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Alexander Aycock
Johannes Gutenberg Universitat, Mainz, aaycock@students.uni-mainz.de

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Answer to a question concerning Euler's paper "Variae considerationes circa series hypergeometricas"

Alexander Aycock, Johannes-Gutenberg University Mainz
Staudinger Weg 9, 55128 Mainz
aaycock@students.uni-mainz.de

Abstract

We solve a problem concerning Euler's paper "Variae considerationes circa series hypergeometricas" ([*E*661]), as suggested by G. Faber in the preface to Volume 16,2 of the first series of Euler's Opera Omnia. Our solution employs methods introduced by Euler at other places.

1 Introduction

This paper is about Euler's work [*E*661]. In that paper, Euler introduced the following function,[1] presented here in Euler's notation[2]:

$$\Gamma_E(x) = a \cdot (a+b) \cdot (a+2b) \cdot (a+3b) \cdots (a+(x-1)b) \quad \text{with} \quad \mathrm{Re(a)}, \mathrm{Re}(b) > 0. \tag{1}$$

Using the Euler-Maclaurin summation formula[3], Euler arrived at the following expression – in his notation – for the function given in Eq. (1)

$$\Gamma_E(x) = A \cdot e^{-x}(a-(b-1)x)^{\frac{a}{b}+x-\frac{1}{2}}. \tag{2}$$

Here, Euler understood x as an infinitely large positive number[4] such that the previous equation has to be understood as an asymptotic equation from a modern

[1]In the same paper, Euler also introduced two other related functions. But since they will not be needed for our purposes, we will not discuss them in the following.

[2]To be completely precise, Euler wrote:

$$\Gamma : i = a \cdot (a+b) \cdot (a+2b) \cdot (a+3b) \cdots (a+(i-1)b)$$

[3]Euler obtained his version of the Euler-Maclaurin summation formula in his treatise [*E*47].

[4]For this reason, Euler might have used the letter i (for infinitus) in his original paper.

perspective. The constant A cannot be defined by means of the Euler-Maclaurin summation formula and Euler did not determine it in any other way[5]. This induced G. Faber (in the introduction to volume 16,2 of the first series of Euler's *Opera Omnia*) to the following statement:

"It might perhaps be worthwhile to take up again Euler's efforts to evaluate this number A".[6]

In the following we shall present a solution to this task. In doing so, we intend to use only methods and ideas that were explicitly explained by Euler at some place.

2 Solution of the Task

2.1 Finding a relation among the functions Γ_E and Γ

In the following, we will always denote Euler's function 1 by the symbol $\Gamma_E(x)$ to distinguish it from the function that is indicated by Γ nowadays (see equation (3))[7]. The main step towards the solution of the task at hand consists in relating the Eulerian function Γ_E (equation (1)) to the familiar Γ-function (equation (3)). The Γ-function is nowadays usually defined by the expression

$$\Gamma(x) := \int\limits_0^\infty t^{x-1}e^{-t}dt, \quad \text{for} \quad \operatorname{Re}(x) > 0. \tag{3}$$

Clearly, for integer values of x, $\Gamma(x)$ is a special case of $\Gamma_E(x)$ for $a = b = 1$. For the Γ-function an asymptotic expansion is provided by the Stirling formula. The final relation we are after is then found in Theorem 2.2.

The first step, therefore, will be to derive an integral representation for the function $\Gamma_E(x)$, using a method introduced by Euler in his papers [*E*123] and [*E*594], papers actually devoted to the theory of continued fractions. In [*E*594, §13, Ex.3], Euler also arrived at the modern expression for the Γ-function. However, in most of his investigations Euler preferred the expression

$$\int\limits_0^1 \log\left(\frac{1}{t}\right)^x dt,$$

[5]The author of this paper was not able to locate a source in which this constant is actually evaluated.

[6]"Es würde sich vielleicht lohnen, die Euler'schen Bemühungen um die Ermittelung dieser Zahl A wieder aufzunehmen." [*Fa*35, p.xliv]

[7]The notation Γ for the integral in equation (3) was introduced by Legendre in his paper [*Le*09] and popularised by his book [*Le*26]. But the author was not able to find a reason, why Legendre used this particular letter. Legendre did not cite Euler's paper [*E*661], but whether he was aware of Euler's work or not and thus inspired by it, is a question the author of this note is unable to answer.

which he discovered in [E19, §14] and used for the interpolation of the factorial. By the substitution $\log(\frac{1}{t}) = u$, Euler's preferred expression reduces to the integral expressing $\Gamma(x+1)$.

We first intend to prove the following:

Theorem 2.1. *The Eulerian function Γ_E has the following integral representation:*

$$\Gamma_E(x) = \frac{b^x}{\Gamma\left(\frac{a}{b}\right)} \int\limits_0^\infty y^{x-1+\frac{a}{b}} e^{-y} dy \quad \textit{for} \quad \operatorname{Re}\left(x - 1 + \frac{a}{b}\right) > 0.$$

Proof. From Euler's representation of $\Gamma_E(x)$ (equation (1)) we deduce the following functional equation

$$\Gamma_E(x+1) = (a+bx)\Gamma_E(x), \tag{4}$$

together with the initial condition $\Gamma_E(1) = a$. This functional equation interpolates the Eulerian expression for $\Gamma_E(x)$, which is only valid for natural numbers x. Furthermore, also based on the representation (1), we will assume that $\Gamma_E(x)$ is logarithmically convex for positive real numbers x satisfying $\operatorname{Re}\left(x - 1 + \frac{a}{b}\right) > 0$. Then we can apply a method of solving homogeneous difference equations with linear coefficients introduced by Euler in his papers [E123] and [E594].

According to this method Eq. (4) is solved by an expression of the form

$$\int\limits_c^d y^{x-1} P(y) dy, \tag{5}$$

where the method also teaches how to find the boundaries of integration c and d and the function $P(t)$, which is assumed to be differentiable.

In order to demonstrate the application of that method to the example at hand, let us consider the following auxiliary equation:

$$\int^y y^x P(y) dy = (a+bx) \int^y y^{x-1} P(y) dy + y^x Q(y), \tag{6}$$

where $Q(y)$ is another differentiable function not known at this moment. By $\int^y f(y)dy$ we denote the integral function of f, i.e., if $F(y)$ is an anti-derivative of $f(y)$, then $\int^y f(y)dy := \int\limits_{y_0}^{y} f(t)dt = F(y) - F(y_0)$, where y_0 is to be chosen in such a way that $F(y_0) = 0$.

Differentiating and dividing equation (6) through by y^{x-1}, we arrive at the following equation:

$$yP(y) = (a+bx)P(y) + xQ(y) + yQ'(y), \tag{7}$$

where, without loss of generality, $y \neq 0$ was assumed. Comparing coefficients of powers of x on each of side of the equation, will lead us to the following system of differential equations for the functions $P(y)$ and $Q(y)$:

$$yP(y) = aP(y) + yQ'(y), \tag{8}$$
$$0 = bP(y) + \; Q(y). \tag{9}$$

The solutions of this system are:

$$P(y) = -\frac{C}{b}e^{-\frac{y}{b}}y^{\frac{a}{b}} \quad \text{and} \quad Q(y) = Ce^{-\frac{y}{b}}y^{\frac{a}{b}},$$

where $C \neq 0$ is a constant of integration.

Going back to the auxiliary equations (6) and (7), we conclude that an ansatz of the form (5) will solve the difference equation (4) if we fix the the boundaries of integration, i.e., c and d, from solutions to the equation:

$$d^x Q(d) - c^x Q(c) = 0. \tag{10}$$

Under the conditions $\operatorname{Re} b > 0$ and $\operatorname{Re}\left(x + \frac{a}{b}\right) > 0$, aside from the trivial solution $c = d$, we find the pair[8]

$$(c, d) = (0, \infty).$$

Inserting everything we discovered into the ansatz (5), we find:

$$\Gamma_E(x) = -\frac{C}{b}\int\limits_0^\infty y^{x-1}e^{-\frac{y}{b}}y^{\frac{a}{b}}dy.$$

It remains to evaluate the constant C. For this, we will need the initial condition $\Gamma_E(1) = a$. Via the last equation and the initial condition we find:

$$(\Gamma_E(1) =)\, a = -\frac{C}{b}\int\limits_0^\infty y^{\frac{a}{b}}e^{-\frac{y}{b}}dy.$$

The integral on the right-hand side can be expressed using the Γ-function (3) by a simple rescaling of the integration variable such that we eventually arrive at the equation:

$$C = -\frac{ab^{-\frac{a}{b}}}{\Gamma\left(\frac{a}{b}+1\right)}.$$

[8]It would be interesting to investigate, whether there are even more solutions to equation (10), but since we only need one non-trivial solution, we will not consider this question here.

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Because of the relation $\Gamma(x+1) = x\Gamma(x)$ this simplifies even further to the expression:

$$C = -\frac{b^{-\frac{a}{b}+1}}{\Gamma\left(\frac{a}{b}\right)}.$$

If we insert this value of the constant of integration C into the ansatz (5), we just have to perform another substitution $\frac{y}{b} \mapsto y$ in the integral. After a quick calculation, eventually we arrive at Theorem 2.1 which we wanted to prove. □

2.1.1 Relation among Γ and Γ_E

Via 2.1 the function $\Gamma_E(x)$ can be expressed in terms of the function $\Gamma(x)$. More precisely, the integral in 2.1 can be expressed by using the definition of the Γ-function (3) such that we have the following

Theorem 2.2. *The following equation holds:*

$$\Gamma_E(x) = \frac{b^x}{\Gamma\left(\frac{a}{b}\right)} \cdot \Gamma\left(x + \frac{a}{b}\right).$$

This relation will be useful for the determination of the constant A, which we will do next.

2.2 Determination of the constant A

To determine the constant A we will need the Stirling formula, i.e., in modern terms the following asymptotic:

$$\Gamma(x+1) \sim \sqrt{2\pi x} \cdot x^x \cdot e^{-x} \quad \text{for} \quad x \to \infty.$$

Writing $x-1$ instead of x, it reads:

$$\Gamma(x) \sim \sqrt{2\pi}(x-1)^{\frac{1}{2}} \cdot (x-1)^{x-1} e^{-(x-1)} \quad \text{for} \quad x \to \infty.$$

Let us apply the Stirling formula in this form to our case, i.e., the formula of Theorem 2.2. A direct application gives:

$$\Gamma_E(x) \sim \frac{b^x}{\Gamma\left(\frac{a}{b}\right)} \cdot \sqrt{2\pi}\left(x - 1 + \frac{a}{b}\right)^{\frac{1}{2}} \left(x + \frac{a}{b} - 1\right)^{x+\frac{a}{b}-1} e^{-\left(x+\frac{a}{b}-1\right)}. \tag{11}$$

If we massage the expression on the right-hand side into a form resembling Euler's expression (2), we end up with the asymptotic:

$$\Gamma_E(x) \sim \frac{\sqrt{2\pi}}{\Gamma\left(\frac{a}{b}\right)} \cdot b^{-\frac{1}{2}-\frac{a}{b}+1} (a + bx - b)^{-\frac{1}{2}+x+\frac{a}{b}} e^{-x} \cdot e^{-\frac{a}{b}+1},$$

and comparison to the Eulerian expression reveals the constant A to be:

$$A = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{a}{b}\right)} \cdot e^{-\frac{a}{b}+1} \cdot b^{\frac{1}{2}-\frac{a}{b}}.$$

Thus, the task given by G. Faber is solved.

2.3 Final Remark

Since we only used methods and results already discovered by Euler without any general changes – we only used modern mathematical language – we achieved our goal to show that Euler himself could have arrived at the solution to Faber's task and he probably would not have been surprised by the result.

A modern solution of Faber's task would quite possibly use the saddle point method, once Theorem 2.1 is established. This way, one would not have to rewrite the integral in terms of the Γ-function and apply the Stirling formula, but would arrive at the final asymptotic expansion (11) immediately. But the saddle point method was invented after Euler. The origin of the method is due to Laplace [*La*74]. Furthermore, we wish to add that Euler's assumption that an equation such as (4) is given as an integral (5) is correct, can be explained by the theory of the Mellin-transform, which was only introduced by Mellin in his paper [*Me*96] in 1895, over 100 years after Euler's death in 1783.

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