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Learning Mathematics from the Master: A Collection of Euler-based Primary Source Projects for Today's Students, Part I

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Abstract

This article and its sequel will together highlight a set of nine classroom-ready projects that draw on the remarkable writing of Leonhard Euler (1707–1783) as a means to help students develop an understanding of standard topics from today's undergraduate mathematics curriculum. Part of a larger collection of primary source projects intended for use in a wide range of undergraduate mathematics courses, these projects are freely available to students and their instructors. We provide a general description of the pedagogical design underlying these projects, more detailed descriptions of the individual projects themselves, and instructions for obtaining downloadable copies for classroom use.

Keywords: Euler, active learning, using history to teach mathematics

1 Introduction

In their 2013 *Convergence* article "Teaching and Research with Original Sources from the Euler Archive" [15], Dominic Klyve, Lee Stemkoski, and Erik Tou wrote:

> If one decides to study original sources in mathematics, one can hardly do better than to read the words of Leonhard Euler, one of the greatest didactic writers in the history of mathematics. More than perhaps any other mathematician, Euler wrote to be understood. His works brim with examples, computations, and even dead-ends in his thinking process—the kind of digression academics are usually taught to avoid in their publications.

In this article and its sequel, we highlight a set of primary source projects that draw on Euler's remarkable writing as a means to help students develop an understanding of standard topics from today's undergraduate mathematics curriculum. We begin with a brief general description of the pedagogical design underlying these projects, which are part of a larger collection of freely available student-ready materials based on primary sources for use in a wide range of undergraduate mathematics courses.

The key design feature of a primary source project is a ''guided reading'' of select excerpts from a primary source. These excerpts are surrounded by a series of student tasks that prompt students to actively interpret the mathematics being developed by the source author. The tasks deliberately interrupt the reading of the source material to offer students various opportunities to engage in activities that model how mathematicians work: asking questions, grappling with uncertainties, making conjectures and testing them, verifying results, and proving theorems. Secondary commentary supplied by the project author ties the whole project together for students by discussing the historical context and mathematical significance of the source material, providing guidance in its interpretation, and connecting its mathematical content to contemporary terminology, notation and standards.[1]

Importantly, primary source projects are intended to replace the standard textbook treatment of core topics in the undergraduate mathematics curriculum. The guided reading approach is also especially well suited to small-group discussions and other student-centered instructional strategies as an alternative to lecture. Each project is limited in scope, usually focusing on one particular topic (e.g., the derivative of the sine function) or a few related concepts that are unified within one particular source (e.g., "Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus" [5]). Classroom implementation of a project may last anywhere from one to two class periods (for shorter "mini" projects) to several weeks (for the longest of the "full-length" projects). The "Notes to Instructors" section provided with each project offers a sample schedule for its full implementation, as well as options for modifying the project to fit within other timeframes and further information about its goals and design. LaTeX code of individual projects is also available from its author upon request to facilitate modifications that instructors may wish to make to better suit their goals for a particular course.

In our concluding section, we provide additional information about the development of this general pedagogical approach, along with details on where to obtain a downloadable copy of a project for classroom use. We first present more detailed descriptions of four particular Euler-based projects designed for use in introductory courses on calculus and differential equations.

[1]The genesis of this particular guided reading approach to learning via primary sources is described in [2]. It and other approaches to using primary sources in mathematics teaching, and the motivation behind each, are compared in [12] and [13].

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2 The Derivatives of the Sine and Cosine Function: A Project for Calculus I

The two-day project "The Derivatives of the Sine and Cosine Function" [14] is based on excerpts from *Institutiones Calculi Differentialis* (E212) [9]. The second part of Euler's trilogy on analysis, the *Calculi Differentialis* is divided into two books (of 9 chapters and 18 chapters respectively) that unify a diversity of topics within a single well-organized treatise. An extremely influential work, it was also the first calculus book to use functions; indeed, seven years earlier Euler himself had been the first mathematician to regularly use an approach that looks like functions to us today, in his great "pre-calculus" book, *Introductio in analysin infinitorum* (E101) [8]. While the use of functions makes the material more accessible to our students today, Euler's approach is different enough from that of modern calculus books to force students to think carefully about the material. The major difference in Euler's approach is the lack of limits in his work. The limit concept would not be formally defined and made a part of mathematics for almost a century; instead, Euler based his calculus (following Leibniz) on the differential dx. Perhaps the most surprising aspect of Euler's approach, however, is his use of Taylor series as the starting point for the derivation of derivatives. In fact, he introduced these in the *Introductio* and thought of them as a pre-calculus idea.

This project has two primary goals: to help students develop a deeper and more intuitive understanding of the limit definition of the derivative, and to understand why the derivative of sine is cosine, using a proof which may seem more straightforward than the version which usually appears in modern calculus books. A secondary goal is to calculate the derivatives of the sine and cosine function using only trigonometric identities and algebra, a process that possibly renders some of calculus less mysterious. Students begin by reading Euler's definition of function [9, p. vi]:

> Those quantities that depend on others in this way, namely, those that undergo a change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity can be determined by others. Hence, if x designates the variable quantity, all other quantities that in any way depend on x or are determined by it are called its functions. Examples are x^2, the square of x, or any other powers of x, and indeed, even quantities that are composed with these powers in any way, even transcendentals, in general, whatever depends on x in such a way that when x increases or decreases, the function changes.

A series of tasks then guides them though Euler's discussion of the example of x^2 to arrive at his definition of differential calculus [9, p. viii]:

> It is a method for determining the ratio of the vanishing increments that any functions take on when the variable, of which they are functions, is given a vanishing increment.

After prompting students to compare this to the limit definition of a derivative, the project turns to Paragraph 201, in which Euler tackled the derivative of the sine function via Taylor series. Although this is likely the first time they have seen such series, students need very little prerequisite knowledge to complete this project. The approximations of sine and cosine via a three-term Taylor series is presented as a *fait accompli*, and students are given an opportunity to convince themselves that the approximations seem valid, even if they can't explain why. It will help if they have seen the trigonometric identities used in the project, but the identities are re-introduced here in case they have not. It would also be useful for students to have been introduced to the limit definition of the derivative, as this project would then provide them with a second lens through which to view the concept, though this is not strictly necessary either. No other background outside of algebra is required.

3 Euler's Calculation of the Sum of the Reciprocals of Squares: A Project for Calculus II

The two-day project "Euler's Calculation of the Sum of the Reciprocals of Squares" [17] is based on short excerpts from E41: *De summis serierum reciprocarum* [6]. In this celebrated paper, Euler launched his scientific career by solving what was arguably the most famous unsolved mathematics problem of his day. More specifically, Euler resolved the so-called Basel problem by showing that the sum of the reciprocals of the squares of the positive integers equals $\frac{\pi^2}{6}$. His technique involved writing the function $\sin(x)/x$ as both an infinite series (using the known power series for sine) and an infinite product (using knowledge of the roots of sine). Multiplying out the infinite product and equating the coefficients of the two series representations' second-degree terms was the final step that resolved the Basel problem. Euler went on to provide a method to extend this technique to any even power and used that method to find the sum of reciprocal even powers up to exponent 12. With slightly different techniques, he later published the values of the sums of reciprocals of even powers up to exponent 26 [Euler, 1748, p. 131]. Or, to put this in more modern terms, Euler succeeded in computing the values of the zeta function $\zeta(s)$ for even values of s up to 26. To this day, no one has yet computed the value of s for any odd value other than $s = 1$; the divergence of the harmonic series associated with $s = 1$ was of course well-known well before Euler.

The primary source project based on E41 is designed to enrich students' understanding of infinite series and power series manipulations. In a typical calculus course, an emphasis is placed on learning to classify infinite series as convergent or divergent via a lengthy list of convergence tests, while the only means demonstrated for actually evaluating an infinite series is to take a known power series (perhaps slightly rearranged) and plug in an input from the interval of convergence. This project aims to broaden a student's perspective on the ways infinite series can be evaluated and the ways power series can be used, as well as introduce the idea of an infinite product.

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Students actually begin the project by reading a non-Euler source: Nicole Oresme's (1323–1382) proof that the harmonic series does not sum to any finite value [18]:

> We shall see this as follows: if there exist infinitely many parts, each of which is longer than half a foot, then the total will be infinite. As is well known, $1/4$ and $1/3$ are more than a half; similarly, [the sum of] $1/5$ through $1/8$, and $1/9$ through $1/16$, etc. to infinity.

After examining Oresme's argument in a more modern form, the first two sections of Euler's paper are then presented [6, pp. 123–124]:

> §.1. So much work has been done on the series of the reciprocals of powers of the natural numbers, that it seems hardly likely to be able to discover anything new about them. ...
>
> §.2. I have recently found, quite unexpectedly, an elegant expression for the sum of this series
>
> $$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + etc.;$$
>
> which depends on the quadrature of the circle.

The remainder of the project then guides students through Euler's clever use of the power series for sine to establish that the sum of this series is indeed given by an elegant expression; namely, $\frac{\pi^2}{6}$.

Although issues related to series convergence were viewed differently in the 18th century, today's standard series convergence tests are used heavily throughout the project. Students are thus assumed to have some familiarity (but not necessarily complete mastery) of those tests, as well as the power series for sine. No other prerequisite knowledge is required beyond some basics from precalculus such as finding zeros of functions and factoring polynomials. While this project is intended for an introductory calculus course, it also makes an ideal starting point for a discussion of the Riemann zeta function or the Weierstrass Factorization Theorem in a complex analysis course, or a discussion of generating functions in a combinatorics course.

4 Leonhard Euler and Johann Bernoulli on Solving Higher Order Linear Differential Equations with Constant Coefficients: A Project for Differential Equations

The two-day project "Leonhard Euler and Johann Bernoulli on Solving Higher Order Linear Differential Equations with Constant Coefficients" [22] begins with an excerpt from a letter written on September 15, 1739 to Johann Bernoulli (1667–1748) in which Euler wrote [4, p. 37]:

> I have recently found a remarkable way of integrating differential equations of higher degrees in one step, as soon as a finite [algebraic] equation has been obtained. Moreover this method extends to all equations which ... are contained in this general form:
>
> $$y + \frac{A\,dy}{dx} + \frac{B\,ddy}{dx^2} + \frac{C\,d^3y}{dx^3} + \frac{D\,d^4y}{dx^4} + \frac{E\,d^5y}{dx^5} + \text{etc.} = 0.$$

In short, Euler declared to his mentor that he had found a general method for solving homogeneous linear differential equations with constant coefficients. Their ensuing conversation about solving this particular type of differential equation played out over two more letters: a December 9, 1739 Bernoulli response letter and a January 19, 1740 return letter from Euler [4, pp. 38–52]. Over the course of this conversation, Euler and Bernoulli hashed out various details and examples, including when the "finite [algebraic] equation" had repeated or complex roots. Several years later, Euler's published his solution method in E62: "De integratione aequationum differentialium altiorum graduum" [7]. Homogeneous linear differential equations with constant coefficients thus became one of the first general class of differential equations to be solved.

This same class of differential equations is also the first type of higher order differential equations solved in an undergraduate Ordinary Differential Equations (ODE) course, and the modern derivation closely mimics that of Euler. Through excerpts from the Bernoulli-Euler correspondence and from Euler's 1743 paper (E 62), this primary source project thus equips students to solve any homogeneous higher order linear differential equations with constant coefficients (with reasonable auxiliary equation). The method itself involves only elementary factoring of polynomials to find roots, along with very basic ODE notation and definitions. Working through what amounts to a modern textbook derivation via the correspondence of two titans of mathematics can be exciting, as can succeeding with examples that we know they did.

The project begins with a preamble that motivates the problem with excerpts from Euler's first letter from to Bernoulli (quoted above). Students then work through the details of Euler's method as he presented it in E62. A separate subsection is dedicated to each of the three basic ways that the auxiliary equation might factor: distinct real roots, repeated real roots of arbitrary multiplicity, and distinct complex roots. A "modern" example of each case is presented in each subsection, with seven additional historical examples extracted from the Euler-Bernoulli correspondence presented as practice in a later section of the project. An example of repeated complex roots is also included for instructors who wish to cover that topic. The project closes with some historical remarks on the role that Euler's work in differential equations played in granting the sine and cosine the status of "function" within analysis.

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5 Solving Linear First Order Differential Equations: Leonard Euler's Integrating Factor: A Project for Differential Equations

The one-day project "Solving Linear First Order Differential Equations: Leonard Euler's Integrating Factor" [21] is based on excerpts from E269: "De integration aequationum differentialium" [10]. Following an example of an exact differential equation, Euler presented a theorem where he showed any differential equation of the form $Mdx + Ndy = 0$ becomes exact when multiplied by a certain integrating factor $L(x, y)$. Seventy years before Euler's contribution, in a letter written to Marquis de l'Hôpital (1661–1704), Gottfried Leibniz (1646–1716) solved first-order linear ordinary differential equations by intuiting a solution then checking that it worked [16]. Strictly speaking, Leibniz did not solve the equation, but asserted a solution and showed it was correct. As such, his method was neither general nor part of a larger context — something that mathematicians always strive for. Somewhat later, Johann Bernoulli considered first-order linear differential equations as special cases of what are now called "Bernoulli differential equations" [3]. He used a method that would decades later be generalized to variation of parameters by Lagrange. Euler's method of integrating factors took even greater strides in terms of presenting a unified theory.

The primary source project itself is part of a series of three independent projects that share the name "Solving Linear First Order Differential Equations," designed to show three solutions to non-homogeneous first-order linear differential equations, each from a different context. The first two in the series (subtitled respectively "Gottfried Leibniz' 'Intuition and Check' Method" [19] and "Johann Bernoulli's (Almost) Variation of Parameters Method" [20]) examine efforts to solve this particular equation prior to Euler's general work on integrating factors. The project based on Euler's work (subtitled "Leonard Euler's Integrating Factor"[21]) opens with a short introduction to what first-order linear differential equations are, along with a description of the way that mathematics often evolves. Some background on exact differential equations, including Euler's first example of such, and the statement of Euler's first theorem follows. The core of the project is then devoted to unpacking Euler's method of solution as presented in Problem 4 of E269 [10, p. 20]:

> Problem 4.
>
> §34. Suppose the differential equation
>
> $$P\,dx + Qy\,dx + R\,dy = 0$$
>
> is proposed, where P, Q and R denote functions of x of any sort, and so that the other variable y has no more than one dimension; to find the factor which allows it to be integrated.

Euler derived his integrating factor L by noticing the partial differential equation

$$L\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)+M\frac{\partial L}{\partial y}-N\frac{\partial L}{\partial x}=0$$

is a separable ordinary differential equation for first-order linear differential equation. The main take-away is that for a first-order linear differential equation, the partial differential equation defining the integrating factor is actually a separable ordinary differential that can be solved, and consequently the proposed equation can be made exact. In the final project task, students show that Euler's method produces the same integrating factor that is typically used today to solve the first-order linear differential equation $\frac{dy}{dx}+P(x)y=Q(x)$.

This project requires some algebraic manipulation of differentials along with knowledge of partial differentiation and integration. The differential equation defining the integrating factor is separable, and knowledge of separable techniques would be helpful. Other techniques of integration needed are dictated by the examples used, where instructors can modify the project itself in order to substitute any example they wish in place of those included in the tasks. Knowledge of exact differential equations (or conservative vector fields) will make this project go much faster but are not necessary based on the background provided.

6 Conclusion

In Part 2 of this article, we will describe a set of five additional Euler-based primary source projects designed for use in various upper-division courses. These and the four lower-division projects described above are part of a larger collection of nearly 120 primary source projects intended for use in a wide range of undergraduate mathematics courses that have been developed and extensively site tested with support from the National Science Foundation. Information about all three grants and digital access to the primary source projects developed with their support are available through the website of the most recent grant: a multi-year, multi-institution collaborative effort entitled TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS): `https://blogs.ursinus.edu/triumphs`.

Of course, given the range, nature, and quality of his writing, the works by Euler currently represented in this collection are but a small fraction of those which are accessible to undergraduate mathematics majors (including first-year students). In this regard, we propose that engaging with his writing during classroom implementation of a primary source projects may serve as an enticement for students to explore Euler's original works more deeply. For example, Klyve, Stemkoski, and Tou [15] described three possibilities for undergraduate research projects that make use of the Euler Archive [1]:

- translation projects for students with the appropriate foreign language skills;
- in-depth studies of the mathematical content contained in a particular text; and

- historical investigations that seek to understand "how Euler's work fit into the general milieux of the times, and how ... that historical moment impact[ed] later understandings of the topic."

Here, we add yet another possibility that combines pedagogy, mathematics, and history into a single research project for students in mathematics education:

- development of a guided-reading primary source project based on a particular Euler text.

Euleriana readers (and their students) who wish to experience the rewards (and challenges!) of writing their own Euler-based primary source project are invited to reach out to us for advice on how to get started. We also encourage readers to send us their top picks for Euler sources that are especially ripe as the foundation of projects that would allow today's students to learn even more mathematics directly from the master.

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