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# Basel Problem: Historical perspective and further proofs from stochastic processes

Uwe Hassler
*Johann Wolfgang Goethe Universitat Frankfurt am Main*, hassler@wiwi.uni-frankfurt.de

Mehdi Hosseinkouchack
*EBS University*, mehdi.hosseinkouchack@ebs.edu



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# Basel Problem: Historical perspective and further proofs from stochastic processes[1]

Uwe Hassler, Goethe University Frankfurt
Theodor-W.-Adorno-Platz 4, 60323 Frankfurt, Germany
hassler@wiwi.uni-frankfurt.de

Mehdi Hosseinkouchack, EBS University
Gustav-Stresemann-Ring 3, 65189 Wiesbaden, Germany
mehdi.hosseinkouchack@ebs.edu

**Abstract**

In this note, we offer a historical perspective on solutions of the Basel problem. In particular, we have a closer look at some of the less famous results by Euler E41 and provide a review of a selection of the assemblage of earlier proofs. Moreover, we show how to generate further proofs using Karhunen-Loève expansions of stochastic processes.

## 1 Introduction

The probability that a randomly chosen natural number is free of squares (not divisible by some integer square) equals the probability that two randomly chosen natural numbers are prime to each other; and this probability is equal to $\frac{6}{\pi^2}$, see Hardy and Wright [23, Thm. 332, 333]. In fact, a statistician encounters the number $\frac{\pi^2}{6}$ on several occasions. First, it equals the variance of the famous Gumbel (or extreme value) distribution, see Gumbel [22]. Second, it shows up in the limiting variance of the GPH estimator, named after the paper by Geweke and Porter-Hudak [20]; confer also Hurvich et al. [35, Thm. 1]. Third, $\frac{6}{\pi^2}$ amounts to the variance in the limiting distribution of the maximum likelihood estimator under so-called fractionally integrated noise, see Hassler [26, Coro. 8.1]. As is known to a much wider audience beyond statistics, the number $\frac{\pi^2}{6}$ solves the so-called Basel problem in that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{1}$$

The solution is of course due to Leonhard Euler [15] and was found in 1735; for an overview of his related publications available on *Euler Archive* see Huffman [34]. A numerical approximation up to 6 digits was already provided in 1731, see p. 104 in "De summatione innumerabilium progressionum" (E20, "The summation of an innumerable progression") by Euler [14]: $\sum_{k=1}^{\infty} \frac{1}{k^2} \approx 1.644934$.

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According to Eneström [12], it was Pietro Mengoli[2] (1625/26 – 1686) of Bologna in Italy who posed the problem to determine the value of $\sum_{k=1}^{\infty} k^{-2}$, for which we nowadays write $\zeta(2)$ with Riemann zeta function. It was clear that $\zeta(2)$ is finite for the following reason. By partial fractions the series of reciprocals of triangular numbers "telescopes",

$$T_N := \sum_{k=1}^{N} \frac{2}{k(k+1)} = 2\sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 2\left(1 - \frac{1}{N+1}\right) \to 2\,,$$

as $N \to \infty$. Because of $2\,k^2 \geq k^2 + k$ it follows that $\zeta(2) \leq 2$. In fact, Eneström [12, pp. 144 – 145] gave three different proofs for $T_N \to 2$ originally published by Mengoli in 1650. It was Jakob Bernoulli (1654 – 1705) from the city of Basel in Switzerland who popularized such results (*Tractatus de seriebus infinitis*, 1689), apparently without being aware of the previous work by Mengoli. We quote his treatise as Bernoulli [2] published posthumously together with his famous *Ars Conjectandi*. Bernoulli's *tractatus* is organized in 60 propositions. Bernoulli [2, Prop. XVII] failed to determine the value of $\zeta(2)$, and he wrote the request that he would be much obliged if someone found what had escaped his efforts and communicated it to him. Hence, the Basel problem was there and resisted the efforts of mathematicians of this time.

Leonhard Euler (1707 – 1783) was born near Basel and studied with Johann Bernoulli (1667-1748). Johann had taken the position as professor at the University of Basel after his elder brother Jacob had passed away. Details about the Bernoulli dynasty of mathematics may be confusing: If we look at the Bernoulli pedigree in Merian [45, p. 4] over 150 years, we count the name Niclaus five times, the name Johann three times, and Jacob and Daniel twice each, and eight of the Bernoullis were mathematicians of high or highest reputation; here we take the spellings of their first names from Merian [45]. The Bernoulli family held the chair of mathematics at the University of Basel from 1687 until 1790. Johann Bernoulli, with whom Euler studied, had three sons: Niclaus (1695 – 1726), Daniel (1700 – 1782) and Johann II (1710 – 1790). Niclaus and Daniel both held positions as mathematicians at the Academy of Sciences in St. Petersburg. When Niclaus died in 1726, Daniel invited Euler to join the Academy in St. Petersburg, which he accepted as his first academic position taken in 1727. Euler's first solutions to the Basel Problem are found in Euler [15] "De summis serierum reciprocarum" (E41, "On the sums of series of reciprocals"). This paper was published in Volume 7 of the Memoirs of the Imperial Academy of Sciences in St. Petersburg (Commentarii Academiae Scientiarum Imperialis Petropolitanae). The title page of this volume, which contains 10 papers by Euler, carries three dates: 1734/1735 and 1740. The first ones are the years when the papers were presented to the Academy, and the latter date refers to the year of printing.

In this note, we provide a historical perspective on the Basel problem and show how to generate further proofs of (1) using the theory of stochastic processes. In the next section we begin with a closer look at some of the less

[2]Born in 1625 according to Boyer [7, p. 406], while Eneström [12] dates his year of birth to 1626. In 1650 Mengoli published *Novae Quadraturae Arithmetica, sev De Additione Fractionum*, which seems to have been largely forgotten until the rediscovery by Eneström [12].

famous results from E41. Section 3 provides a short review and a classification of the multitude of earlier proofs, without claim to be complete. Section 4 uses the so-called Karhunen-Loève expansion from the theory of stochastic processes to produce further proofs. We give three illustrative examples: the Wiener process or standard Brownian motion, the demeaned Brownian motion and the Brownian bridge. The final section provides some concluding remarks.

## 2 Euler's proofs

There are three publications by Euler containing different proofs of (1). Euler's first paper (E41 – we largely profited from the English translation by Jordan Bell available at *Euler Archive*) builds on the Taylor expansion of the sine function, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. In fact, the paper contains two different routes of proof. The second route relies on a factorization of

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \pm \cdots ,$$

see also Euler [16] "De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera, in qua eaedem summationes ex fonte maxime diverso derivantur" (E61, "On sums of series of reciprocals from powers of natural numbers from another discussion, in which the sums are derived principally from another source") for a more rigorous argument. More easily accessible is this proof e. g. in Dunham [10, Ch. 3].

Here, we turn to the route treated first in Euler [15, §5]. Consider some fixed $r$ $(0 < r < \pi)$ with $y = \sin r$ and determine the roots $\lambda_{n,r}$ of the infinite polynomial $1 - \frac{\sin s}{y}$ in order to factorize accordingly:

$$1 - \frac{s}{y} + \frac{s^3}{y \cdot 3!} - \frac{s^5}{y \cdot 5!} \pm \cdots \quad = \quad \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_{n,r}}\right).$$

The roots depend on $y = \sin r$, we omit details. For brevity, we use the Riemann zeta function $\zeta(p) = \sum_{k=1}^{\infty} k^{-p}$ for natural numbers $p > 1$, and similarly $\xi(p) = \sum_{k=1}^{\infty} (2k-1)^{-p}$ with

$$\zeta(p) = \xi(p) + \frac{1}{2^p}\zeta(p). \tag{2}$$

By comparison of coefficients and employing Newton's identities, Euler [15] established

$$\frac{1}{y^2} = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n,r}^2}, \tag{3}$$

which he evaluated for three cases. First, he considered $y = 1$ with $r = \pi/2$, see E41(§11); a close look at the roots $\lambda_{n,r}$ yields

$$\xi(2) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}, \tag{4}$$

which solves the Basel problem by (2). Second, picking $y = \sqrt{2}/2$ with $r = \pi/4$ it follows again in E41(§14) that $\xi(2) = \pi^2/8$. Third, with the choice of $y = \sqrt{3}/2$ and $r = \pi/3$, (3) implies

$$\frac{4\pi^2}{3^3} = \sum_{k=1}^{\infty} \left( \frac{1}{(3k-2)^2} + \frac{1}{(3k-1)^2} \right).$$

Similarly to (2), Euler [15, §15] observed that $\zeta(2) = \frac{4\pi^2}{3^3} + \frac{1}{3^2}\zeta(2)$, which amounts to (1).

On top of solving the Basel problem, Euler's first paper answers questions with respect to alternating series that had not been asked before. More generally than (3), Euler obtained

$$\Sigma_{m,r} = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n,r}^m}, \tag{5}$$

for appropriate values of $\Sigma_{m,r}$, $m = 1, 2, 3, \ldots$. Evaluating (5) for different values of $m$ and $r$, he found for instance

$$\begin{aligned} \frac{\pi^3}{32} &= 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} \pm \cdots, \\ \frac{5\pi^5}{1536} &= 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} \pm \cdots, \end{aligned}$$

that generalize the well known Leibniz series. Further alternating schemes found in E41 are

$$\begin{aligned} \frac{\pi^5}{2\sqrt{2}} &= 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \pm \cdots, \\ \frac{2\pi}{3\sqrt{3}} &= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} \pm \cdots. \end{aligned}$$

Euler [17] "Démonstration de la somme the cette suite $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} +$etc." (E63, "Demonstration of the sum of the series $1 + 1/4 + 1/9 + 1/16 + \ldots$") added a different proof of (1) in a second paper building on an expansion of the arcsine function. It had been published in a journal called *Journal littéraire d'Allemagne, de Suisse et du Nord* and received little or no attention until the discovery by Stäckel [52] who called this paper a "forgotten treatise by Leonhard Euler" and provided a historical perspective including a reprint. For "Euler's other proof" in a nutshell we refer to Kimble [37] or footnote 63 in Knopp [38, p. 376]; see also Choe [8], where the last author does not seem to be aware of reproducing Euler's proof. Dunham [10, p. 55] provided a detailed derivation along the lines of E63. Third, in the textbook "Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum, volume 1" ("E212, Foundations of differential calculus, with applications to finite analysis and series, volume 1") Euler [18, Ch. 15] presented a novel proof of (1) via l'Hôpital's rule; see also Dunham [11].

## 3 Further proofs of (1)

An early proof by Niclaus Bernoulli (1687 – 1759, a cousin of the Niclaus mentioned in our introductory section) is nowadays rarely associated with him.

As with Euler's paper E41, it was published in the *Commentarii*, Volume 10. This time the title page carries two dates, 1738 and 1747, where 1747 is the year of printing. The paper "Inquisitio in summam seriei $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+$etc" ("Inquiry into the sum of the series $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+$etc") is fully cited in [3]. Denote the sum over reciprocals of odd squares as $Z_N$, and consider the alternating Leibniz series $q_N$:

$$Z_N = \sum_{k=1}^{N} \frac{1}{(2k-1)^2} \quad \text{and} \quad q_N = \sum_{k=1}^{N} \frac{(-1)^{k-1}}{(2k-1)}\,.$$

Bernoulli's idea was to square $q_N$ and define $y_N = Z_N - q_N^2$. Then Bernoulli [3] argued that $q_N \to \pi/4$ and $y_N \to \pi^2/16$ as $N \to \infty$, which amounts to $\xi(2) = \pi^2/8$ and completes the argument again by (2). Similarly, with more rigorous arguments but without referring to Bernoulli [3], Estermann [13, p. 12] established $Z_N = 2\,q_N^2 + u_N$, $u_N \to 0$, which proves (4) again; see also Knopp [38, pp. 322 – 324].

Ever since Euler cracked the Basel problem, many other proofs appeared. Some of them also culminate in establishing (4), e.g. in chronological order: Giesy [21], Hofbauer [30], Harper [24], Ivan [36], Marshall [43], Hirschhorn [29], Muzaffar [46] and Ritelli [50]. More proofs that do not rely on (4) in order to show $\zeta\,(2) = \frac{\pi^2}{6}$ were given by Knopp and Schur [39] in German (also found in English in Knopp [38, pp. 266 – 267]), by Yaglom and Yaglom [55], Matsuoka [44], Stark [53], and Papadimitriou [48]. A note following Papadimitriou [48] says, that this paper was translated from a Greek manuscript, and that the proof coincides with the one given in Norwegian by Holme [31]; we checked that it is actually identical with the one by Yaglom and Yaglom [55].[3] Further, more or less elementary proofs have been published by Apostol [1], Beukers et al. [4], Kortram [40], Borwein et al. [5], Passare [49], Daners [9] (similar to Matsuoka [44], but more straightforward), Xu and Zhou [54] and Lord [42].

The above list consists of proofs that are more or less elementary and does not include proofs that rely on Fourier analysis. Most students will encounter (1) as an example of Parseval's identity: with the Fourier expansion of $x$ with sine coefficients $b_k = (-1)^{k+1}2/k$ and $a_k = 0$ it holds that

$$4\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x\,,$$

which gives (1).

There are recent proofs that stand out against the ones above in that they use results from probability theory: Hesse and Meister [28] (in German), Bourgade et al. [6] and Holst [32], see also Pace [47]. In the next section we give additional proofs for $\zeta(2) = \frac{\pi^2}{6}$ that are rooted in probability theory, too, relying on results from the theory of stochastic processes.

[3] For simplicity, we quote Yaglom and Yaglom [55] with an English translation of the title although the paper is in Russian.

## 4 Karhunen-Loève expansions and more proofs

We begin with the example of a so-called standard Brownian motion or Wiener process denoted by $W(t)$, $t \in [0,1]$. It is defined by three properties: non-overlapping increments that are independent and Gaussian, and a starting value of zero (with probability one). For existence of $W(t)$ see e. g. Shorack and Wellner [51, Sect. 2.2], who more formally assume:

- If $t_0 < t_1 < \ldots < t_{n+1}$, then $W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_{n+1}) - W(t_n)$ are independent for all $n \geq 1$, and
- $W(t_{k+1}) - W(t_k)$ follows a normal distribution with expectation 0 and variance $t_{k+1} - t_k$ for all $k = 0, \ldots, n$: $\mathsf{E}[W(t_{k+1}) - W(t_k)] = 0$, $\mathsf{Var}[W(t_{k+1}) - W(t_k)] = t_{k+1} - t_k$;
- further, $W(0) = 0$.

From these defining properties it follows that $\mathsf{E}[W(t)] = 0$ and for the covariance kernel $k(s,t) := \mathsf{E}[W(s)W(t)]$ that $k(s,t) = \min(s,t)$. Note that $k(s,t)$ is symmetric, continuous, and positive definite in that

$$\int_0^1 \int_0^1 g(s)g(t)k(s,t)\mathsf{d}s\mathsf{d}t > 0\,,$$

where $g(\cdot)$ is square-integrable ($L^2$) and not identical to 0.

Let us turn now to more general stochastic processes $X(t)$, $t \in [0,1]$. We assume that $\mathsf{E}[X(t)] = 0$ and that the covariance kernel $k(s,t) := \mathsf{E}[X(s)X(t)]$ is symmetric, continuous, and positive definite. Under Gaussianity (and some continuity assumption) $X(t)$ is endowed with a so-called Karhunen-Loève (KL) expansion (see e. g. Loève [41, Sect. 37.5]),

$$X(t) = \sum_{j=1}^{\infty} \frac{f_j(t)}{\sqrt{\lambda_j}}\, Z_j\,. \tag{6}$$

In (6), $\{Z_j\}$ is a sequence of independent standard normal random variables, and $\lambda_j > 0$ and $f_j(t)$ are the eigenvalues and eigenfunctions of $k(s,t)$; the latter are the solutions ($j \in \mathbb{N}$) of the so-called Fredholm integral equation:

$$f(t) = \lambda \int_0^1 k(s,t)f(s)\mathsf{d}s.$$

It is worth noting that $f_j(t)$, $j \in \mathbb{N}$, form an orthonormal base for $L^2$. From (6) we have with $\mathsf{E}[Z_i Z_j] = \delta_{ij}$ that

$$k(t,t) = \sum_{j=1}^{\infty} \frac{f_j^2(t)}{\lambda_j}\,, \tag{7}$$

see also Mercer's Theorem, e. g. Shorack and Wellner [51, Sect. 5.2].

Consider three examples of how (7) can be related to (1). First, let us return to the Wiener process $W(t)$ where $k(s,t) = \min(s,t)$ has the well-known eigenstructure $\lambda_j = (j-1/2)^2\pi^2$ and $f_j(t) = \sqrt{2}\sin((j-1/2)\pi t)$. By (7),

$$t = 8\sum_{j=1}^{\infty} \frac{\sin^2\left[(j-1/2)\,\pi t\right]}{(2j-1)^2\,\pi^2}\,.$$

Evaluation for $t=1$ amounts to (4), which gives $\zeta\,(2) = \frac{\pi^2}{6}$ by (2).

Second, consider the so-called Brownian bridge defined as $W(t) - tW(1)$, $t \in [0,1]$; here, $k(s,t) = \min(s,t) - st$, see e. g. Hassler [25, Prob. 7.6]. It follows that $\lambda_j = j^2\pi^2$ and $f_j\,(t) = \sqrt{2}\sin(j\pi t)$, see e. g. Shorack and Wellner [51, pp. 213 – 214]. For $t = 1/2$, (7) provides

$$\frac{1}{4} = \frac{2}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\,,$$

which establishes again (4) and hence $\zeta\,(2) = \frac{\pi^2}{6}$. After circulating the working paper Hassler and Hosseinkouchack [27], we found out that the proof via the Brownian bridge has been proposed independently in Foster and Habermann [19, p. 12].

Third, consider the so-called demeaned Wiener process, $W(t) - \int_0^1 W(s)\mathrm{d}s$. Here, the kernel turns out to be $k(s,t) = \min(s,t) - (s+t) + \frac{1}{2}(s^2+t^2) + \frac{1}{3}$, and it follows that $\lambda_j = j^2\pi^2$ and $f_j(t) = \sqrt{2}\cos(j\pi t)$, see Hosseinkouchack and Hassler [33, Remark 1] who treat the more general case of a so-called demeaned Ornstein-Uhlenbeck process. For $t=1$, we immediately obtain the required result:

$$\frac{1}{3} = \frac{2}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}.$$

## 5 Concluding remarks

First, we looked into the less well-known proof in E41 and, second, classified a multitude of historical proofs of the Basel problem. Third, we demonstrated that more solutions can be produced along the route of stochastic processes as in the previous section; we discussed the cases of a (demeaned) Brownian motion and a Brownian bridge. Similarly, one may work with the Karhunen-Loève expansion of the so-called detrended Brownian motion as the orthogonal component of the projection of a Brownian motion on a constant and a linear time trend; see Hassler and Hosseinkouchack [27] for details.

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