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On the Motion of the Nodes of the Moon and the Variation of its Inclination to the Ecliptic (an English translation of *De Motu Nodorum Lunae Eiusque Inclinationis Ad Eclipticam Variatione*[a])

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Abstract

In this paper Euler attempts to explain some features of the motion of the Moon using Newton's inverse-square law of gravity. He describes the evidence in favor of Newton's theory but also the lack of progress in the study of lunar motion due to the difficulty of the three-body problem, arising here since both the Sun and the Earth have large effects on the Moon. He proceeds to investigate the line of intersection between the planes of the Earth's orbit and the Moon's orbit, as well as the angle between the two planes.

Translator's Introduction

Euler's goal in this paper is to investigate the relationship between the plane containing the orbit of the Moon around the Earth and the plane containing the orbit of the Earth around the Sun. He begins with the gravitational forces between the three bodies decomposed into their rectangular components, with fictitious forces included so that an Earthbound observer can be considered to be at rest. As a test case, he shows that a complete description of two-body motion can be derived from these components. He proceeds to consider the motion of the line of nodes, which is the line along which the planes of the two orbits intersect. He is able to calculate a mean angular speed for this line of nodes which is approximately correct, as well as deriving the equations, or

[a]Euler, L.(1750)."De Motu Nodorum Lunae Eiusque Inclinationis Ad Eclipticam Variatione"(E138), *Novi Commentarii Academiae Scientiarum Petropolitanae* 1750(1): 387-427. Reprinted in Opera Omnia: Series 2, Volume 23, pp.11-48. Original text available online at https://scholarlycommons.pacific.edu/euler-works/138/ .

correction terms, appearing in contemporary tables. Still, he is unable to explain discrepancies with observation and with the conclusions that Newton claimed to have reached. Finally, he investigates the variation in the angle, known as the inclination, between the lunar orbit and Earth's orbit. He derives maximum and minimum values for the inclination, but these also differ somewhat from observed values.

In spite of his claims to the contrary here, Euler had begun to suspect that Newton's inverse-square law was insufficient. However, in 1749 Alexis Clairaut announced that he could derive the motion of the Moon's perigee from the inverse-square law alone. Clairaut's paper on this topic won a contest judged by Euler in 1751 [b]. Euler then used Clairaut's ideas to refine his own methods and write E187 *Theory of the Motion of the Moon Which Exhibits All Its Irregularities*[c].

In this translation, several small typographical errors in the original text have been corrected, but no attempt has been made to correct repeated inconsistencies in Euler's use of signs, or to judge the appropriateness of his approximation methods.

All footnotes are comments by the translator.

1. The Moon is the nearest to us of all celestial bodies, and its exact distance to Earth can be found at any given time without perceptible error on account of its sufficiently large parallax. This calculation cannot be performed for the Sun, planets, and especially the fixed stars. Nevertheless, the lunar motion is so intricate, and subject to so many perturbations, that at this point no rules for its motion can be established with any certainty, and it cannot be described accurately in tables. Since any of the principal planets moves within its own plane according to the rules observed by Kepler, its true location can be found at any time with respect to a mean position using a single equation derived from the orbital eccentricity[d]. But the Moon strays far from this kind

[b] For more details, see Calinger, Ronald S. *Leonhard Euler: Mathematical Genius in the Enlightenment,* (Princeton: Princeton University Press, 2016), pp. 303-7, 320-2, 336-7, 376-8, and Linton, C.M. *From Eudoxus to Einstein: A History of Mathematical Astronomy,* (Cambridge: Cambridge University Press, 2004), pp. 292-304.

[c] Euler, L.(1753)."Theoria motus lunae exhibens omnes eius inaequalitates"(E187), St. Petersburg: Imperial Academy of Sciences, pp.1-347. Reprinted in Opera Omnia: Series 2, Volume 23, pp.64-336. Original text available online at https://scholarlycommons.pacific.edu/euler-works/187/ .

[d] The mean position is an approximation based on the assumption that the planet moves at a constant speed in a circular orbit, and an *equation* is the term, or one of the terms, that

of uniform motion. First, its motion is not confined to a single plane. If, at any given time, the plane through the center of the Earth containing the path of the Moon is known, not only does the intersection of this plane with the ecliptic (commonly called the line of nodes) continually change, sometimes moving forward, sometimes backward, but the inclination of this plane with the ecliptic is also variable, and at different times is seen to be larger or smaller. Next, the Moon does not move uniformly in this variable orbit, and it does not remain at the same distance from the center of the Earth. This is certainly true of the primary planets as well, but in the orbits of the planets the points at which the Sun is closest or furthest away stay at constant positions in space[e]. The Moon requires a very different theory, since the points of the lunar orbit that are nearest and furthest from the Earth do not remain at rest. Also, the Moon's minimum distance from the Earth, corresponding to the point at which the Moon is said to pass through perigee, is not constant, nor is its maximum distance, corresponding to the point called the apogee. Such inconsistency is not observed in any of the primary planets. Furthermore, the motion of the Moon away from the moving apogee and perigee is not bound by the rules that apply at all times to the planets. Instead, at the same elongation from apogee the true position is sometimes nearer and sometimes further away from the mean position. Because of this, when astronomers try to represent the motion of the Moon with an ellipse as they do for the primary planets, with the center of the Earth at one of the foci, not only must the position of the ellipse (or its apse line) continually change, but even the size and eccentricity must be made to vary. Nor can the motion be recovered by this method through a single correction, depending only on the eccentricity and size of this ellipse. Many tables of equations must be constructed, so that calculations for the Moon are extremely tiresome, and yet still not in perfect accord with observation.

2. The more the perturbations of the lunar motion are studied, the more the theory of celestial motion first brought to light by the great Newton is corroborated and confirmed. After Newton had derived by calculation the laws that Kepler had deduced from observation, he then explored the true laws of motion, demonstrating that all planets move in the same way, as if drawn toward

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must be added or subtracted to provide the true position.

[e]Readers familiar with the history of general relativity will know that this is not quite true, since Einstein's successful explanation of the precession of the perihelion of Mercury was one of the earliest pieces of evidence in favor of the theory. But this precession is at a rate of only about 10 arc minutes per century, while the major axis of the Moon's orbit makes a complete rotation in 8.85 years, and the line of nodes in 18.6 years. All relativistic effects on the motion of the Moon are far too small to have been detected by eighteenth-century observations.

the Sun by forces that are proportional to the reciprocal of the square of the distance to the Sun. He showed that it follows from this that the planets move in ellipses with the Sun at one of the foci, that the areas passed through by this motion are proportional to the times, and, furthermore, that the square of the orbital period is proportional to the cube of the major axis of the ellipse. As these conclusions agree with observations with great accuracy, Newton did not hesitate to state with certainty the principle that all planets are in fact drawn toward the Sun by forces reciprocal to the square of the distance, and he then found the motion of comets to be determined by the same law, thus providing more confirmation of the principle. Since space is a vacuum, and the motion of the planets is not impeded by a resistant medium, this force, by which the planets are attracted to the Sun, cannot admit a physical cause. For this reason, Newton tacitly presumed the force to be given instantaneously to the Sun by the Creator, and that all heavenly bodies are attracted to it. His followers dared to proclaim this openly. And, since they claimed that no body can attract another unless it is, at the same time, attracted by an equal force, they attributed a similar force to each planet and comet. The reason that the Sun is not noticeably affected by the forces of the planets, they claimed, is that it consists of so much more mass than the planets that the effect of these forces is minimal. The stupendous size of the Sun, which greatly exceeds that of the planets, provides jusitification for this conjecture. But gravity, by which all bodies are observed to be drawn toward Earth, and by which it is argued that the Moon is evidently drawn toward Earth, overcomes this force on Earth, and, in a similar way, the satellites of Jupiter and Saturn demonstrate that these planets also possess an attracting force. Finally, it appears to be quite clear from the phenomenon of ocean tides that, while the Earth attracts the Moon to itself, all parts of the Earth are attracted to the Moon in turn. In this way they argued that all celestial bodies are mutually attracted, ventured to extend this same force to all earthly bodies, and even to count this attraction among the properties of matter. This last conclusion is too speculative, but the arguments presented above support it, as does the extraordinary applicability of Newton's Philosophy of Astronomy. As the rest is confirmed by observations and indisputable proofs, surely one cannot doubt that all heavenly bodies are attracted to each other, even if the cause of this force is unknown. Much progress can be made in astronomy just from knowing that such forces exist and observing their effects, while the cause remains hidden.

3. With the principle established that all celestial bodies mutually attract one another, the determination of all motion in the heavens is reduced to a

problem in mechanics, since mechanics is the subject by which knowledge of the forces from two or more bodies acting upon each other is applied in order to determine the motion of one of the bodies. In determining the motion of the primary planets, although they are attracted to all of the other planets and not just the Sun, the forces from the planets are so small compared with those extending from the Sun that for this purpose they can be ignored without perceptible error. Because of this, the study of the motion of a primary planet leads to the problem in which the velocity and position of two bodies, mutually attracted in proportion to the reciprocal of the square of their distance, are found at any given time. As this problem is not difficult to solve, the motion of the primary planets can easily be calculated, and the tables used in astronomy constructed. For the Moon, however, this theory leads to a calculation that is so troublesome, and subject to so many difficulties, that almost nothing about its motion can be deduced with any certainty. For the Moon is attracted not only to the Earth but also to the Sun, and neither of these forces is small enough that it can be ignored as in the previous instance. Thus, a problem of the greatest difficulty arises: the investigation of the motion of three mutually attracting bodies. This must take into account three forces: first, that by which the Earth is drawn to the Sun; second, that by which the Moon is drawn to the Earth; and, third, that by which the Moon is drawn to the Sun. If this problem could be conveniently solved, it would immediately determine the motion of the Moon. But in this case the failure of analysis and the intricate calculations of the available methods are such that little more is revealed about the motion of the Moon than may be gathered from observation. Still, whatever astronomers have been able to bring to light from this theory accords with observation so accurately that no doubt remains - the motion of the Moon will correspond exactly to all of the conclusions that calculation can draw from it. Newton, who himself first attacked this problem, seems to have put incredible effort into untangling it, and this has been of benefit to astronomy, since the astronomical tables constructed using his ideas show lunar positions that are much closer to the truth than the others. Nevertheless, Newton was far from completing this work. He brilliantly illuminates the great difficulties posed by these calculations, but some of his writing is extremely obscure, and geometric rigor is lacking, especially when addressing the motion of the line of nodes and the variation of the inclination to the ecliptic. Those who have attempted this work after Newton have failed to make progress, or in truth distinguish themselves at all, considering that they were fortunate enough to have Newton come before them.

4. I myself had started into this work, but the difficulties of the tedious calculations had discouraged and obstructed me, so that I had barely advanced beyond Newton. I had not at this point considered the angle between the lunar orbit and the ecliptic, which appeared to immediately lead to greater obstacles. I was under the impression that introducing the variability of the plane of the lunar orbit into my calculations would make the work insurmountable. With the method I was using at the time, the difficulties multiplied tremendously. I was studying the forces acting on the Moon the way they are usually expressed, using tangents and normals. From these I investigated the increase or decrease in the velocity of the Moon, and from this the curvature of the orbit. This approach leads to differential equations. These equations are hard to solve, and, even if they were simple, an accurate and convenient determination of the motion would still be far off, since, in astronomy, neither the lunar velocity nor the curvature of the orbit are desired for their own sake. Instead, these values permit the Moon's position and distance to the Earth to be found for any given time, but not without a difficult calculation. Considering all of these obstacles, I wondered if there was another method that could determine motion of this kind, taking a path to the desired outcome that did not require the velocity and curvature. I had discovered a method for solving several problems of unusual complexity in mechanics, and these, for the most part, had similar obstacles. I perceived that the same method could be applied to the present work without immense calculations. In particular, the motion of the line of nodes and the variation of the inclination to the ecliptic can be determined conveniently, when these can only be found with great difficulty using other methods. But I do not doubt that the remaining phenomena of lunar motion can be explained more successfully along similar lines.

5. In order to see the power and utility of this method more clearly, it is worthwhile to test it first in solving the easier problem of the motion of just two mutually attracting bodies. Since in this case other methods can be invoked, it will be easier to see how much help we can expect from the new method in a more intricate problem. Moreover, since the motion of the Moon cannot be known without the motion of the Sun, it will be necessary to determine the solar motion by this same method before tackling the much more complicated lunar motion. In this way, not only will the nature of the method be shown through example, but the determination of the motion of the Sun will also prepare the way for determining the motion of the Moon. The Earth travels around the Sun, but in astronomy it is not the true motion but the apparent motion that is observed. This raises the question of whether the relative motion of the Sun

as seen from the Earth, which we perceive to be at rest, is what ought to be determined. If so, it is necessary, following the rules of mechanics, that we first transfer the motion with which the Earth actually proceeds to the Sun, but in the opposite direction. Or, we can conceive of the whole of space as having a motion equal and opposite to that of the Earth; with this done, the Earth is brought back to rest. So that the Earth is not disturbed by the forces continually acting upon it, it is similarly necessary to imagine opposite and equal forces acting upon the whole of space at any given moment, or that we always mentally transfer the forces we know to affect the Earth to the Sun, but in the opposite direction. The same rules will be observed if we then extend our investigation to the Moon. That is to say, since we determine all motion from the perspective of an observer on Earth, it is necessary to provide the Moon, like the Sun, with the motion of the Earth in the opposite direction, and, for each of the forces acting on the Earth, to apply equal and opposite forces to the Moon as well as the Sun. In this way we will obtain the motion of both the Moon and the Sun, not as they actually move, but as they would appear for an observer taken to be stationary at the center of the Earth.

6. Therefore, let G be the center of the Earth, and, as G is considered to be fixed, let the Sun move along curve AFf, with the plane of the figure representing the plane of the ecliptic[f]. Assume GA is a fixed line in this plane, Fig. 1
so that the position of the Sun at any given time can be represented by the angle AGF; the most convenient choice for GA is through the apogee or perigee. Let T be the time that elapses as the Sun proceeds from A to F, and let the angle AGF equal r, which will be the true anomaly. The mean anomaly will be the angle which is to $360°$ as T is to the orbital period, or sidereal year, which is $365d., 6h., 8m., 30s.$. Let v be the distance FG from the Sun to the Earth, and drop the perpendicular FP from F to the line GA. If the whole sine[g] is set to 1, then $FP = v \sin r$ and $GP = v \cos r$. For the sake of brevity, let $FP = v \sin r = y$ and $GP = v \cos r = x$. If the Sun now moves through Ff during the infinitesimally small interval dT of time, another perpendicular fp is dropped from f to AG, and Fr and fs are drawn[h] parallel to AG, then

[f]The figures for this article are on page 43 of the Euler Archive pdf of the original text and are also reproduced at the end of this translation.

[g]In the trigonometric tables of Euler's time, the *whole sine* was set to some convenient but arbitrary number (usually a large power of 10), and a "sine" shown in the table corresponded to what would now be considered the product of this whole sine with the sine of the angle. Thus, setting the whole sine to 1 brings the usage of sine in this paper into line with modern practice.

[h]Note that r has been given two different definitions in this section, once as an angle and

$Pp = -dx = -dv\cos r + v\,dr\sin r$ and $fr = dy = dv\sin r + v\,dr\cos r$, so that $(Ff)^2 = (dx)^2 + (dy)^2 = (dv)^2 + v^2(dr)^2$. If the line Gf is drawn, the infinitesimal triangle FGf has area $\frac{1}{2}vv\,dr$.

7. We now examine the forces acting on the Sun at a given point F, first considering that caused by the Earth. Since the effect of gravity on the surface of the Earth is well-known, the accelerations caused by all other forces will be measured in relation to it. Therefore I will set the radius of the Earth equal to g, and the acceleration caused by the Earth at that distance will be set equal to 1. Then at any other distance v the acceleration is $\frac{gg}{vv}$, since it is proportional to the reciprocal of the square of the distance to the center. Thus, in the case under consideration, the Sun at F is attracted to the Earth at G in direction FG with acceleration $\frac{gg}{vv}$. As the distances are equal, the force from the Sun is to the force from the Earth as the mass of the Sun is to the mass of the Earth. If we let the Earth's mass be G, and the Sun's mass be F, then the acceleration of the Earth caused by the Sun is $\frac{Fgg}{Gvv}$. Since we are considering the Earth to be at rest, the acceleration of the Earth toward the Sun should be transferred to the Sun in the opposite direction, so that the Sun has acceleration $\frac{Fgg}{Gvv}$ in the direction FG. With the previous acceleration of $\frac{gg}{vv}$ in the same direction, the total acceleration in direction FG is $\frac{(F+G)gg}{Gvv}$. For the rest of this article, it should be noted that forces will always be understood in terms of the acceleration caused by the force, and that the acceleration caused by the force of gravity will always be indicated by 1. This is in contrast to motive forces, which must be divided by the mass of the body in motion to produce the acceleration[i]. Since we obtain the accelerations directly, it is not necessary to know the masses of the bodies in motion. All bodies, however large or small, are accelerated equally by these forces.

8. The effect of each acceleration in altering the motion of other bodies is easily understood from first principles of mechanics. For, if a body is moving at the speed acquired by a body falling from height V, and it is attracted in the same direction with acceleration P (that is, P times the acceleration of gravity) while moving through the spatial infinitesimal dX, then $dV = P\,dX$. If the time is also taken into account, and the body moves through dX during

once as a point.

[i]Following Newton, Euler uses *vis motrix* ("motive force") and *vis acceleratrix* ("accelerative force"), respectively, for what would now just be called force and acceleration. Since Euler only needs to use the acceleration in this work, *vis* refers to acceleration and not force in most of the original text.

the infinitesimal dT of time, then $\frac{dX}{dT}$ will be proportional to the speed, which can be expressed using the square root of the altitude V. Since the units of time are arbitrary, let $\frac{dX}{dT} = \sqrt{V}$, so that dT can be expressed as $\frac{dX}{\sqrt{V}}$, and the time T itself as $\int \frac{dX}{\sqrt{V}}$. I demonstrated in my treatise on motion that, if $\int \frac{dX}{\sqrt{V}}$ is expressed as a length in thousandths of Rhenish feet, dividing by 125 will produce the time in seconds[j]. Expressing the time this way, $dT = \frac{dX}{\sqrt{V}}$, and thus $\sqrt{V} = \frac{dX}{dT}$, so that $V = \left(\frac{dX}{dT}\right)^2$. If the infinitesimal dT is assumed to be constant, then $dV = 2\,\frac{dX\,ddX}{(dT)^2}$. Substituting from the equation $dV = P\,dX$, we have $2\,\frac{dX\,ddX}{(dT)^2} = P\,dX$, and thus[k] $2\,ddX = P\,(dT)^2$. That is, twice the second differential of space is equal to the product of the acceleration P and the square of the infinitesimal representing the time elapsed. This, therefore, is what we have if the body is attracted in the same direction in which it is moving. However, if the attraction is in the opposite direction, then $2\,ddX = -P(dT)^2$. In either case, the direction of the body in relation to the attracting force does not change. If an oblique force acts on the body, not just the speed but also the direction of motion is affected. But we do not need to consider this case in the present work, since the motion of the body and forces acting on it can be resolved into fixed directions, so that the motion is not affected by any forces other than those in the same direction.

9. Thus, if the Sun is moving in direction Ff with speed $\frac{Ff}{dT}$, we will resolve this motion in the directions Fr and Fs, the speed relative to the first being $\frac{Fr}{dT} = -\frac{dx}{dT}$, and relative to the second $\frac{Fs}{dT} = \frac{dy}{dT}$. During the infinitesimal time interval dT the Sun moves through $-dx$ in the direction Fr and dy in the direction Fs. Now in a similar way the acceleration $\frac{(F+G)gg}{Gv^2}$ is resolved in directions Fr and FP, being $\frac{(F+G)ggx}{Gv^3}$ in direction Fr and $\frac{-(F+G)ggy}{Gv^3}$ in direction FP. From these the argument in the preceding section leads to the equations $-2\,ddx = \frac{(F+G)ggx(dT)^2}{Gv^3}$ and $2\,ddy = \frac{-(F+G)ggy(dT)^2}{Gv^3}$. If the first

[j]The "treatise on motion" is Euler's *Mechanica*, and the relevant passage is Section 222 on page 88 of Volume 1. It can be found in the Euler Archive at https://scholarlycommons.pacific.edu/euler-works/15/ . The divisor is actually given as 250 in the *Mechanica*. The Rhenish foot is 0.314 m, or 1.03 ft.

[k]Note that Euler's choice of space and time units, which he first introduced in E177 "Découverte d'un nouveau principe de mécanique"(https://scholarlycommons.pacific.edu/euler-works/177/), will consistently require a factor of 2 when relating the second derivatives of spatial coordinates with respect to T to gravitational acceleration. For further discussion, see the footnote on page 198 from Langton, S.G., "Euler on Rigid Bodies," pages 195-212 of *Leonhard Euler: Life, Work and Legacy,* eds. Robert E. Bradley and C. Edward Sandifer, Elsevier, 2007.

equation is multiplied by y, the second by x, and the two equations are then added, the result is $y\,ddx - x\,ddy = 0$, for which the integral is $y\,dx - x\,dy = CdT$. But, since $y = v\sin r$ and $x = v\cos r$, we have $y\,dx - x\,dy = -v^2\,dr$ from $\sin^2 r + \cos^2 r = 1$. Thus, we have arrived at the equation[1]

$$vv\,dr = C\,dT.$$

Next, if the first equation is multiplied by dx and the second by dy, and then one is subtracted from the other, the result is

$$\frac{2\,dx\,ddx + 2\,dy\,ddy}{(dT)^2} = \frac{-(F+G)gg}{Gv^3}(x\,dx + y\,dy).$$

Since $vv = xx + yy$, we have $x\,dx + y\,dy = v\,dv$, and thus

$$\frac{2\,dx\,ddx + 2\,dy\,ddy}{(dT)^2} = \frac{-(F+G)gg\,dv}{Gv^2},$$

with integral $\frac{(dx)^2+(dy)^2}{(dT)^2} = \frac{(F+G)gg}{Gv} + a$. We found above that $(dx)^2 + (dy)^2 = (dv)^2 + v^2(dr)^2$, from which we get the equation

$$(dv)^2 + v^2(dr)^2 = a(dT)^2 + \frac{(F+G)gg(dT)^2}{Gv}.$$

Together with the earlier $vv\,dr = C\,dT$, this will determine both unknowns v and r at any given time, and these are the values desired in astronomy. Since, also, $\frac{1}{2}vv\,dr$ is the differential of the area AGF, the area itself is $\frac{1}{2}\int vv\,dr = \frac{1}{2}CT$, establishing that the areas the Sun appears to pass through are proportional to the times, which property Kepler first observed for the Sun around the Earth, and then for all of the primary planets around the Sun.

10. Having found the two equations $vv\,dr = C\,dT$ and $(dv)^2 + v^2(dr)^2 = (a + \frac{(F+G)gg}{Gv})(dT)^2$, we write the first as $dr = \frac{C\,dT}{vv}$, which, substituted into the second, shows that

$$(dv)^2 + \frac{C^2(dT)^2}{v^2} = a(dT)^2 + \frac{(F+G)gg}{Gv}(dT)^2.$$

For the sake of brevity, let $\frac{(F+G)gg}{G} = cc$, and then

$$v^2(dv)^2 + C^2(dT)^2 = av^2(dT)^2 + ccv(dT)^2,$$

[1]Note that, since Euler has dropped the minus sign in front of the v^2, C is now the opposite of its original value. Also note that the use of both v^2 and vv for the square of v is found in the original.

or

$$dT = \frac{v\,dv}{\sqrt{-C^2 + ccv + av^2}},$$

and, from this,

$$dr = \frac{C\,dv}{v\sqrt{-C^2 + ccv + av^2}}.$$

To find the constants, consider the cases in which $dv = 0$, which must occur at apogee and perigee. In these cases $av^2+ccv-C^2 = 0$. This equation could have one positive and one negative root. However, the distance to the Sun can never be negative, from which it is clear that, if the positive root corresponded to the perigee, the Sun would never reach apogee, and this case would correspond to a hyperbolic orbit. This happens if a is positive; in order to obtain an ellipse, a must be negative, since the terms cc and C^2 are squares and cannot be negative. So let a become $-a$, and the equation $avv = ccv - CC$ has the solutions $v = \frac{cc\pm\sqrt{c^4-4aCC}}{2a}$, the smaller of which gives the distance $v = \frac{cc-\sqrt{c^4-4aCC}}{2a}$ from the Sun to the Earth at perigee, with the larger solution $v = \frac{cc+\sqrt{c^4-4aCC}}{2a}$ the distance from the Sun to the Earth at apogee. In summary, the major axis is $\frac{cc}{a}$ and the distance between the foci is $\frac{\sqrt{c^4-4aCC}}{a}$, so that the eccentricity is $\frac{\sqrt{c^4-4aCC}}{cc}$. The minor axis is $\frac{2C}{\sqrt{a}}$, and, thus, the latus rectum is $\frac{4CC}{cc}$. If we employ $2a$ for the major axis and $2b$ for the latus rectum, then a as originally defined will become $\frac{-cc}{2a}$, and $4CC = 2b\,cc$, and thus $C = c\sqrt{\frac{b}{2}}$. Therefore, the differential equations found at first become

$$vv\,dr = c\,dT\sqrt{\frac{b}{2}}$$

and

$$(dv)^2 + v^2(dr)^2 = \frac{-cc(dT)^2}{2a} + \frac{cc(dT)^2}{v}.$$

From these can be derived

$$dT = \frac{v\,dv\sqrt{2a}}{c\sqrt{-ab + 2av - vv}}$$

and

$$dr = \frac{dv\sqrt{ab}}{v\sqrt{-ab + 2av - vv}},$$

with $cc = \frac{(F+G)gg}{G}$, and the eccentricity being $\sqrt{\frac{a-b}{a}}$.

11. If the equation $dr = \frac{dv\sqrt{ab}}{v\sqrt{-ab+2av-vv}}$ is integrated, we get $r = \arccos \frac{(b-v)\sqrt{a}}{v\sqrt{a-b}}$, and, thus, $\cos r = \frac{(b-v)\sqrt{a}}{v\sqrt{a-b}}$. The angle r will be the angle described by the Sun as measured from the perigee, for, if $r = 0$, then $\cos r = 1 = \frac{(b-v)\sqrt{a}}{v\sqrt{a-b}}$, and $v = \frac{b\sqrt{a}}{\sqrt{a}+\sqrt{a-b}} = a - \sqrt{aa-ab}$, which is the distance to the Earth at perigee. Thus, if A denotes the point of perigee in the orbit of the Sun, the distance from the Sun to the Earth when angle AGF (the true anomaly) is r is found from $v = \frac{b\sqrt{a}}{\sqrt{a}+\cos r\sqrt{a-b}}$. If the eccentricity $\sqrt{\frac{a-b}{a}}$ is set equal to n, then $v = \frac{b}{1+n\cos r}$. Also, $\sqrt{\frac{a-b}{a}} = n$ leads to $a = \frac{b}{1-nn}$, and the equation for dT becomes

$$dT = \frac{v\,dv\sqrt{2b}}{c\sqrt{-bb+2bv-vv+nnvv}},$$

and, thus,

$$T = \frac{-\sqrt{2b}}{c(1-nn)}\sqrt{-bb+2bv-(1-nn)vv} + \frac{b\sqrt{2b}}{(1-nn)c}\int \frac{dv}{\sqrt{-bb+2bv-(1-nn)vv}}.$$

But $\int \frac{dv}{\sqrt{-bb+2bv-(1-nn)vv}} = \frac{1}{\sqrt{1-nn}} \arcsin \frac{v(1-nn)-b}{nb}$, or, alternatively, $\int \frac{dv}{\sqrt{-bb+2bv-(1-nn)vv}} = \frac{1}{\sqrt{1-nn}} \arccos \frac{\sqrt{1-nn}}{nb}\sqrt{-bb+2bv-(1-nn)v^2}$. Let $\arcsin \frac{v(1-nn)-b}{nb} = \omega$. Then $v = \frac{nb\sin\omega+b}{1-nn}$ and $\sqrt{-bb+2bv-(1-nn)v^2} = \frac{nb\cos\omega}{\sqrt{1-nn}}$, from which $T = \frac{b\omega\sqrt{2b}}{(1-nn)^{3/2}c} - \frac{nb\sqrt{2b}}{(1-nn)^{3/2}c}\cos\omega$, or $T = \frac{b\sqrt{2b}}{(1-nn)^{3/2}c}(\omega - n\cos\omega)$, in which a constant must be included. If $r = 0$, or $v = \frac{b}{1+n}$, then T is also 0, and, since $v = \frac{b}{1+n}$, $\omega = \arcsin(-1) = -\frac{\pi}{2}$, and thus $T = \frac{b\sqrt{2b}}{(1-nn)^{3/2}c}(\frac{\pi}{2}+\omega-n\cos\omega)$. If $\frac{\pi}{2}+\omega = \Phi$, then $\omega = -\frac{\pi}{2}+\Phi$ and $\cos\omega = \sin\Phi$, and therefore

$$T = \frac{b\sqrt{2b}}{(1-nn)^{3/2}c}(\Phi - n\sin\Phi) = \frac{a\sqrt{2a}}{c}(\Phi - n\sin\Phi).$$

Since $\sin\omega = -\cos\Phi$, we get $v = \frac{b-nb\cos\Phi}{1-nn} = a(1-n\cos\Phi)$ and $\cos r = \frac{\cos\Phi-n}{1-n\cos\Phi}$, from which a solar table is easily constructed.

12. Having seen this way to look at the motion of the Sun, and thus the power of the method employed, it is clear how to proceed to the motion of

the Moon. As before, the plane of the ecliptic is the same as the plane of the figure[m], G in this plane is the center of the Earth (which is still assumed to be at rest), and GA is a fixed line. Let T be the time in which the Sun moves from its initial point to F. Let E be the position of the Moon, which is outside of the ecliptic, and let EM be the perpendicular dropped from E to the ecliptic plane. Then the normal MP is dropped from M to GA, and the line segments GE and GM are drawn. The angle MGE thus constructed will give the ecliptic latitude of the Moon, and the angles AGF and AGM are the solar and lunar longitudes as calculated from the fixed point A on the ecliptic. Now let the distance GF to the Sun from the Earth be f, and the distance GE to the Moon from the Earth be v. Also, let $\angle AGF = r$, $\angle AGM = q$, and $\angle EGM = p$, so that $EM = v \sin p$ and $GM = v \cos p$. It follows that $PM = v \cos p \sin q$ and $GP = v \cos p \cos q$. To view these segments in terms of coordinates, we have $GP = v \cos p \cos q = x$, $PM = v \cos p \sin q = y$, and $ME = v \sin p = z$, with $xx + yy + zz = vv$. Suppose the Moon advances through infinitesimal Ee during the infinitesimal time interval dT, the perpendicular em is dropped to the ecliptic plane from e, and the normal mp is dropped from m to GA. Also suppose the rectangles $Mtem$ and $Psmp$ are completed, and Mr is drawn parallel to GA. The lunar motion is resolved into three components, two of which lie in the ecliptic plane: in the direction Mr the speed is $\frac{Mr}{dT} = \frac{-dx}{dT}$, and in the direction Ms the speed is $\frac{Ms}{dT} = \frac{dy}{dT}$. The third direction, in which the Moon recedes from the ecliptic plane, is the direction Et, in which the speed is $\frac{Et}{dT} = \frac{dz}{dT}$. Fig. 2

13. We now consider the forces acting on the Moon. First, the Sun is attracted to the Earth in direction FG with acceleration $\frac{gg}{ff}$, and the Moon in direction EG with acceleration $\frac{gg}{vv}$, as explained before. Next, if the mass of the Earth is set to G, and the mass of the Sun to F, the Earth is attracted to the Sun in the direction GF with acceleration $\frac{Fgg}{Gff}$. If the length of EF is set to u, then the Moon is attracted to the Sun in the direction EF with acceleration $\frac{Fgg}{Guu}$. Finally, if the mass of the Moon is set to E, then the Earth is attracted to the Moon in the direction GE with acceleration $\frac{Egg}{Gvv}$, and the Sun to the Moon in direction FE with acceleration $\frac{Egg}{Guu}$. Thus, we have forces by which the Sun, Earth, and Moon act mutually upon each other, of which we ignore those that act on the Sun, thus proceeding as if the Sun is not perturbed by the Moon. Also, since the Earth is assumed to be at rest, we treat the forces acting on the Earth as being transferred to the Moon in the opposite

[m]Figure 2 represents the positions of the Earth, Moon, and Sun in three dimensions.

direction, just as we have done previously, so that the Moon is considered to be affected by four forces. That is to say, the Moon is attracted first in direction EG with acceleration $\frac{gg}{vv}$ and second in direction EF with acceleration $\frac{Fgg}{Guu}$, corresponding to the forces actually acting on the Moon. Third, the Moon is attracted in the direction EG with acceleration $\frac{Egg}{Gvv}$, and, fourth, if the line HEI is drawn through E parallel to FG, then the Moon is attracted in the direction EI with acceleration $\frac{Fgg}{Gff}$. Resolving the forces acting on the Moon in this way in three directions, first, the acceleration in direction EG is $\frac{(E+G)gg}{Gvv}$; second, that in direction EF is $\frac{Fgg}{Guu}$; and, third, that in direction EI is $\frac{Fgg}{Gff}$. But the middle one, in direction EF, can be resolved again in directions EG and EH, with the component in direction EG being $\frac{Fggv}{Gu^3}$, and in direction EH being $\frac{Fggf}{Gu^3}$. Thus, the forces affecting the Moon are reduced to two directions. Specifically, in direction EG its acceleration is $\frac{(E+G)gg}{Gv^2} + \frac{Fggv}{Gu^3}$, and in direction EH it is $\frac{Fggf}{Gu^3} - \frac{Fgg}{Gff} = \frac{Fgg}{G}\left(\frac{f}{u^3} - \frac{1}{ff}\right)$.

14. With $\angle AGM = q$ and $\angle AGF = r$, we let $\angle FGM = q - r = s$, which is the difference in longitude from the Sun to the Moon. Since $\angle EGM = p$, trigonometry shows that $\cos\angle EGF = \cos p \cos s$. Thus, the sides FG and EG and the included angle FGE in triangle FGE produce the third side $FE = u = \sqrt{ff - 2fv\cos p\cos s + vv}$. Since f is so much larger than v, $\frac{1}{u^3} = (ff - 2fv\cos p\cos s + vv)^{-3/2}$ is approximately $\frac{1}{f^3} + \frac{3v\cos p\cos s}{f^4} + \frac{3vv(5\cos^2 p\cos^2 s - 1)}{2f^5}$, in which expression the last term is so small that it can be omitted in the calculation of the Moon without error, for, if we take the horizontal solar parallax as $12''$, then the mean distance[n] to the Earth from the Sun is $17189g$, and, with the distance to the Moon from the Earth being about $60g$, the ratio $v : f = 1 : 286$ is so small that all of its higher powers can be dropped. Accounting for this, the acceleration of the Moon in direction EG is $\frac{(E+G)gg}{Gv^2} + \frac{Fggv}{Gf^3}$, and in direction EH it is $\frac{3Fggv\cos p\cos s}{Gf^3}$. We can ignore terms as needed, as long as we retain the earlier expressions in which u appears.

15. We can now resolve these forces using the directions in which we have decomposed the motion of the Moon. The acceleration $\frac{(E+G)gg}{Gv^2} + \frac{Fggv}{Gu^3}$ in di-

[n]The distance given here is not at all accurate: the true distance is approximately $23455g$. Observations of the 1761 and 1769 transits of Venus made it finally possible to obtain an accurate value. Since the distance to the Moon from the Earth was known with reasonable accuracy, Euler's value for the ratio $v : f$ has a similar relative error. The true value is approximately $1 : 389$.

rection EG will resolve into three components:
first, that in direction Mr is $\frac{(E+G)ggx}{Gv^3}+\frac{Fggx}{Gu^3}$;
second, that in direction MP is $\frac{(E+G)ggy}{Gv^3}+\frac{Fggy}{Gu^3}$;
third, that in direction EM is $\frac{(E+G)ggz}{Gv^3}+\frac{Fggz}{Gu^3}$.
The other acceleration, $\frac{Fggf}{Gu^3}-\frac{Fgg}{Gff}$ in direction EH, which is parallel to the ecliptic plane, does not affect the motion in direction EM. Thus, it is transferred to the ecliptic plane and will have direction ML parallel to GF. It resolves into
$\frac{-Fggf\cos r}{Gu^3}+\frac{Fgg\cos r}{Gff}$ in direction Mr;
$\frac{Fggf\sin r}{Gu^3}-\frac{Fgg\sin r}{Gff}$ in direction Ms.
Thus, the motion of the Moon is altered by these forces taken together, and these three equations arise from the results given above:

$$\frac{2\,ddx}{(dT)^2}=\frac{-(E+G)ggx}{Gv^3}-\frac{Fggx}{Gu^3}+\frac{Fggf\cos r}{Gu^3}-\frac{Fgg\cos r}{Gff},$$

$$\frac{2\,ddy}{(dT)^2}=\frac{-(E+G)ggy}{Gv^3}-\frac{Fggy}{Gu^3}+\frac{Fggf\sin r}{Gu^3}-\frac{Fgg\sin r}{Gff},$$

$$\frac{2\,ddz}{(dT)^2}=\frac{-(E+G)ggz}{Gv^3}-\frac{Fggz}{Gu^3}.$$

Eliminating the terms involving $\frac{(E+G)gg}{Gv^3}$ produces the following three equations:

$$\frac{2(z\,ddx-x\,ddz)}{(dT)^2}=\frac{Fggfz\cos r}{Gu^3}-\frac{Fggz\cos r}{Gff},$$

$$\frac{2(z\,ddy-y\,ddz)}{(dT)^2}=\frac{Fggfz\sin r}{Gu^3}-\frac{Fggz\sin r}{Gff},$$

$$\frac{2(y\,ddx-x\,ddy)}{(dT)^2}=\frac{Fggf(y\cos r-x\sin r)}{Gu^3}-\frac{Fgg(y\cos r-x\sin r)}{Gff}.$$

With $x=v\cos p\cos q$ and $y=v\cos p\sin q$, we have $y\cos r-x\sin r=v\cos p(\sin q\cos r-\cos q\sin r)=v\cos p\sin s$, since $q-r=s$. And from $z=v\sin p$, we have $z\cos r=v\sin p\cos r$ and $z\sin r=v\sin p\sin r$. From these the equations just found are transformed in this way:

$$\frac{2d(z\,dx-x\,dz)}{(dT)^2}=\frac{Fggv\sin p\cos r}{G}\left(\frac{f}{u^3}-\frac{1}{ff}\right),$$

$$\frac{2d(z\,dy-y\,dz)}{(dT)^2}=\frac{Fggv\sin p\sin r}{G}\left(\frac{f}{u^3}-\frac{1}{ff}\right),$$

$$\frac{2d(y\,dx - x\,dy)}{(dT)^2} = \frac{Fggv\cos p\sin s}{G}\left(\frac{f}{u^3} - \frac{1}{ff}\right).$$

16. With $x = v\cos p\cos q$, $y = v\cos p\sin q$, and $z = v\sin p$, it follows that

$$dx = dv\cos p\cos q - v\,dp\sin p\cos q - v\,dq\cos p\sin q,$$

$$dy = dv\cos p\sin q - v\,dp\sin p\sin q + v\,dq\cos p\cos q,$$

$$dz = dv\sin p + v\,dp\,\cos p.$$

From these are produced

$$z\,dx - x\,dz = -vv\,dp\cos q - vv\,dq\sin p\cos p\sin q,$$

$$z\,dy - y\,dz = -vv\,dp\sin q + vv\,dq\sin p\cos p\cos q,$$

$$y\,dx - x\,dy = -vv\,dq\cos^2 p.$$

If these expressions are substituted into the equations found before, three equations are produced in the four variables T, v, p, and q, by which three can be determined from the fourth. Furthermore, it should be noted that the sum of the squares of the infinitesimals dx, dy, and dz is $(dx)^2 + (dy)^2 + (dz)^2 = (dv)^2 + v^2(dp)^2 + v^2(dq)^2\cos^2 p$, from which a new equation can be inferred from the three found before. For, if the first equation is multiplied by dx, the second by dy, and the third by dz, then from $x\,dx + y\,dy + z\,dz = v\,dv$ we will have the equation

$$\frac{2\,dx\,ddx + 2\,dy\,ddy + 2\,dz\,ddz}{(dT)^2} = \frac{d((dv)^2 + v^2(dp)^2 + v^2(dq)^2\cos^2 p)}{(dT)^2}$$

$$= -\frac{(E+G)gg\,dv}{Gv^3} - \frac{Fggv\,dv}{Gu^3} + \frac{Fgg}{G}\left(\frac{f}{u^3} - \frac{1}{ff}\right)(dx\,\cos r + dy\,\sin r).$$

But $dx\,\cos r + dy\,\sin r = dv\,\cos p\cos s - v\,dp\sin p\cos s - v\,dq\cos p\sin s$, and this, when combined with the above, will make it easier to investigate the lunar orbit.

17. Since I have decided here to explore the motion of the line of nodes and the variation of the inclination to the ecliptic, but not the motion of the Moon itself, these are the two things that I will primarily be considering. Therefore, while the Moon travels through the infinitesimal Ee of its orbit, let $G\Omega$ be the line of nodes, which is the intersection of the ecliptic with the plane determined

by the point G and Ee. Let $\angle AG☊ = \Phi$. If the normal MQ is drawn from M to $G☊$ and joined to EQ, then angle EQM is equal to the inclination of the lunar orbit to the ecliptic. Let $\angle EQM = \theta$. From $\angle ☊GM = q - \Phi$ will follow $MQ = v \cos p \sin(q - \Phi)$ and $GQ = v \cos p \cos(q - \Phi)$. Thus, $\frac{ME}{MQ} = \frac{v \sin p}{v \cos p \sin(q-\Phi)} = \tan\theta$, or $\tan\theta = \frac{\tan p}{\sin(q-\Phi)}$. Since the position of the line of nodes and the inclination will remain the same at points E and e, it is clear that the angles Φ and θ should be invariant when differentiating by p and q. Thus, by differentiating the equation $\tan\theta = \frac{\tan p}{\sin(q-\Phi)}$ we obtain

$$0 = \frac{dp}{\cos^2 p \sin(q - \Phi)} - \frac{dq \tan p \cos(q - \Phi)}{\sin^2(q - \Phi)},$$

from which $dp = \frac{dq \sin p \cos p \cos(q-\Phi)}{\sin(q-\Phi)}$, and this, when substituted into previous equations, gives

$$z\,dx - x\,dz = -\frac{vv\,dq \sin p \cos p \cos\Phi}{\sin(q - \Phi)},$$

$$z\,dy - y\,dz = -\frac{vv\,dq \sin p \cos p \sin\Phi}{\sin(q - \Phi)},$$

$$y\,dx - x\,dy = -vv\,dq \cos^2 p.$$

18. These values are substituted into earlier equations to give

$$d.\frac{vv\,dq \sin p \cos p \cos\Phi}{\sin(q - \Phi)} = \frac{Fggv(dT)^2 \sin p \cos r}{2G}\left(\frac{1}{ff} - \frac{f}{u^3}\right),$$

$$d.\frac{vv\,dq \sin p \cos p \sin\Phi}{\sin(q - \Phi)} = \frac{Fggv(dT)^2 \sin p \sin r}{2G}\left(\frac{1}{ff} - \frac{f}{u^3}\right),$$

$$d.vv\,dq \cos^2 p = \frac{Fggv(dT)^2 \cos p \sin s}{2G}\left(\frac{1}{ff} - \frac{f}{u^3}\right).$$

Differentiating and then dividing throughout by v produces

$$\frac{2\,dv\,dq\ \sin p \cos p \cos\Phi}{\sin(q - \Phi)} + v\,d.\frac{dq\ \sin p \cos p \cos\Phi}{\sin(q - \Phi)}$$

$$= \frac{Fgg(dT)^2 \sin p \cos r}{2G}\left(\frac{1}{ff} - \frac{f}{u^3}\right),$$

$$\frac{2\,dv\,dq\ \sin p \cos p \sin\Phi}{\sin(q - \Phi)} + v\,d.\frac{dq\ \sin p \cos p \sin\Phi}{\sin(q - \Phi)}$$

$$= \frac{Fgg(dT)^2 \sin p \sin r}{2G} \left(\frac{1}{ff} - \frac{f}{u^3} \right),$$

$$2\,dv\,dq\,\cos^2 p + v\,d(dq\,\cos^2 p) = \frac{Fgg(dT)^2 \cos p \sin s}{2G} \left(\frac{1}{ff} - \frac{f}{u^3} \right),$$

which transform to these:

$$\frac{2\,dv}{v} + d.\ln \frac{dq \sin p \cos p \cos \Phi}{\sin(q-\Phi)} = \frac{Fgg(dT)^2 \cos r \sin(q-\Phi)}{2Gv\,dq \cos p \cos \Phi} \left(\frac{1}{ff} - \frac{f}{u^3} \right),$$

$$\frac{2\,dv}{v} + d.\ln \frac{dq \sin p \cos p \sin \Phi}{\sin(q-\Phi)} = \frac{Fgg(dT)^2 \sin r \sin(q-\Phi)}{2Gv\,dq \cos p \sin \Phi} \left(\frac{1}{ff} - \frac{f}{u^3} \right),$$

$$\frac{2\,dv}{v} + d.\ln\ dq\ \cos^2 p = \frac{Fgg(dT)^2 \sin s}{2Gv\,dq \cos p} \left(\frac{1}{ff} - \frac{f}{u^3} \right).$$

The results of two subtractions among these equations are

$$d.(\ln \tan \Phi) = \frac{Fgg(dT)^2 \sin(r-\Phi) \sin(q-\Phi)}{2Gv\,dq \cos p \sin \Phi \cos \Phi} \left(\frac{1}{ff} - \frac{f}{u^3} \right),$$

$$d.\ln \frac{\tan p \sin \Phi}{\sin(q-\Phi)} = \frac{Fgg(dT)^2 (\sin r \sin(q-\Phi) - \sin s \sin \Phi)}{2Gv\,dq \cos p \sin \Phi} \left(\frac{1}{ff} - \frac{f}{u^3} \right).$$

Since $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$, we have $\sin r \sin(q-\Phi) = \frac{1}{2}\cos(s-\Phi) - \frac{1}{2}\cos(q+r-\Phi)$ and $\sin s \sin \Phi = \frac{1}{2}\cos(s-\Phi) - \frac{1}{2}\cos(q-r+\Phi)$, since $s = q-r$. Thus, $\sin r \sin(q-\Phi) - \sin s \sin \Phi = \frac{1}{2}\cos(q-r+\Phi) - \frac{1}{2}\cos(q+r-\Phi) = \sin q \sin(r-\Phi)$, since, in turn, $\frac{1}{2}\cos A - \frac{1}{2}\cos B = \sin\frac{A+B}{2}\sin\frac{B-A}{2}$. On account of this, the last equation is transformed to

$$d.\ln \frac{\tan p \sin \Phi}{\sin(q-\Phi)} = \frac{Fgg(dT)^2 \sin q \sin(r-\Phi)}{2Gv\,dq \cos p \sin \Phi} \left(\frac{1}{ff} - \frac{f}{u^3} \right).$$

19. Since $d.(\ln \tan \Phi) = \frac{d\Phi}{\sin \Phi \cos \Phi}$, substituting this into the first of the two equations above leads to

$$d\Phi = \frac{Fgg(dT)^2 \sin(r-\Phi) \sin(q-\Phi)}{2Gv\,dq \cos p} \left(\frac{1}{ff} - \frac{f}{u^3} \right).$$

At this point we refer to the approximation found earlier that $\frac{1}{u^3}$ is nearly equal to $\frac{1}{f^3} + \frac{3v \cos p \cos s}{f^4}$, and thus $\frac{1}{ff} - \frac{f}{u^3} = -\frac{3v \cos p \cos s}{f^3}$. When this expression is substituted, we have

$$d\Phi = -\frac{3Fgg(dT)^2 \cos s \sin(r-\Phi) \sin(q-\Phi)}{2Gf^3\,dq}.$$

Since the speed of the line of nodes is expressed by $\frac{d\Phi}{dT}$, we find

$$\frac{d\Phi}{dT} = -\frac{3Fgg(dT)\cos s\sin(r-\Phi)\sin(q-\Phi)}{2Gf^3\,dq}.$$

Note that $\frac{dq}{dT}$ is the speed of the Moon in longitude. Thus, the retrograde speed of the line of nodes is directly proportional to the cosine of the distance from the Sun to the Moon, the sine of the distance from the node to the Sun, and the sine of the distance from the node to the Moon together, and inversely proportional to the cube of the distance from the Earth to the Sun and the longitudinal speed of the Moon, so that the motion of the Moon depends on these five things mentioned[o]. This amazing expression is in accord with the conclusion of Newton as shown in Proposition XXX, Book III of the *Principia*, and, since it can determine the speed of the line of nodes at any given moment, it will also find the hourly motion of the nodes, since the infinitesimal dT can be assumed to be an hour without error. Suppose the Sun is moving at its mean motion and at its mean distance from Earth. We set the mean distance to a and let $\frac{(F+G)gg}{G} = cc$. Let $d\omega$ be the angle traversed during the infinitesimal time interval dT. Then, from Section 11, $dT = \frac{a\,d\omega\sqrt{2a}}{c}$ and $(dT)^2 = \frac{2a^3(d\omega)^2}{cc} = \frac{2Ga^3(d\omega)^2}{(F+G)gg}$, which, when substituted into the equation above, will give $d\Phi = -\frac{3Fa^3(d\omega)^2}{(F+G)f^3\,dq}\cos s\sin(r-\Phi)\sin(q-\Phi)$. In this expression, if $d\omega$ is assumed to be the mean hourly motion of the Sun, which[p] is $2', 27'', 50''', 37''''$, and dq the hourly motion of the Moon according to longitude, then $d\Phi$ will give the hourly motion of the line of nodes.

20. According to Newton, the ratio[q] of F to G is $227512 : 1$, so that the fraction $\frac{F}{F+G}$ can be written as 1. Thus, $d\Phi = -\frac{3a^3(d\omega)^2}{f^3 dq}\cos s\sin(r-\Phi)\sin(q-\Phi)$. In order to more easily extract the motion of the nodes from this, we first assume that both Sun and Moon move around the Earth uniformly, so that $f = a$, and dq denotes the mean hourly motion of the Moon in longitude, so that $dq = 32', 56'', 27''', 13''''$. With $d\omega = 2', 27'', 50''', 37''''$, we get $d\omega = 532237''''$ and $dq = 7115233''''$, so that $\frac{3(d\omega)^2}{dq} = 119437'''' = 33'', 10''', 37''''$, and the

[o]Note that *distantia*, or distance, is used in this sentence for both differences in ecliptic longitude and actual linear distance. The meaning in each instance should be clear from the context.

[p]This angle measure is given in minutes, seconds, thirds, and fourths, where a third is one-sixtieth of a second, and a fourth is one-sixtieth of a third.

[q]Since this calculation relies on an accurate comparison of the distances to the Moon and the Sun, the value is again quite far from the truth, with the actual ratio being about $332946 : 1$.

hourly motion of the nodes is $d\Phi = -\cos s \sin(r-\Phi)\sin(q-\Phi)\, 33'', 10''', 37''''$. Thus, the nodes move most rapidly if each of the sines is equal to the whole sine, which will happen if the Sun and Moon are in conjunction, and the line of nodes forms a right angle with the line GF from the Earth to the Sun. The same occurs if the Sun and Moon are at opposition, and the line of nodes is normal to GF. In each of these cases the line of nodes moves backward in one hour by $33'', 10''', 37''''$, and this is the fastest possible retrograde motion of the line of nodes. The motion of the nodes vanishes in three cases: first, if the Sun and Moon are at a 90° angle with respect to each other, second, if the Sun is passing through the line of nodes, and third, if the Moon is passing through the line of nodes. In order for the nodes to move forward, the value of $\cos s \sin(r-\Phi)\sin(q-\Phi)$ must be negative; the method of maxima and minima can be used to find its extreme value. This will occur, first, if the Sun and Moon are at a 60° angle from each other, and the line of nodes bisects the angle FGE, and, second, if the Sun and Moon are at a 120° angle and the line of nodes is at a right angle to the bisector of FGE. In either case the speed of the nodes will be at its maximum, and, since each sine will be one-half, the maximum hourly motion will be one-eighth of the fastest retrograde motion, or about $4'', 8''', 59''''$.

21. The nodes move backward faster and more often than they move forward, producing the retrograde notion of the nodes. To determine this motion accurately, it will be necessary to investigate the integrals of the equations found previously. From this, it will be easy to find the position of the line of nodes for any given time. Here it is finally necessary to introduce the true motions of the Sun and the Moon to the calculations. Therefore, let the mean distance from the Earth to the Sun be a, the eccentricity be n, and the eccentric anomaly of the Sun at the given time be ϱ. Since the angle ω above was defined with respect to the mean motion of the Sun, we have $d\varrho(1 - n\cos\varrho) = d\omega$, and thus $d\varrho = d\omega(1 + n\cos\varrho)$, neglecting the terms in which n has higher dimensions. Furthermore, since the true anomaly is approximately $\varrho + n\sin\varrho$, we have $dr = d\varrho(1 + n\cos\varrho)$, and, thus, $dr = d\omega(1 + 2n\cos\varrho)$, and also $f = a(1 - n\cos\varrho)$. Then, although the motion of the Moon is not known exactly, we assume it moves around the Earth with uniform elliptical motion, for the difference between this assumption and reality would not lead to a noticeable error in the present argument. Therefore, let the mean distance from the Earth to the Moon be α, the eccentricity be m, the eccentric anomaly be ξ, and the true distance from the Earth to the Moon be v. If the mean motion of the Moon is to that of the Earth or Sun as λ is to 1, then $\lambda = 13.3685$

from observation. From this it follows that $d\xi(1 - m\cos\xi) = \lambda\, d\omega$, and thus $d\xi = \lambda\, d\omega(1 + m\cos\xi)$. The true anomaly is $\xi + m\sin\xi$. If the mean motion of the apsides is to the mean motion of the Sun as x is to 1, then $x = 0.112996$ from the mean motion of the apogee, and the motion of the Moon can be found from $dq = \lambda\, d\omega + 2(\lambda - x)m\, d\omega\cos\xi$. Substituting this into the equation shown previously, we get

$$d\Phi = \frac{-3\, d\omega}{(1 - n\, cos\varrho)^3(\lambda + 2(\lambda - x)m\cos\xi)}\cos(q - r)\sin(r - \Phi)\sin(q - \Phi),$$

where it should be noted that the eccentric anomalies ϱ and ξ are not measured from the apogee as is customary, but from the perigee.

22. By raising factors out of the denominator and neglecting the terms with higher powers of the eccentricities m and n, we have

$$d\Phi = -\frac{3\, d\omega}{\lambda}(1 + 3n\cos\varrho)\left(1 - \frac{2(\lambda - x)m}{\lambda}\cos\xi\right)\cos(q - r)\sin(r - \Phi)\sin(q - \Phi).$$

In order to investigate the integral, we will treat the quantity $(1 + 3n\cos\varrho)(1 - \frac{2(\lambda-x)m}{\lambda}\cos\xi)$ as constant, since it is never perceptibly different from 1. For sake of brevity, let $(1 + 3n\cos\varrho)(1 - \frac{2(\lambda-x)m}{\lambda}\cos\xi) = i$, so that

$$d\Phi = \frac{-3i\, d\omega}{\lambda}\cos(q - r)\sin(r - \Phi)\sin(q - \Phi).$$

Since, as we saw before, $\sin A\sin B = \frac{1}{2}\cos(B - A) - \frac{1}{2}\cos(A + B)$, we have $\sin(r - \Phi)\sin(q - \Phi) = \frac{1}{2}\cos(q - r) - \frac{1}{2}\cos(q + r - 2\Phi)$, and, when this is substituted, we have

$$d\Phi = \frac{-3i\, d\omega}{2\lambda}(\cos(q - r)\cos(q - r) - \cos(q - r)\cos(q + r - 2\Phi)).$$

Furthermore, since $\cos A\cos B = \frac{1}{2}\cos(B - A) + \frac{1}{2}\cos(B + A)$, we have

$$\cos(q - r)\cos(q - r) = \frac{1}{2} + \frac{1}{2}\cos 2(q - r),$$

$$\cos(q - r)\cos(q + r - 2\Phi) = \frac{1}{2}\cos 2(r - \Phi) + \frac{1}{2}\cos 2(q - \Phi),$$

and thus we will have

$$d\Phi = \frac{-3i\, d\omega}{4\lambda}(1 + \cos 2(q - r) - \cos 2(r - \Phi) - \cos 2(q - \Phi)).$$

Since we know the rate of change of Φ to be much smaller than that of q and r, we assume at first that Φ is constant in these cosines. Since dq is approximately $\lambda\, d\omega$, and dr is approximately $d\omega$, the resulting integral is

$$\Phi = C - \frac{3i}{4\lambda}\left(\omega + \frac{\sin 2(q-r)}{2(\lambda-1)} - \frac{\sin 2(r-\Phi)}{2} - \frac{\sin 2(q-\Phi)}{2\lambda}\right).$$

This requires multiple corrections, first because Φ was assumed to be constant, next because we assumed $dq = \lambda\, d\omega$ and $dr = d\omega$ when actually $dq = \lambda\, d\omega + 2(\lambda - x)m\, d\omega \cos\xi$ and $dr = d\omega + 2n\, d\omega \cos\varrho$, and, thirdly, because we assumed the quantity i was constant when it is actually variable.

23. Let the approximation found for Φ now be called P, so that

$$P = C - \frac{3i}{4\lambda}\left(\omega + \frac{\sin 2(q-r)}{2(\lambda-1)} - \frac{\sin 2(r-\Phi)}{2} - \frac{\sin 2(q-\Phi)}{2\lambda}\right).$$

In order to find the correction arising from the variability of Φ itself, differentiate P with the assumption that Φ is the only variable, and let the differential be $Q\, d\Phi$, so that $\Phi = P - \int Q\, d\Phi$ by the properties of integration. Differentiating produces

$$Q\, d\Phi = \frac{3i\, d\Phi}{4\lambda}\left(\cos 2(r-\Phi) - \frac{\cos 2(q-\Phi)}{\lambda}\right).$$

If the expression found above is substituted for $d\Phi$, then

$$\begin{aligned} Q\, d\Phi = \frac{-9ii}{16\lambda^2}\, d\omega(&\cos 2(r-\Phi) + \cos 2(r-\Phi)\cos 2(q-r) \\ &- \cos 2(r-\Phi)\cos 2(r-\Phi) - \cos 2(r-\Phi)\cos 2(q-\Phi)) \\ +\frac{9ii}{16\lambda^3}\, d\omega(&\cos 2(q-\Phi) + \cos 2(q-\Phi)\cos 2(q-r) - \cos 2(q-\Phi)\cos 2(r-\Phi) \\ &- \cos 2(q-\Phi)\cos 2(q-\Phi)). \end{aligned}$$

But, by the reduction $\cos A \cos B = \frac{1}{2}\cos(B-A) + \frac{1}{2}\cos(B+A)$ applied above,

$$\begin{aligned} Q\, d\Phi = \frac{9ii}{16\lambda^2} d\omega &\left(\frac{1}{2} + \frac{1}{2}\cos 4(r-\Phi) - \cos 2(r-\Phi) - \frac{1}{2}\cos 2(q-2r+\Phi)\right. \\ &\left.-\frac{1}{2}\cos 2(q-\Phi) + \frac{1}{2}\cos 2(q-r) + \frac{1}{2}\cos 2(q+r-2\Phi)\right) - \frac{9ii}{16\lambda^3}\, d\omega\left(\frac{1}{2}\right. \\ &+\frac{1}{2}\cos 4(q-\Phi) - \cos 2(q-\Phi) - \frac{1}{2}\cos 2(r-\Phi) - \frac{1}{2}\cos 2(2q-r-\Phi) \end{aligned}$$

$$+\frac{1}{2}\cos 2(q-r)+\frac{1}{2}\cos 2(q+r-2\Phi)\biggr).$$

If now, as before, Φ is treated as a constant in the cosines in which it occurs, and we assume $dq=\lambda\,d\omega$ and $dr=d\omega$, the integral is

$$\int Q\,d\Phi=\frac{9ii}{16\lambda^2}\left(\frac{1}{2}\omega+\frac{\sin 4(r-\Phi)}{8}-\frac{\sin 2(r-\Phi)}{2}-\frac{\sin 2(q-2r+\Phi)}{4(\lambda-2)}\right.$$

$$\left.-\frac{\sin 2(q-\Phi)}{4\lambda}+\frac{\sin 2(q-r)}{4(\lambda-1)}+\frac{\sin 4(q+r-2\Phi)}{4(\lambda+1)}\right)-\frac{9ii}{16\lambda^3}\left(\frac{1}{2}\omega\right.$$

$$+\frac{\sin 4(q-\Phi)}{8\lambda}-\frac{\sin 2(q-\Phi)}{2\lambda}-\frac{\sin 2(r-\Phi)}{4}-\frac{\sin 2(2q-r-\Phi)}{4(2\lambda-1)}+\frac{\sin 2(q-r)}{4(\lambda-1)}$$

$$\left.+\frac{\sin 2(q+r-2\Phi)}{4(\lambda+1)}\right).$$

This quantity should be added to the expression for P itself given above in order to produce the value of Φ corrected for the variability of Φ. It is clear that, since $\lambda=13.3685$, most of the terms will be extremely small. The maximum value reached among the sine terms is $\frac{9ii}{32\lambda^2}\sin 2(r-\Phi)$ when this sine is equal to 1, and this contributes about $5'$. But, since a given ω can be increased by this quantity, these terms cannot be neglected. Therefore, after dropping terms that are extremely small, and assuming $i=1$ in the smaller terms, we get

$$\Phi=C-\frac{3i\omega}{4\lambda}\left(1-\frac{3}{8\lambda}-\frac{3}{8\lambda^2}\right)-\frac{3i\sin 2(q-r)}{8\lambda(\lambda-1)}\left(1-\frac{3}{8\lambda}-\frac{3}{8\lambda^2}\right)$$

$$+\frac{3i\sin 2(r-\Phi)}{8\lambda}\left(1-\frac{3}{4\lambda}-\frac{3}{8\lambda^2}\right)+\frac{9}{128\lambda^2}\sin 4(r-\Phi)$$

$$+\frac{3i\sin 2(q-\Phi)}{8\lambda^2}\left(1-\frac{3}{8\lambda}-\frac{3}{4\lambda^2}\right).$$

24. Since the complete forms of the differentials of q and r have not been used thus far, we will now consider how much error results from these. We will first differentiate P taking q for the only variable. We write $2(\lambda-x)m\,d\omega\cos\xi$ for dq, but, since x is sufficiently small with respect to λ, let $dq=2\lambda m\,d\omega\cos\xi$, and we will have

$$dP=\frac{-3i}{4\lambda}\,d\omega\left(\frac{2\lambda m\cos\xi}{\lambda-1}\cos 2(q-r)+2m\cos\xi\cos 2(q-\Phi)\right).$$

The integral of this should be subtracted from the one found before. Reducing the cosine products to simple cosines gives

$$dP = \frac{-3im\,d\omega}{4\lambda}\left(\frac{\lambda}{\lambda-1}\cos(2q-2r-\xi)+\frac{\lambda}{\lambda-1}\cos(2q-2r+\xi)\right.$$

$$\left.+\cos(2q-\xi-2\Phi)+\cos(2q+\xi-2\Phi)\right).$$

The approximations $dq=\lambda\,d\omega$ and $d\xi=\lambda\,d\omega$ make the integral equal

$$\frac{-3im}{4\lambda}\left(\frac{\sin(2q-2r-\xi)}{\lambda-1}+\frac{\sin(2q-2r+\xi)}{3(\lambda-1)}+\frac{\sin(2q-\xi-2\Phi)}{\lambda}\right.$$

$$\left.+\frac{\sin(2q+\xi-2\Phi)}{3\lambda}\right).$$

With $m=0.1414$, these expressions produce little more than two minutes at maximum. Taking i to be 1, the expression

$$\frac{3m}{4\lambda}\left(\frac{4\sin 2(q-r)\cos\xi-2\cos 2(q-r)\sin\xi}{3(\lambda-1)}\right.$$

$$\left.+\frac{4\sin 2(q-\Phi)\cos\xi-2\cos 2(q-\Phi)\sin\xi}{3\lambda}\right)$$

should be added to the one for Φ found above. Similarly, if P is differentiated with respect to r alone, and we put $2n\,d\omega\cos\varrho$ in place of dr, we get

$$dP=\frac{-3i}{4\lambda}\left(\frac{-2n\,d\omega\cos\varrho}{\lambda-1}\cos 2(q-r)+2n\,d\omega\cos\varrho\cos 2(r-\Phi)\right),$$

or

$$dP=\frac{-3in\,d\omega}{4\lambda}\left(\frac{-\cos(2q-2r-\varrho)-\cos(2q-2r+\varrho)}{\lambda-1}+\cos(2r-\varrho-2\Phi)\right.$$

$$\left.+\cos(2r+\varrho-2\Phi)\right),$$

for which the integral, with $dr=d\omega$ and $d\varrho=d\omega$, will be

$$\frac{-3in}{4\lambda}\left(\frac{\sin(2q-2r-\varrho)}{3(\lambda-1)}+\frac{\sin(2q-2r+\varrho)}{\lambda-1}+\sin(2r-\varrho-2\Phi)\right.$$

$$\left.+\frac{\sin(2r+\varrho-2\Phi)}{3}\right).$$

Thus, from this source the following should be added to the expression for Φ found above:

$$\frac{3n}{4\lambda}\left(\frac{4\sin 2(q-r)\cos\varrho + 2\cos 2(q-r)\sin\varrho}{3(\lambda-1)} + \frac{4\sin 2(r-\Phi)\cos\varrho - 2\cos 2(r-\Phi)\sin\varrho}{3}\right).$$

25. It remains to investigate the error arising from the variability of i. Since[r] $i = 1 + 3n\cos\varrho - \frac{2(\lambda-x)}{\lambda} m\cos\xi$, we will have $di = -3n\,d\omega\sin\varrho + 2(\lambda - x)m\,d\omega\sin\xi$. Differentiating P with respect to i alone produces

$$dP = \frac{3\,d\omega}{4\lambda}\left(3n\omega\sin\varrho - \frac{3n\sin\varrho\sin 2(q-r)}{2(\lambda-1)} + \frac{3n\sin\varrho\sin 2(r-\Phi)}{2} + \frac{3n\sin\varrho\sin 2(q-\Phi)}{2\lambda}\right) - \frac{3(\lambda-x)m\,d\omega}{4\lambda}\left(2\omega\sin\xi + \frac{\sin\xi\sin 2(q-r)}{\lambda-1} + \sin\xi\sin 2(r-\Phi) + \frac{\sin\xi\sin 2(q-\Phi)}{\lambda}\right),$$

for which the integral also should be subtracted from the expression for Φ found above. Using $d\varrho = d\omega$, we get $\int \omega\,d\omega\sin\varrho = -\omega\cos\varrho + \sin\varrho$ and $\int \omega\,d\omega\sin\xi = -\frac{\omega}{\lambda}\cos\xi + \frac{\sin\xi}{\lambda\lambda}$. Therefore, with

$$dP = \frac{9n\,d\omega}{4\lambda}\left(\omega\sin\varrho + \frac{\cos(2q-2r-\varrho)-\cos(2q-2r+\varrho)}{4(\lambda-1)} + \frac{\cos(2r-\varrho-2\Phi)-\cos(2r+\varrho-2\Phi)}{4} + \frac{\cos(2q-\varrho-2\Phi)-\cos(2q+\varrho-2\Phi)}{4\lambda}\right)$$
$$-\frac{3(\lambda-x)m\,d\omega}{2\lambda}\left(\omega\sin\xi + \frac{\cos(2q-2r-\xi)-\cos(2q-2r+\xi)}{4(\lambda-1)} + \frac{\cos(2r-\xi-2\Phi)-\cos(2r+\xi-2\Phi)}{4} + \frac{\cos(2q-\xi-2\Phi)-\cos(2q+\xi-2\Phi)}{4\lambda}\right),$$

[r] This formula for i does not match the one given in Section 22, but, presumably, Euler has dropped a term he considers insignificant.

the integral will be[s]

$$\frac{9n}{4\lambda}\left(-\omega\cos\varrho+\sin\varrho+\frac{\sin(2q-2r-\varrho)}{4(\lambda-1)(2\lambda-3)}-\frac{\sin(2q-2r+\varrho)}{4(\lambda-1)(2\lambda-1)}\right.$$

$$+\frac{\sin(2r-\varrho-2\Phi)}{4}-\frac{\sin(2r+\varrho-2\Phi)}{12}+\frac{\sin(2q-\varrho-2\Phi)}{4\lambda(2\lambda-1)}$$

$$\left.-\frac{\sin(2q+\varrho-2\Phi)}{4\lambda(2\lambda+1)}\right)$$

$$-\frac{3(\lambda-x)m}{2\lambda}\left(\frac{-\omega\cos\xi}{\lambda}+\frac{\sin\xi}{\lambda\lambda}+\frac{\sin(2q-2r-\xi)}{4(\lambda-1)(\lambda-2)}-\right.$$

$$\frac{\sin(2q-2r+\xi)}{4(\lambda-1)(3\lambda-2)}+\frac{\sin(2r-\xi-2\Phi)}{4(2-\lambda)}-\frac{\sin(2r+\xi-2\Phi)}{4(2+\lambda)}+\frac{\sin(2q-\xi-2\Phi)}{4\lambda\lambda}$$

$$\left.-\frac{\sin(2q+\xi-2\Phi)}{12\lambda\lambda}\right).$$

With these rearranged and the smaller terms discarded, we find[t]

$$\Phi=C-\frac{3\omega}{4\lambda}\left(1-\frac{3}{8\lambda}-\frac{3}{8\lambda\lambda}\right)-\frac{9n\sin\varrho}{4\lambda}+\frac{3m\sin\xi}{2\lambda^3}$$

$$-\frac{3\sin 2(q-r)}{8\lambda(\lambda-1)}\left(1-\frac{3}{8\lambda}-\frac{3}{8\lambda^2}\right)+\frac{3\sin 2(r-\Phi)}{8\lambda}\left(1-\frac{3}{4\lambda}-\frac{3}{8\lambda\lambda}\right)$$

$$+\frac{9}{128\lambda^2}\sin 4(r-\Phi)+\frac{3\sin 2(q-\Phi)}{8\lambda\lambda}\left(1-\frac{3}{8\lambda}-\frac{3}{4\lambda\lambda}\right).$$

26. The part $C-\frac{3\omega}{4\lambda}(1-\frac{3}{8\lambda}-\frac{3}{8\lambda\lambda})$ of the prior expression depends only on the time elapsed at a given epoch. Thus, it gives the mean motion of the nodes. The rest of the terms, which depend on the anomalies of the Sun and Moon, as well as their positions with respect to each other and the line of nodes, will determine the corrections to the mean position of the nodes after the mean position is found. If ω is taken to be $360°$, the mean motion of the nodes in one sidereal year is produced. With $\lambda=13.3685$, the value of $1-\frac{3}{8\lambda}-\frac{3}{8\lambda\lambda}$ will be 0.9698506, making the annual retrograde motion of the nodes equal to 19.5878

[s]The original has $\frac{-\omega\cos\xi}{x}$ as the first term in the long parenthetical expression after $-\frac{3(\lambda-x)m}{2\lambda}$, but this is clearly a typographical error.

[t]In the original, Euler has accidentally omitted the 1 from the parenthetical expression after $-\frac{3\sin 2(q-r)}{8\lambda(\lambda-1)}$.

degrees, which is $19^\circ, 35', 16''$. Astronomical tables for the current time do not show a value above $19^\circ, 20', 32''$, and, thus, the motion defined from theory exceeds that observed by $14', 44''$, or nearly $\frac{1}{79}$ of its value. The difference is certainly not small, and this could lead to doubt about the theory. However, it would be worthwhile to study this discrepancy carefully, since Newton himself, through calculation, arrived at the same mean motion of the nodes as shown by observation. He first treats the lunar orbit as circular, and from this he derives nearly the same excessively large mean motion as we found here. Then, in Proposition XXX, Book III and the proposition that follows, the motion is decreased by the ratio of the major axis to the minor axis, or 70 to 69, and this is nearly in accord with observation. But the Moon is assumed to move in an ellipse with the Earth at the center and not at one of the foci, and this is not the case. Our integration clearly shows that the mean motion is not influenced by the ellipticity of the lunar orbit if the Earth is located at one focus of the ellipse. Nor indeed do the terms omitted in the integration make the mean motion any smaller. Instead, if the quantity $\int Q\, d\Phi$ is determined more accurately, the mean motion of the nodes increases by small, if imperceptible, amounts. The cause of the difference between our calculations and observation can only be the value of dq, which was found from an ellipse and not the natural motion of the Moon. This defect cannot be perfectly corrected in advance, since the motion of the Moon itself in its orbit must be brought into the calculation. Thus, it must suffice to say that the mean motion of the nodes found here must be diminished by $\frac{1}{79}$ of its value in order to be in agreement with the truth. The coefficient $1 - \frac{3}{8\lambda} - \frac{3}{8\lambda\lambda}$, which is 0.9698506, must be decreased by its $\frac{1}{79}$th part, leaving 0.957693, with logarithm 9.9812263.

27. The mean motion of the line of nodes at any given time can be found by the equation $\Phi = C - \frac{3\omega}{4\lambda}(1 - \frac{3}{8\lambda} - \frac{3}{8\lambda\lambda})$, and, thus, a table of the mean motion of the line of nodes can be constructed. The position must be corrected by multiple equations in order to be accurate. The first arises from the term $\frac{-9n\sin\varrho}{4\lambda}$ and depends on the eccentric anomaly of the Sun, which is approximately the arithmetic mean of the mean and true anomalies. But, since the difference between the mean and true anomalies of the Sun is extremely small, we can accurately let ϱ be the mean anomaly of the Sun, calculated from the perigee. If the mean anomaly is computed from the apogee, as is customary, its sine should be taken to be negative. Thus, if ϱ denotes the mean anomaly of the Sun, the angle arising from the expression $\frac{9n\sin\varrho}{4\lambda}$ should be used to correct the mean location of the nodes; this equation is added as the Sun travels from the apogee to the perigee, and subtracted from the perigee to the apogee. This

equation is therefore maximal if the mean anomaly is $90°$ or $270°$, and, then, with $n = 0.01690$ and $\lambda = 13.3685$, it will be $586''$, or $9', 46''$. For all other anomalies it is smaller on account of their sines. In Leadbetter's tables[u] this equation proportional to the sine of the mean anomaly also appears, but the maximum there is just $9', 30''$, smaller than ours by $16''$.

28. The second equation $\frac{3m\sin\xi}{2\lambda^3}$ is proportional to the eccentric or mean anomaly of the Moon. If calculated from the apogee, it should be subtracted as the Moon proceeds from apogee to perigee, but added as the Moon returns from perigee to apogee. The maximum equation arising from this is just $18''$. Since it can thus be omitted from astronomical calculations without perceptible error, it is not mentioned in astronomical tables.

29. The third equation comes from the term $\frac{-3\sin 2(q-r)}{8\lambda(\lambda-1)}(1 - \frac{3}{8\lambda} - \frac{3}{8\lambda\lambda})$, which is proportional to the sine of twice the distance to the Moon from the Sun. That is to say, the position of the Sun is subtracted from the position of the Moon, and then doubling this difference gives the angle with sine proportional to this equation. Therefore, the maximum of this equation occurs when the difference is $45°$, and then it will be $475''$, or $7', 55''$, from which the equation is easy to determine for other angles. From new Moon to first quarter this equation should be subtracted, and then it should be added until opposition. It should again be subtracted from opposition to last quarter, and added from last quarter to conjunction. Or, more succinctly, it should be subtracted as the Moon proceeds from syzygy to quadrature, and added as it moves from quadrature to syzygy. An equation appears in Leadbetter's tables[v] which is said to be proportional to twice the distance to the Moon from the Sun, with a maximum of $1°, 45', 0''$. But this equation appears to be confused with the one that follows, which depends on the distance to the Sun from the node, as we will see shortly. The actual equation is commonly ignored, although it can alter the position of the node by nearly $8'$. But since this equation vanishes at the syzygies, where the position of the nodes can be measured most accurately, and it does not perceptibly affect the position of the Moon at other times, the error caused by its omission is not noticed.

[u] see Leadbetter, Charles. *Uranoscopia, or the Contemplation of the Heavens*. London, 1735. The tables referenced here are on pages 92 and 93. The book is available online at Google Books.

[v] see Leadbetter, Charles. *Compleat System of Astronomy, Vol.2*, 2nd edition. London, 1742. The table is on page 56. The book is available online at Google Books.

30. The fourth equation for the lunar nodes is supplied by the term $\frac{3\sin 2(r-\Phi)}{8\lambda}(1-\frac{3}{4\lambda}-\frac{3}{8\lambda\lambda})$, which can be combined with $\frac{9}{128\lambda^2}\sin 4(r-\Phi)$, since both depend on the distance from the node to the Sun - the first on twice the distance, the second on four times the distance. The first part of this equation should be added as the Sun proceeds from a node to a 90° angle, and subtracted as it moves from a 90° angle to the other node. The maximum occurs when the Sun is at a 45° angle to the line of nodes, and then it is $5449'$, or $1^\circ, 30', 49''$, and, thus, this equation is without doubt the one which Leadbetter associates to the distance from the Sun to the Moon. The other part of this equation, which can be combined with the first part in the same table, should be added as the Sun travels from a node, or quadrature with a node, through an angle of 45°, and subtracted in the remaining cases. The maximum occurs when the Sun is $22^\circ, 30'$ from the line of nodes or a normal to that line, in which case the equation is $1', 21''$.

31. The fifth equation for the lunar nodes is supplied by the term $\frac{3\sin 2(q-\Phi)}{8\lambda\lambda}(1-\frac{3}{8\lambda}-\frac{3}{4\lambda\lambda})$, and thus depends on the distance from the node to the Moon. Since it is proportional to the sine of twice this distance, it should be added to the mean position as the Moon moves away from a node and toward maximum inclination, and subtracted as the Moon moves from quadrature with the nodes to the line of nodes itself. This equation is maximal when the Moon is at a 45° angle with the line of nodes, in which case it is $6', 58''$. Thus, if each of these last three equations are at their maxima, they combine to reach $1^\circ, 45', 42''$, very similar to that in Leadbetter's table under the heading of double the distance from the Sun to the Moon. This error can be tolerated provided that the heading is changed to double the distance from the node to the Sun, since the equation arising from this is the largest. Many other equations could also be included, but, since they only amount to seconds, they may reasonably be omitted; seconds have already been neglected in the differential and integral formulas derived for the sake of greater accuracy. For this reason the correction resulting from the mean anomaly of the Moon can also be safely omitted. The remaining four equations must be retained, since they may alter the position of the nodes by several minutes. Of these four equations we have found only two included in the most recent astronomical tables. In this area astronomical tables need considerable revision.

32. The position of the nodes having been determined, we now investigate the variation of the inclination of the lunar orbit to the ecliptic. We will call the

inclination θ and refer to the final equation of Section 18, which was

$$d.\ln\frac{\tan p\sin\Phi}{\sin(q-\Phi)}=\frac{Fgg(dT)^2\sin q\sin(r-\Phi)}{2Gv\,dq\cos p\sin\Phi}\left(\frac{1}{ff}-\frac{f}{u^3}\right).$$

With the approximation $\frac{1}{ff}-\frac{f}{u^3}=\frac{-3v\cos p\cos s}{f^3}$, this will become

$$d.\ln\frac{\tan p\sin\Phi}{\sin(q-\Phi)}=\frac{-3Fgg(dT)^2\sin q\cos s\sin(r-\Phi)}{2Gf^3\,dq\sin\Phi}.$$

Earlier we showed that $\frac{\tan p}{\sin(q-\Phi)}=\tan\theta$, and thus

$$d.\ln(\tan\theta\sin\Phi)=d.\ln(\tan\theta)+\frac{d\Phi\cos\Phi}{\sin\Phi}$$

$$=\frac{-3Fgg(dT)^2\sin q\cos s\sin(r-\Phi)}{2Gf^3\,dq\sin\Phi}.$$

If we assume that the Sun moves at its mean motion around the Earth at distance a, then the time dT is found from the angle $d\omega$ through $(dT)^2=\frac{2Ga^3(d\omega)^2}{Fgg}$, and thus

$$d.\ln(\tan\theta)=\frac{-d\Phi\cos\Phi}{\sin\Phi}-\frac{3a^3(d\omega)^2\sin q\cos s\sin(r-\Phi)}{f^3\,dq\sin\Phi}.$$

But in Section 20 it was shown that

$$d\Phi=\frac{-3a^3(d\omega)^2\cos s\sin(r-\Phi)\cos(q-\Phi)}{f^3\,dq},$$

and from this we obtain

$$d.\ln(\tan\theta)=\frac{3a^3(d\omega)^2\cos s\sin(r-\Phi)}{f^3\,dq\sin\Phi}(\cos\Phi\sin(q-\Phi)-\sin q)$$

but $\sin q=\sin(q-\Phi)\cos\Phi+\cos(q-\Phi)\sin\Phi$, and when this is substituted we get

$$d.\ln(\tan\theta)=\frac{-3a^3(d\omega)^2\cos s\sin(r-\Phi)\cos(q-\Phi)}{f^3\,dq}.$$

Since $\sin A\cos B=\frac{1}{2}\sin(A+B)-\frac{1}{2}\sin(B-A)$, we have $\sin(r-\Phi)\cos(q-\Phi)=\frac{1}{2}\sin(q+r-2\Phi)-\frac{1}{2}\sin(q-r)$. When multiplied by $\cos s=\cos(q-r)$, this gives $\frac{1}{4}\sin 2(q-\Phi)+\frac{1}{4}\sin 2(r-\Phi)-\frac{1}{4}\sin 2(q-r)$. From this, $d.\ln(\tan\theta)=\frac{-3a^3(d\omega)^2}{4f^3\,dq}(\sin 2(q-\Phi)+\sin 2(r-\Phi)-\sin 2(q-r))$. Let the integral of this formula be R, so that $\ln(\tan\theta) = C + R$ and $\tan\theta = Ce^R$. Since R will be small, $\tan\theta$ is approximately $C(1+R)$.

33. If, as above, we let $\lambda : 1$ be the ratio of the mean lunar motion to the mean solar motion, and we assume $dr = d\omega$ and $dq = \lambda\, d\omega$, ignoring small discrepancies from these values, then

$$d.\ln(\tan\theta) = \frac{-3\,d\omega}{4\lambda}(\sin 2(q-\Phi) + \sin 2(r-\Phi) - \sin 2(q-r)),$$

for which the integral, if Φ is taken to be constant, will be

$$\ln(\tan\theta) = \ln C + \frac{3}{8\lambda}\left(\frac{\cos 2(q-\Phi)}{\lambda} + \cos 2(r-\Phi) - \frac{\cos 2(q-r)}{\lambda-1}\right).$$

The variability of Φ alters this only a little, since the angle θ is so small. But, if we want to account for this, we differentiate the expression assuming Φ to be variable, and we get $\frac{3\,d\Phi}{4\lambda}(\frac{\sin 2(q-\Phi)}{\lambda} + \sin 2(r-\Phi))$. Since

$$d\Phi = \frac{3\,d\omega}{4\lambda}(1 + \cos 2(q-r) + \cos 2(q-\Phi) + \cos 2(r-\Phi)),$$

the differential will expand in this form:

$$\frac{9\,d\omega}{16\lambda\lambda}\left(\frac{\sin 2(q-\Phi)}{\lambda} + \sin 2(r-\Phi) + \frac{\sin 2(2q-r-\Phi)}{2\lambda} + \frac{\sin 2(r-\Phi)}{2\lambda}\right.$$
$$+\frac{\sin 2(q-\Phi)}{2} - \frac{\sin 2(q-2r+\Phi)}{2} + \frac{\sin 4(q-\Phi)}{2\lambda} + \frac{\sin 2(q+r-2\Phi)}{2}$$
$$\left.-\frac{\sin 2(q-r)}{2} + \frac{\sin 2(q+r-2\Phi)}{2\lambda} + \frac{\sin 2(q-r)}{2\lambda} + \frac{\sin 4(r-\Phi)}{2}\right).$$

The integral, which should be subtracted from the value of $\ln(\tan\theta)$, is

$$\frac{-9}{16\lambda\lambda}\left(\frac{\cos 2(q-\Phi)}{4\lambda} + \frac{\cos 2(q-\Phi)}{2\lambda\lambda} + \frac{\cos 2(r-\Phi)}{2} + \frac{\cos 2(r-\Phi)}{4\lambda}\right.$$
$$\left.-\frac{\cos 2(q-r)}{4(\lambda-1)} + \frac{\cos 2(q-r)}{4\lambda(\lambda-1)}\right),$$

omitting the remaining terms, which are extremely small. From this, therefore

$$\ln(\tan\theta) = \ln C + \frac{3}{8\lambda}\cos 2(r-\Phi)\left(1 + \frac{3}{4\lambda} + \frac{3}{8\lambda\lambda}\right)$$

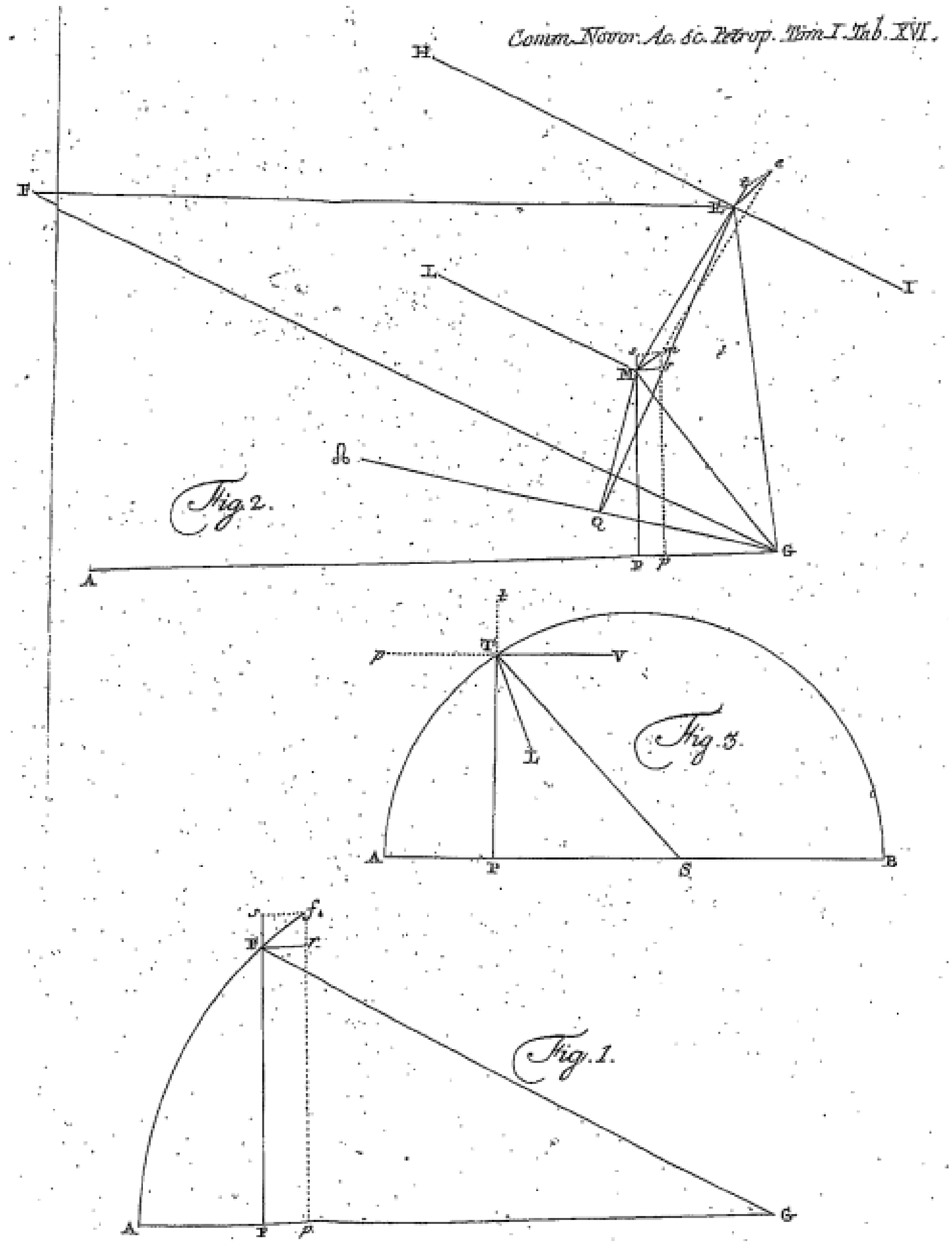
$$+\frac{3}{8\lambda\lambda}\cos 2(q-\Phi)\left(1+\frac{3}{8\lambda}+\frac{3}{4\lambda\lambda}\right)$$
$$-\frac{3}{8\lambda(\lambda-1)}\cos 2(q-r)\left(1+\frac{3}{8\lambda}-\frac{3}{8\lambda\lambda}\right).$$

Assume for sake of brevity that $\ln(\tan\theta) = \ln C + R$. Since R is small, $\tan\theta = C(1+R)$. Let the inclination θ be k when $R=0$, and let $\theta = k+u$ otherwise. Then $C = \tan k$ and $\tan\theta = \tan k + \frac{u}{\cos^2 k} = \tan k + R\tan k$. Thus, $u = R\sin k\cos k = \frac{1}{2}R\sin 2k$. So the correction which should be added or subtracted from the known mean value of the inclination at any given time is u, which will be

$$u = \frac{3}{16\lambda}\left(1+\frac{3}{4\lambda}+\frac{3}{8\lambda\lambda}\right)\sin 2k\cos 2(r-\Phi)$$
$$+\frac{3}{16\lambda\lambda}\left(1+\frac{3}{8\lambda}+\frac{3}{4\lambda\lambda}\right)\sin 2k\cos 2(q-\Phi)$$
$$-\frac{3}{16\lambda(\lambda-1)}\left(1+\frac{3}{8\lambda}-\frac{3}{8\lambda\lambda}\right)\sin 2k\cos 2(q-r)$$
$$= 0.014831\sin 2k\cos 2(r-\Phi) + 0.001082\sin 2k\cos 2(q-\Phi)$$
$$-0.001164\sin 2k\cos 2(q-r).$$

34. If the Sun and Moon are both crossing the line of nodes, all of these angles vanish, so that $u = 0.014749\sin 2k$, and the inclination of the orbit to the ecliptic is at its maximum. But, if the Sun and Moon are at right angles to the line of nodes, so that $q-\Phi = 90°$, $r-\Phi = 90°$, and $q-r = 0°$, the inclination will be at its minimum, with $u = -0.017077\sin 2k$. Thus, the difference between the maximal and minimal inclination is $0.031826\sin 2k$. In most astronomical tables, the minimal inclination of the Moon is stated as $4°, 59', 35''$. If $k - 0.017077\sin 2k = 4°, 59', 35''$, then $k = 5°, 10', 7''$, and $\ln(\sin 2k) = 9.2539340$. Therefore, the maximal inclination, when both the Sun and Moon are crossing the line of nodes, will be $5°, 19', 13''$. Otherwise, three equations will be needed to determine the inclination at any given time, and these will be added or subtracted from the mean inclination $5°, 10', 7''$. The first equation, which is much larger than the other two, depends on the distance from the node to the Sun, and is proportional to the cosine of twice this distance. At its maximum it will be $9', 9''$. The second equation is proportional to the cosine of twice the distance from the node to the Moon, and is $40''$ at maximum. The third equation is proportional to the cosine of twice the distance from the

Moon to the Sun, and is $43''$ at maximum. Thus, in practice these last two equations can be omitted without noticeable error, and the first, depending on the distance from the node to the Sun, can suffice. If the maximum inclination of the lunar orbit is $5°, 17', 20''$, as shown in tables, then k must be smaller. If we set k to $5°, 8', 45''$, so that $\ln \sin 2k = 9.2520250$, then the maximum will be $5°, 17', 48''$, and the minimum will be $4°, 58', 16''$. Although this difference between the maximal and minimal inclination is greater than that shown in tables by nearly two minutes, it should not for that reason be suspect. The other two equations are neglected in tables, and these values are extremely difficult to determine from observation.



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