergo $a = 22$ et $c = 37$, fietque $b = 31$ et $d = 22$, consequenter habebimus

$$22 \cdot 31 \left(22^2 + 3 \cdot 31^2\right) = 37 \cdot 22 \left(37^2 + 3 \cdot 22^2\right) .$$

Hic scilicet erit

$$22^2 + 3 \cdot 31^2 = 3367 = 7 \cdot 13 \cdot 37 \quad \text{et} \quad 37^2 + 3 \cdot 22^2 = 2821 = 7 \cdot 13 \cdot 31 ;$$

sicque utrinque idem oritur productum, scilicet $2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 37$.

Alia solutio ejusdem exempli.

§ 21. Sumatur $q = \frac{1}{2}$, eritque $\alpha = -3$ et $\beta = -4$, hincque porro $s = -\frac{13}{2}$ et $z = \frac{3}{688}$, quocirca habebimus $p = \frac{691}{1376}$ et $\frac{a}{c} = \frac{1337}{1376}$. Capiatur ergo $a = 1337$ et $c = 1376$, unde fit $b = 691$ et $d = \frac{1337}{2}$. Duplicatis ergo valoribus habebimus $a = 2674$; $b = 1382$; $c = 2752$ et $d = 1337$.

Adhuc alia solutio ejusdem exempli.

§ 22. Sumamus hic $q = 2$, eritque $\alpha = \frac{12}{11}$ et $\beta = \frac{1}{11}$,[g] unde fit $s = \frac{37}{22}$ et $z = -\frac{3993}{6977}$; ex quo fit $p = \frac{5968}{6977}$, porro vero $sz = -\frac{13431}{13954}$, ideoque $\frac{a}{c} = \frac{523}{13954}$. Sicque sumi poterit $a = 523$ et $c = 13954$, unde tandem fit $b = 11936$ et $d = 1046$.

§ 23. Casum principalem, quo $m = 1$ et $n = 1$, quandoquidem hinc inventio binorum biquadratorum pendet, quorum summam in alia duo biquadrata resolvere liceat, peculiari tractationi reservamus, quae ergo continebit evolutionem hujus formulae $ab\,(aa + bb) = cd\,(cc + dd)$.

§ 24. Cum igitur hic sit $m = 1$ et $n = 1$, pro quolibet numero ad lubitum assumpto q capiatur $\alpha = \frac{qq}{qq-1}$ et $\beta = \frac{1}{qq-1}$, unde porro accipiatur $s = \frac{3\alpha+\beta}{2}$ atque $z = \frac{-3}{4\alpha+\beta(3\alpha+\beta)^2}$, quibus valoribus inventis erit $p = q\,(1 + z)$ et $\frac{a}{c} = 1 + sz$. Denique vero erit $b = cp$ et $d = a = aq$,[h] quibus praemissis ad exempla descendamus.

[g] Euler made a mistake here. The correct values are $s = \frac{37}{22}$, $z = -\frac{3993}{7177}$, $a = 923$, $b = 12736$, $c = 14354$ and $d = 1846$.

[h] This should simply be $d = aq$.

https://scholarlycommons.pacific.edu/euleriana/vol3/iss2/3
DOI: 10.56031/2693-9908.1041

Exemplum I.

quo $q = 3$.

§ 24.[i] Evidens scilicet est hoc casu sumi non posse $q = 1$, quia litterae α et β fierent infinitae; deinde vero facile est praevidere, positionem $q = 3$ simpliciorem solutionem suppeditare quam $q = 2$. Sumpto igitur $q = 3$ erit $\alpha = \frac{9}{8}$ et $\beta = \frac{1}{8}$, unde fit $s = \frac{7}{4}$ atque $z = -\frac{96}{193}$. Hinc ergo colligitur $p = \frac{291}{193}$, tum vero $sz = -\frac{168}{193}$, ideoque $\frac{a}{c} = 1 + sz = \frac{25}{193}$; quocirca capiatur $a = 25$ et $c = 193$, eritque $b = 291$ et $d = 75$. Sicque pertingimus ad hanc solutionem

$$25 \cdot 291 \left(25^2 + 291^2\right) = 193 \cdot 75 \left(193^2 + 75^2\right)$$

ubi notetur esse[j]

$$\begin{aligned} 25^2 + 291^2 &= 85306 = 2 \cdot 13 \cdot 17 \cdot 193 \quad \text{et} \\ 193^2 + 75^2 &= 42879 = 2 \cdot 13 \cdot 17 \cdot 97 \end{aligned}$$

Nunc autem facile perspicitur idem productum utrinque per factores eosdem prodire

$$2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 17 \cdot 97 \cdot 193.$$

§ 25. Transferamus nunc hanc solutionem ad biquadrata, atque in genere si fuerit $ab\,(aa + bb) = cd\,(cc + dd)$, statuatur $a = p + q$ et $b = p - q$, similique modo $c = r + s$ et $d = r - s$, prodibitque ista aequatio: $2\left(p^4 - q^4\right) = 2\left(r^4 - s^4\right)$. Hoc ergo modo erit $p^4 + s^4 = q^4 + r^4$, ubi notetur esse

$$p = \frac{a+b}{2}\,,\; q = \frac{a-b}{2}\,,\; r = \frac{c+d}{2}\,,\; \text{ et } \; s = \frac{c-d}{2}\,,$$

sive duplicando, sumere licebit

$$p = a + b\,,\; q = a - b\,,\; r = c + d\,,\; \text{ et } \; s = c - d\,.$$

Ex casu igitur modo invento colligimus $p = 158$; $q = 133$; $r = 134$ et $s = 59$.

§ 26. Hi numeri prorsus conveniunt cum iis, quos initio commemoravimus, atque adeo sine dubio simplicissimam solutionem illius problematis, quod olim tractaveram, suppeditant, cum sit

[i] The original publication repeats the paragraph number 24 and continues the following paragraphs with their numbering off by one. This has not been corrected in order to keep internal references intact.

[j] There is a typo here, $193^2 + 75^2 = 42874 = 2 \cdot 13 \cdot 17 \cdot 97$.

$$158^4 + 59^4 = 133^4 + 134^4 \,.$$

Hic autem praeter expectationem se offert alia solutio nostrae formulae $ab\,(aa+bb) = cd\,(cc+dd)$, hinc enim sequitur fore $158^4 - 134^4 = 133^4 - 59^4$, unde in factores resolvendo fit

$$24 \cdot 292\left(158^2 + 134^2\right) = 74 \cdot 192\left(133^2 + 59^2\right) \;;$$

ubi notetur ob $pp+qq = \frac{1}{2}\,(p+q)^2 + \frac{1}{2}\,(p-q)^2$, fore $158^2 + 134^2 = \frac{1}{2}\,(292)^2 + \frac{1}{2}\,(24)^2$, similique modi[k] $133^2 + 59^2 = \frac{1}{2}\,(192)^2 + \frac{1}{2}\,(74)^2$, quibus substitutis prodit forma nostrae similis,[l]

$$24 \cdot 292\left(292^2 + 24^2\right) = 74 \cdot 192^2\left(74^2 + 192^2\right)$$

singulisque factoribus per 2 divisis erit

$$12 \cdot 146\left(146^2 + 12^2\right) = 37 \cdot 96\left(37^2 + 96^2\right) \,.$$

§ 27. Ecce ergo deducti sumus ad solutionem adhuc simpliciorem nostrae formulae $ab\,(aa+bb) = cd\,(cc+dd)$, qua est $a = 12$; $b = 146$; $c = 37$ et $d = 96$. Siquidem hi numeri notabiliter sunt minores, quam supra inventi; unde sequitur methodus generalis ex qualibet solutione nostrae formulae aliam solutionem derivandi. Cum enim posito $a+b = p$, $a-b = q$, $c+d = r$ et $c-d = s$, prodeat haec aequalitas: $p^4 + s^4 = q^4 + r^4$, inde vicissim erit $p^4 - r^4 = q^4 - s^4$, unde sumptis factoribus erit

$$(p+r)\,(p-r)\,(pp+rr) = (q+s)\,(q-s)\,(qq+ss) \;, \quad \text{sive}$$

$$(p+r)\,(p-r)\left[(p+r)^2 + (p-r)^2\right] = (q+s)\,(q-s)\left[(q+s)^2 + (q-s)^2\right] \,.$$

Quamobrem si statuamus $p+r = a'$; $p-r = b'$; $q+s = c'$ et $q-s = d'$, habebimus

$$a'b'\,(a'a' + b'b') = c'd'\,(c'c' + d'd') \,,$$

quae ergo aequalitas locum habebit, si sumatur $a' = (a+b) + (c+d)$; $b' = (a+b) - (c+d)$; $c' = (a-b) + (c-d)$ et $d' = (a-b) - (c-d)$.

§ 28. Cum igitur invenissemus

[k] It should be *modo*.

[l] The equation below should be $24 \cdot 292\left(292^2 + 24^2\right) = 74 \cdot 192\left(74^2 + 192^2\right)$.

$$a = 291\ ,\ b = 25\ ,\ c = 193\ ,\ \text{ et }\ d = 75\ ,\ \text{ erit}$$
$$a + b = 316\ ,\ a - b = 266\ ,\ c + d = 268\ ,\ \text{ et }\ c - d = 118\,.$$

Hinc colligitur fore $a' = 584$; $b' = 48$; $c' = 384$ et $d' = 148$, qui numeri per 4 depressi praebent solutionem simplicissimam, scilicet: $a' = 146$; $b' = 12$; $c' = 96$ et $d' = 37$, ita ut sit

$$146 \cdot 12\left(146^2 + 12^2\right) = 96 \cdot 37\left(96^2 + 37^2\right)\ ,$$

ubi nempe est

$$146^2 + 12^2 = 21460 = 2^2 \cdot 5 \cdot 29 \cdot 37 \ \text{ et }\ 96^2 + 37^2 = 10585 = 5 \cdot 29 \cdot 73,$$

et nunc utrinque resultat idem productum $2^5 \cdot 3 \cdot 5 \cdot 29 \cdot 37 \cdot 73$. Ob insignem igitur usum hujus regulae eam sequenti theoremate complectamur.

§ 29. **Theorema.** *Si quatuor numeri a, b, c, d ita fuerint comparati, ut sit*

$$ab\,(aa + bb) = cd\,(cc + dd)\ ,$$

tum si inde formentur isti quaterni numeri: $A = (a+b)+(c+d)$; $B = (a+b)-(c+d)$; $C = (a-b)+(c-d)$ et $D = (a-b)-(c-d)$, erit quoque

$$AB\left(A^2 + B^2\right) = CD\left(C^2 + D^2\right).$$

Hinc igitur patet pro qualibet solutione semper dari adhuc aliam ipsi conjugatam, ope hujus theorematis inveniendam, ita ut perpetuo binae solutiones conjugatae exhiberi queant, quae ita invicem sunt connexae, ut ope theorematis altera ex altera definiatur.

§ 30. Quo nunc hinc alias solutiones facilius invenire liceat, formulas repertas ita adornemus, ut inde statim valores integri pro numeris a, b, c, d exhiberi queant. Hunc in finem, quoniam etiam pro q fractiones accipere licet, statim ponamus $q = \frac{f}{g}$, eritque

$$\alpha = \frac{ff}{ff - gg}\ \text{ et }\ \beta = \frac{gg}{ff - gg}\ ,\ \text{ unde fit }\ s = \frac{3ff + gg}{2\,(ff - gg)}\,.$$

Hinc cum porro sit

$$z = \frac{-3}{4\alpha + \beta\,(3\alpha+\beta)^2}\ , \quad \text{erit} \quad z = \frac{-3\,(ff-gg)^3}{4f^6 + f^4gg + 10ffg^4 + g^6}\ ,$$

unde fit

$$1 + z = \frac{f^6 + 10f^4gg + ffg^4 + 4g^6}{4f^6 + f^4gg + 10ffg^4 + g^6}\ ,$$

tum vero erit

$$sz = -\frac{3\,(ff+gg)\,(ff-gg)^2}{2\,(4f^6 + f^4gg + 10ffg^4 + g^6)}\ ,$$

ergo

$$\begin{aligned} 1 + sz &= -\frac{-f^6 + 17f^4gg + 17ffg^4 - g^6}{2\,(4f^6 + f^4gg + 10ffg^4 + g^6)} \\ &= \frac{(ff+gg)\left(-f^4 + 18ffgg - g^4\right)}{2\,(4f^6 + f^4gg + 10ffg^4 + g^6)}\ . \end{aligned}$$

§ 31. Cum igitur posuissemus

$$p = q\,(1+z) = \frac{f}{g}\,(1+z)\ , \quad \text{erit} \quad p = \frac{f\left(f^6 + 10f^4gg + ffg^4 + 4g^6\right)}{g\,(4f^6 + f^4gg + 10ffg^4 + g^6)}\ .$$

Denique habebamus $\frac{a}{c} = 1 + sz$, unde si statuamus

$$\begin{aligned} a &= (ff+gg)\left(-f^4 + 18ffgg - g^4\right) \ \text{et} \\ c &= 2\left(4f^6 + f^4gg + 10ffg^4 + g^6\right)\ , \end{aligned}$$

cum fuerit

$$b = pc \ \text{ et } \ d = aq\ , \quad \text{erit} \quad b = \frac{2f\left(f^6 + 10f^4gg + ffg^4 + 4g^6\right)}{g}$$

et tandem

$$d = \frac{f\,(ff+gg)\left(-f^4 + 18ffgg - g^4\right)}{g}\ .$$

Multiplicemus igitur omnes hos valores per g, eritque ut sequitur:

$$\begin{aligned} a &= g\,(ff+gg)\left(-f^4+18ffgg-g^4\right)\\ b &= 2f\left(f^6+10f^4gg+ffg^4+4g^6\right)\\ c &= 2g\left(4f^6+f^4gg+10ffg^4+g^6\right)\\ d &= f\,(ff+gg)\left(-f^4+18ffgg-g^4\right). \end{aligned}$$

§ 32. Hae quidem formulae numeros vehementer magnos producunt, qui autem plerumque per communem divisorem ad numeros multo minores redigi possunt. Veluti si, ut supra fecimus, sumamus $f=3$ et $g=1$, nostrae formulae dabunt $a=800$; $b=9312$; $c=6176$ et $d=2400$; hi autem numeri omnes divisionem admittunt per 32, quo pacto ad ipsos numeros supra inventos deprimuntur.

Exemplum II.

quo $f=2$ et $g=1$.

§ 33. Substitutis his valoribus reperiemus $a=5\cdot 55=275$; $b=928$; $c=626$ et $d=550$, qui numeri ulterius ad minores deprimi non possunt. Hujus solutionis etiam exhibeamus suam conjugatam secundum theorema supra datum; erit ergo $A=2379$; $B=27$; $C=729$ et $D=577$. Sicque duas nacti sumus novas solutiones.

Alia Solutio

aequationis $ab\,(aa+bb)=cd\,(cc+dd)$.

§ 34. Posito ut ante $b=cp$ et $d=aq$ adepti sumus hanc aequationem: $\frac{aa}{cc}=\frac{p^3-q}{q^3-p}$, ita ut hanc fractionem ad quadratum redigi oporteat, ad quod praestandum hic alia via utamur, quam in forma generali adhibere non licuisset. Statuamus scilicet

$$p=1+\alpha z \text{ et } q=1+\beta z\,, \quad \text{reperieturque} \quad \frac{aa}{cc}=\frac{3\alpha-\beta+3\alpha\alpha z+\alpha^3 zz}{3\beta-\alpha+3\beta\beta z+\beta^3 zz}\,,$$

sicque pervenimus ad fractionem, in qua quantitas incognita z tam in numeratore quam in denominatore non ultra quadratum assurgit, cujusmodi formulae satis commode tractari possunt, si fuerint vel primae partes absolutae quadrata, vel si partes postremae fuerint quadrata.

Evolutio casus prioris

quo formulae $3\alpha - \beta$ et $3\beta - \alpha$ sunt quadrata.

§ 35. Statuamus igitur $3\alpha - \beta = ff$ et $3\beta - \alpha = gg$, eritque

$$\alpha = \frac{3ff + gg}{8} \text{ et } \beta = \frac{ff + 3gg}{8} \,,$$

hocque modo habebimus

$$\frac{aa}{cc} = \frac{ff + 3\alpha\alpha z + \alpha^3 zz}{gg + 3\beta\beta z + \beta^3 zz} \,,$$

quamobrem statuamus

$$aa = ff + 3\alpha\alpha z + \alpha^3 zz \text{ et } cc = gg + 3\beta\beta z + \beta^3 zz \,.$$

Jam ut has formulas ad quadrata reducamus, formemus hanc aequalitatem:

$$aagg - ccff = 3\left(\alpha\alpha gg - \beta\beta ff\right) z + \left(\alpha^3 gg - \beta^3 ff\right) zz \,,$$

et quia hic membrum sinistrum duos habet factores

$$ag + cf \text{ et } ag - cf \,,$$

etiam membrum dextrum in duos factores resolvi oportet, quorum alter statuatur

$$ag + cf = \frac{z}{\lambda} \,,$$

eritque alter

$$ag - cf = 3\lambda\left(\alpha\alpha gg - \beta\beta ff\right) + \lambda\left(\alpha^3 gg - \beta^3 ff\right) z \,,$$

ubi ad valorem λ inveniendum primo fiat $z = 0$ et addantur quadrata horum duorum factorum, quorum summa erit

$$2\left(aagg + ccff\right) = 9\lambda\lambda\left(\alpha\alpha gg - \beta\beta ff\right)^2 \;;$$

at vero ex ipsis formulis propositis, posito $z = 0$, ob $aa = ff$ et $cc = gg$, erit

$$2\,(aagg+ccff)=4ffgg=9\lambda\lambda\,(\alpha\alpha gg-\beta\beta ff)^2\ ,$$

unde reperitur $\lambda=\frac{2fg}{3(\alpha\alpha gg-\beta\beta ff)}$, quae forma, ob $ff=3\alpha-\beta$ et $gg=3\beta-\alpha$, abit in hanc

$$\lambda=\frac{2fg}{3\,(\beta-\alpha)^2}\,.$$

§ 36. Invento jam hoc valore pro λ bini factores erunt

$$ag+cf=\frac{3\,(\beta-\alpha)\,3z}{2fg}\ \text{et}\ ag-cf=2fg+\frac{2fg\left(\alpha^3gg-\beta^3ff\right)z}{3\,(\alpha\alpha gg-\beta\beta ff)}\ ,\ \text{sive}$$

$$ag-cf=2fg+\frac{2fg\,(\beta+\alpha)\,(\beta\beta-3\alpha\beta+\alpha\alpha)\,z}{3\,(\beta-\alpha)^2}\,.$$

Statuamus autem brevitatis gratia

$$\frac{(\alpha+\beta)\,(\beta\beta-3\alpha\beta+\alpha\alpha)}{3\,(\beta-\alpha)^2}=\Delta\,,\quad\text{ut sit}\ \ ag-cf=2fg+2fg\Delta z\,.$$

Addantur nunc iterum quadrata harum formarum, et prodibit:

$$2\,(aagg+ccff)=4ffgg+8ffgg\Delta z+4ffgg\Delta^2zz+\frac{9\,(\beta-\alpha)^6\,zz}{4ffgg}\,.$$

Ex ipsis autem formulis propositis colligimus

$$2\,(aagg-ccff)=4ffgg+6\,(\alpha\alpha gg+\beta\beta ff)\,z+2\left(\alpha^3gg+\beta^3ff\right)zz\ ,$$

ubi ergo prima membra se mutuo tollunt, reliqua per z divisa dabunt hanc aequationem:

$$\begin{aligned}&6\,(\alpha\alpha gg+\beta\beta ff)+2\left(\alpha^3gg+\beta^3ff\right)z\\=\,&8ffgg\Delta+4ffgg\Delta^2z+\frac{9\,(\beta-\alpha)^6\,z}{4ffgg}\ ,\end{aligned}$$

ex qua aequatione facile deducitur valor ipsius z. Has autem formulas ulterius non evolvamus, quoniam evolutio quovis casu speciali facilius instituetur.

Evolutio casus,

quo $f = 3$ et $g = 1$.

§ 37. Hic igitur erit $\alpha = \frac{7}{2}$ et $\beta = \frac{3}{2}$, hinc ergo fiet

$$aa = 9 + \frac{147}{4}z + \frac{343}{8}zz \text{ et } cc = 1 + \frac{27}{4}z + \frac{27}{8}zz; \text{ porro vero erit } \Delta = -\frac{25}{48}.$$

Hinc ergo pro quantitate z invenienda habebimus hanc aequationem:

$$195 + \frac{293}{2}z = -\frac{75}{2} + \frac{625}{64}z + 16z = -\frac{75}{2} + \frac{1649}{64}z, \text{ sive}$$

$$\frac{465}{2} = -\frac{7727}{64}z, \text{ unde fit } z = -\frac{14880}{7727}.$$

Hoc modo perveniremus ad nimis magnos numeros, quos ultra evolvere opere non est pretium.

§ 38. Interim tamen haud difficile erit hinc adeo in genere valores idoneos pro litteris a, b, c, d exhibere. Sumptis scilicet pro lubitu binis quadratis ff et gg, capiantur numeri α et β ita, ut sit $\frac{\alpha}{\beta} = \frac{3ff+gg}{3gg+ff}$; scilicet postquam fractio $\frac{3ff+gg}{3gg+ff}$ ad minimos terminos fuerit reducta, numerator pro α, denominator vero pro β accipiatur, quo facto erunt numeri quaesiti

$$\begin{aligned}
a &= f(\alpha+\beta)(\alpha\alpha - 3\alpha\beta + \beta\beta)\\
b &= g(\beta^3 - 5\alpha\beta\beta + 4\alpha\alpha\beta - 2\alpha^3)\\
c &= g(\alpha+\beta)(\alpha\alpha - 3\alpha\beta + \beta\beta)\\
d &= f(\alpha^3 - 5\alpha\alpha\beta + 4\alpha\beta\beta - 2\beta^3).
\end{aligned}$$

Vel si ponatur

$$(\alpha+\beta)(\alpha\alpha - 3\alpha\beta + \beta\beta) = \Delta, \text{ erit}$$

$$\begin{aligned}
a &= f\Delta, \; b = g\left(\Delta - 3\alpha(\alpha-\beta)^2\right)\\
c &= g\Delta, \; d = f\left(\Delta - 3\beta(\alpha-\beta)^2\right).
\end{aligned}$$

Harum formularum ope exempla multo facilius evolvi poterunt.

§ 39. Ita si sumamus $f = 3$ et $g = 1$, erit $\frac{\alpha}{\beta} = \frac{28}{12} = \frac{7}{3}$, erit ergo $\alpha = 7$ et $\beta = 3$, hincque $\Delta = -50$, quocirca ipsi numeri quaesiti reperiuntur $a = 150$; $b = 386$; $c = 50$ et $d = 582$, qui porro per binarium depressi fiunt $a = 75$; $b = 193$; $c = 25$ et $d = 291$.

§ 40. Consideremus nunc quoque casum, quo $f = 2$ et $g = 1$, eritque $\frac{\alpha}{\beta} = \frac{13}{7}$, quare capiatur $\alpha = 13$ et $\beta = 7$, unde fit $\Delta = -1100$, hincque ipsi numeri quaesiti prodeunt $a = 2200$, $b = 2504$, $c = 1100$, $d = 3712$, qui per 4 depressi evadunt $a = 550$, $b = 626$, $c = 275$, $d = 928$, quae solutio convenit cum solutione supra § 33 inventa.

§ 41. Fundamentum hujus Analyseos sequenti nititur problemati:

Problema. *Propositis his duabus formulis:* $xx + 2fxy + hyy$, *nec non* $xx + 2gxy + kyy$, *invenire rationem inter numeros* x *et* y, *ut ambae iste*[m] *formulae evadant quadrata.*

§ 42. **Solutio.** Ponatur $xx+2fxy+hyy = P^2$ et $xx+2gxy+kyy = Q^2$, ac differentia dabit $PP - QQ = 2(f-g)xy + (h-k)yy$. Statuatur alter factor $P - Q = (f-g)y$, eritque alter $P + Q = 2x + \frac{(h-k)y}{f-g}$. Jam quadrata horum duorum factorum addantur, prodibitque

$$2P^2 + 2Q^2 = 4xx + \frac{4(h-k)xy}{f-g} + \frac{(h-k)^2 yy}{(f-g)^2} + (f-g)^2 yy\ ;$$

ex ipsis autem formulis propositis erit

$$2P^2 + 2Q^2 = 4xx + 4(f+g)xy + 2(h+k)yy\ ,$$

unde quia primi termini $4xx$ se destruunt, reliqui per y divisi dant hanc aequationem:

$$4(f+g)x + 2(h+k)y = \frac{4(h-k)x}{f-g} + \frac{(h-k)^2 y}{(f-g)^2} + (f-g)^2 y\ , \quad \text{sive}$$

$$4(f-g)[ff - gg - h + k]x =$$
$$\Big((h-k)^2 + (f-g)^2\Big[(f-g)^2 - 2(h+k)\Big]\Big)y\ ;$$

[m] It should be *istae*.

hinc igitur erit[n]

$$\frac{x}{y} = \frac{(f-g)^4 - 2\,(h+k)\,(f-g)^2 + (h-k)^2}{4\,(f-g)\,(ff-gg-h+k)}\,.$$

§ 43. Supra adhuc mentionem fecimus alius casus, quo ambae formulae pro aa et cc ad quadrata redigi queant (§ 34) ubi, ut partes postremae fiant quadrata, necesse est ut ambo numeri α et β sint quadrata. Sit igitur $\alpha = mm$ et $\beta = nn$, eritque

$$aa = m^6zz + 3m^4z + 3mm - nn \;\text{ et }\; cc = n^6zz + 3n^4z + 3nn - mm\,.$$

§ 44. Has aequalitates ut ad formam superioris problematis revocemus, ita repraesentemus:

$$\begin{aligned} aan^6 &= m^6n^6zz + 3m^4n^6z + 3mmn^6 - n^8 \quad \text{et} \\ ccm^6 &= m^6n^6zz + 3n^4m^6z + 3nnm^6 - m^8 \end{aligned}$$

et jam facta comparatione erit $x = m^3n^3z$, sumptoque $y = 1$, erit pro hoc casu $2fm^3n^3z = 3m^4n^6z$, unde fit $f = \frac{3}{2}mn^3$ et $g = \frac{3}{2}nm^3$, denique vero $h = 3nnm^6 - m^8$ et $k = 3mmn^6 - n^8$, unde colligitur

$$\begin{aligned} f-g &= \tfrac{3}{2}mn\,(nn-mm) \quad \text{et} \\ h-k &= 3nnm^6 + n^8 - 3mmn^6 - m^8 = \left(n^4-m^4\right)\left(n^4+m^4-3mmnn\right)\,. \end{aligned}$$

Jam problema nobis praebet valorem[o]

$$\frac{x}{y} = m^3n^3z = \frac{(f-g)^4 - 2\,(h+k)\,(f-g)^2 + (h-k)^2}{4\,(f-g)\,(ff-gg-h+k)}\,.$$

§ 45. Coronidis loco subjungam hic adhuc aliam Analysin pro resolutione formulae $ab\,(aa+bb) = cd\,(cc+dd)$. Quoniam res perducta est ad hanc aequationem:

$$\frac{aa}{cc} = \frac{p^3-q}{q^3-p}\,,\;\text{ ponatur }\; p = nn\,(q+1) - 1\,,\text{ fietque}$$

[n]The original has erroneous brackets in the numerator which have been removed.
[o]The original has erroneous brackets in the numerator which have been removed.

$$\frac{aa}{cc} = \frac{n^6\,(q+1)^3 - 3n^4\,(q+1)^2 + 3nn\,(q+1) - 1 - q}{q^3 - nn\,(q+1) + 1}\ ,$$

quae fractio per factorem $q+1$ depressa dabit

$$\frac{aa}{cc} = \frac{n^6\,(q+1)^2 - 3n^4\,(q+1) + 3nn - 1}{qq - q + 1 - nn}\ , \quad \text{sive}$$

$$aa = n^6 qq + n^4\,(2nn-3)\,q + (nn-1)^3$$

tum vero

$$cc = qq - q + 1 - nn\ , \ \ \text{sive} \ \ n^6 cc = n^6 qq - n^6 q - n^6\,(nn-1)\ .$$

Pro problemate igitur capi oportebit $x = n^3 q$ et $y = 1$; reliqua sunt manifesta.