Euleriana

Volume 2 | Issue 1 Article 8

2022

Euler Archive Spotlight

Cynthia Huffman Ph.D.
Pittsburg State University, cjhuffman@pittstate.edu

Recommended Citation
Huffman, Cynthia Ph.D. (2022) "Euler Archive Spotlight," *Euleriana*: *2*(1), p. 56, Article 8.
DOI: https://doi.org/10.56031/2693-9908.1024
Available at: https://scholarlycommons.pacific.edu/euleriana/vol2/iss1/8

Euler Archive Spotlight

Cynthia Huffman
Pittsburg State University
cjhuffman@pittstate.edu

In this issue, we highlight the Basel Problem in the Euler Archive. This important problem brought Euler's mathematical prowess into the limelight. Specifically, the Basel Problem was to find the exact sum of the infinite series of reciprocals of the squares of positive integers, namely

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots$$

Mentioned by Pietro Mengoli in the mid-17th century, the problem was popularized by Jakob Bernoulli, who was unable to find a closed form, but did show that the series converged to a sum which was less than 2. In "Tractatus de seriebus infinitis", published in 1689 in Basel, Bernoulli wrote, "If anyone finds and communicates to us that which thus far has eluded our efforts, great will be our gratitude." (Translation by Sandifer [2, p. 157].) The best mathematicians of the day, including the Bernoulli brothers, Leibniz, de Moivre, and others, accepted the challenge, but all were unsuccessful. A young Euler achieved immediate fame in the mathematical world by proving the series converges to $\pi^2/6$.

Before solving the Basel Problem, Euler approximated the series to six decimal places in 1730. Below in Figure 1 is Euler's approximation of the series from Section 22 of E20 "De summatione innumerabilium progressionum" ("The summation of an innumerable progression"), which was published in 1738. The Euler Archive contains a German translation of E20, but at this time, there is no English translation.

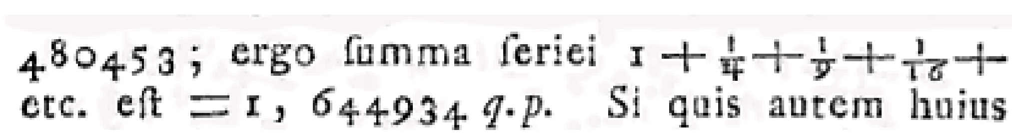

480453; ergo ſumma ſeriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+$ etc. eſt $=1,644934$ *q.p.* Si quis autem huius

Figure 1 *Euler's six decimal place approximation of the infinite series of the reciprocals of squares from Section 22 of E20.*

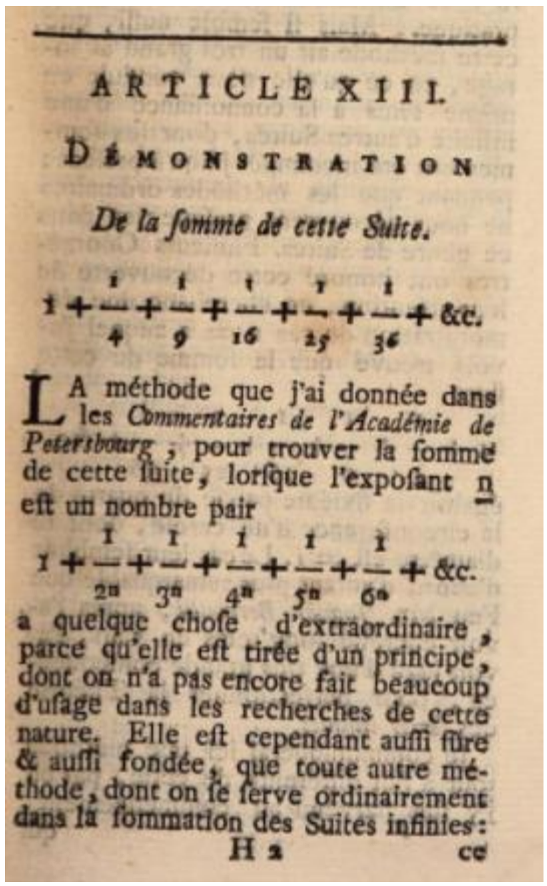
Then, in 1734, Euler announced he had solved the Basel Problem, demonstrating to the mathematical world his brilliance and amazing ability. Euler's solution was published in 1740 as "De summis serierum reciprocarum" (E41). Figure 2 below is from an English translation, "On the sums of series of reciprocals", by Jordan Bell, which is available in the Euler Archive. This clip shows Euler's announcement of not only solution of the Basel Problem, but also, more general results, namely the sum of a series of reciprocals of even powers of the positive integers. It is interesting to note that Euler is not yet using π to denote the ratio of the circumference of a circle to its diameter.

§2. I have recently found, quite unexpectedly, an elegant expression for the sum of this series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc., which depends on the quadrature of the circle, so that if the true sum of this series is obtained, from it at once the quadrature of the circle follows. Namely, I have found for six times the sum of this series to be equal to the square of the perimeter of a circle whose diameter is 1; or by putting the sum of this series $= s$, then $\sqrt{6s}$ will hold to 1 the ratio of the perimeter to the diameter. Indeed I recently showed for the sum of this series to be approximately $1,644934066842264364$; multiplying this number by six, and then taking the square root, the very same number $3,141592653589793238$ is found which expresses the perimeter of a circle whose diameter is 1. Following again the same steps by which I had arrived at this sum, I have discovered that the sum of the series $1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} +$ etc. also depends on the quadrature of the circle. Namely, the sum of this multiplied by 90 gives the biquadrate of the perimeter of a circle whose diameter is 1. And by similar reasoning I have likewise been able to determine the sums of the subsequent series in which the exponents are even numbers.

Figure 2 *Euler's announcement of a solution to the Basel Problem and more general results from an English translation of E41.*

Euler included three solutions to the Basel Problem in E41, along with generalizations, and approximations to π. There were criticisms of his use of an infinite product that he claimed was equal to $(\sin x)/x$. Euler attempted to address these concerns in "De seriebus quibusdam considerations" ("Considerations on certain series", E130, written in 1739 but not published until 1750, no English translation available yet in the Euler Archive) and "De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera, in qua eaedem summationes ex fonte maxime diverso derivantur" ("On sums of series of reciprocals from powers of natural numbers from another discussion, in which the sums are derived principally from another source", E61, published in 1743, German translation available in the Euler Archive). Then Euler presented a fourth solution to the Basel Problem in E63, also published in 1743, in French, "Demonstration de la somme de cette suite 1 + 1/4 + 1/9 + 1/16 + ..." ("Demonstration of the sum of the series 1 + 1/4 + 1/9 + 1/16 + ..."). Figure 3 is the beginning of E63.

Solving the challenge of the Basel Problem brought Euler to the attention of top European mathematicians at the beginning of his illustrious career. To read more about Euler and his solutions to the Basel Problem see [1] and [3], visit the Euler Archive to read about the Basel Problem in Euler's own words, and if you're up to the challenge, translate one or more of the articles (E20, E130, E61, E63) into English for the Euler Archive to make them readable for a wider audience.



ARTICLE XIII.

DÉMONSTRATION

De la ſomme de cette Suite.

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\&c.$$

LA méthode que j'ai donnée dans les *Commentaires de l'Académie de Petersbourg*, pour trouver la ſomme de cette ſuite, lorſque l'expoſant n eſt un nombre pair

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\&c.$$

a quelque choſe d'extraordinaire, parce qu'elle eſt tirée d'un principe, dont on n'a pas encore fait beaucoup d'uſage dans les recherches de cette nature. Elle eſt cependant auſſi ſûre & auſſi fondée, que toute autre méthode, dont on ſe ſerve ordinairement dans la ſommation des Suites infinies:

H 2 ce

Figure 3 *The beginning of E63 in which Euler presented his fourth solution of the Basel Problem.*

References

[1] Dunham, William (1999), *Euler: The Master of Us All*, Washington, D.C.: Mathematical Association of America.

[2] Sandifer, C. Edward (2007). *The Early Mathematics of Leonhard Euler*, Washington, D.C.: Mathematical Association of America.

[3] Sandifer, C. Edward (2007). “Euler’s Solution of the Basel Problem – The Longer Story”, Bradley, Robert E. et al (editors) (2007), *Euler at 300: An Appreciation,* Washington, D.C.: Mathematical Association of America.

[E20] Euler, Leonhard (1738). “De summatione innumerabilium progressionum,” *Commentarii academiae scientiarum Petropolitanae*, Volume 5, pp. 91-105. Reprinted in *Opera Omnia*: Series 1, Volume 14, pp. 25-41. Original text available online, along with a German translation by Alexander Aycock, at the Euler Archive.

[E41] Euler, Leonhard (1740). “De summis serierum reciprocarum,” *Commentarii academiae scientiarum Petropolitanae*, Volume 7, pp. 123-134. Reprinted in *Opera Omnia*: Series 1, Volume 14, pp. 74-86. Original text available online, along with an English translation by Jordan Bell, at the Euler Archive.

[E61] Euler, Leonhard (1743). “De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera, in qua eaedem summationes ex fonte maxime diverso derivantur,” *Miscellanea Berolinensia*, 1743(7): 172-192. Reprinted in *Opera Omnia*: Series 1, Volume 14, pp. 138-155. Original text available online, along with a German translation by Alexander Aycock, at the Euler Archive.

[E63] Euler, Leonhard (1743). “Demonstration de la somme de cette suite 1 + 1/4 + 1/9 + 1/16 + ...,” *Journal litteraire d'Allemagne, de Suisse et du Nord*, 1743(2): 115-127. Reprinted in *Opera Omnia*: Series 1, Volume 14, pp. 177-186. Original text available online at the Euler Archive and Google Books.

[E130] Euler, Leonhard (1739). “De seriebus quibusdam considerationes,” *Commentarii academiae scientiarum Petropolitanae*. 1739(12): 53-96. Reprinted in *Opera Omnia*: Series 1, Volume 14, pp. 407-462. Original text available online at the Euler Archive.

DOI: 10.56031/2693-9908.1024