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On

Rectifiable Algebraic Curves

And

Reciprocal Algebraic Trajectories

By The Author Leonhard Euler

Although it is very easy to give innumerable algebraic curves, that can be rectified, in either the desired evolutes or caustics of algebraic curves; if we were to consider the orders of curves, very rarely they occur in them, that admit rectification. In the second order of linears, which consists of conical sections, nothing is of this kind; in the third there are two rectifiables, indeed it consists as much. However, since I was occupied with finding reciprocal algebraic trajectories several years before, I searched for rectifiable curves very diligently following the first method of the Celebrated Johann Bernoulli, in order that I may use them for the purpose. I detected a rectifiable curve in the sixth order, which to me rendered an algebraic trajectory of the fourth order, with which I satisfied the question deliberated at the time, by exhibiting simpler reciprocal algebraic trajectories. I also found many general equations giving rectifiable curves, from which it was easy to draw out all simpler rectifiable curves. Now again, examining these, I was led to one most general equation containing all rectifiable curves in itself. Indeed there are many universal quantities within it, in place of which any might be substituted, a rectifiable curve would appear. If a certain variable quantity would be \( z \), whose differential would be set as constant, \( P \) and \( Q \) too would at least be whatever algebraic functions of this variable \( z \). If now in this manner a curve were to be constructed such that its abscissa, which were set as \( x \), it would be taken equal to \( P + \frac{dQ(dP^2-\frac{dQ^2}{dP})}{dQdP-dPdQ} \) and its applicate which I call \( y = \frac{(dP^2-\frac{dQ^2}{dP})^3}{dQdP-dPdQ} \). The length of this curve named \( s \) will be equal to \( Q + \frac{dP(dP^2-\frac{dQ^2}{dP})}{dQdP-dPdQ} \). Then whatever values always granted with the letters \( P \) and \( Q \), the curve will be rectifiable and algebraic. I do not deem the demonstration of this to be necessary, indeed for anything, if it will have taken differentials of coordinates \( x \) and \( y \) and of the curve \( s \), it is known that \( dx^2 + dy^2 = ds^2 \) needs to be done merely with hard work but with no skill.

This form extends indeed widely, but yet I have another one hitherto much more general, indeed the most general one as the following. With designators such that before the majuscule letters \( L \), \( M \) and \( N \) any function of the variable \( z \), if they are taken to be

\[
x = L + \frac{(dL^2 + dM^2 - dN^2)(dLdN + dM \sqrt{(dL^2 + dM^2 - dN^2)})}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + (dLddM - dMddL)\sqrt{(dL^2 + dM^2 - dN^2}}}.
\]
\[ x = M + \frac{(dl^2 + dM^2 - dN^2)(dMdN + dl\sqrt{(dl^2 + dM^2 - dN^2)})}{dlNd dl + dMdNdM - dl^2 dN - dM^2 dN + (dl dM - dM dl)\sqrt{(dl^2 + dM^2 - dN^2)}} \]

the length of the respondent curve will be

\[ s = N + \frac{(dl^2 + dM^2 - dN^2)(dl^2 + dM^2)}{dlNd dl + dMdNdM - dl^2 dN - dM^2 dN + (dl dM - dM dl)\sqrt{(dl^2 + dM^2 - dN^2)}} \]

These formulae change in the preceding, if it is set that \( M = 0 \), therefore they are bound within these.

It is easy to see that, if the letters \( L \), \( M \) and \( N \) do not only signify algebraic quantities, but also whatever transcendent ones, all curves both algebraic as well as transcendent are directly contained in these formulae. Indeed, because these functions \( L \), \( M \) and \( N \) are not in any way dependent on one another, no equation can be named between \( x \) and \( y \), whether it be algebraic or transcendent, which would not be reducible to the formulae written above.

Similarly if \( N \) remaining the most universal function, such that thus \( \frac{dN}{dz} \) would be an algebraic function indeed \( N \) and \( M \) would denote algebraic functions, all algebraic curves are directly comprehended in the given formulae; however they will be rectifiable if \( N \) is taken as an algebraic function, and indeed, if \( N \) is not an algebraic function, but transcendent or dependent on the quadrature of some curve, the resultant curve will not be rectifiable, instead its rectification will depend on the quadrature of its curve. Therefore it is in this manner that the famous problem much debated among Geometers is solved, postulating the method of reducing the quadrature of curves into rectifications of algebraic curves, whose two solutions have been given in the Acta Lipsiensia by the Most Celebrated Gentlemen Jakob Hermann and Johann Bernoulli. Indeed, it will be solved by these formulae thus; the curves, whose quadrature ought to be reduced to a rectification of algebraic curve, would be coordinates \( t \) and \( v \) of which each algebraic function would be of the \( z \) itself. Let it be assumed \( N = R + a \int v dt \), where \( R \) would designate any algebraic function of \( z \) itself. This being posited, the delivered formulae will give all algebraic curves, the rectification of which on the quadrature of the proposed curve, certainly they depend on \( \int v dt \), if indeed \( L \) and \( M \) are assumed, just as it has already been pointed out, to be the algebraic functions of \( z \) itself. Another usage of these formulae, which I have set to expound here, looks towards finding reciprocal trajectories; from them indeed the most universal equation encompassing all reciprocal trajectories is found, it is thus very easily solved in order to only represent all those very same algebraics.

This finding depends on Bernoulli’s theorem, by which the reciprocal trajectories are constructed from the rectification of the curves holding the diameter, in this way: \( MAM \) is such a curve holding the diameter \( AP \), and on which the tangent \( AQ \) at the peak \( A \) is perpendicular towards the diameter \( AP \). Lines parallel to the diameter are drawn through each \( M \) point of this curve, in them \( MN \) would be taken as equal to the arcs \( AM \), the points \( N \) constitute the reciprocal trajectory curve \( NAE \), whose axis of conversion is the diameter \( PAB \) itself. If the abscissa and the applicate of this curve are taken as \( AQ \) and \( QN \), it will be that \( AQ = PM \) and \( QN = AM-AP \). Wherefore if the coordinates of the curve \( MAM \) are taken as the very same \( AP \) and \( PM \), which before were called \( x \) and \( y \), the equation for the reciprocal trajectory \( EAN \) will immediately be obtained. Since all lines \( AP \), \( PM \) and \( AM \) are functions of \( z \) itself, they are determined in \( z \), such that when \( z \) taken as positive the right branch of the curve \( MAM \) would appear, and when \( z \) taken as negative the left branch would appear. To this it is required that \( AP \) remains the same in each case, the function of \( z \) itself would be even, or the unchanged function which remains, even if \( z \) is negative. But \( PM \) and \( AM \) need to be uneven functions of \( z \) itself, i.e. which would become negatives with \( z \) changed into \(-z\). This is why \( L \) has to be an even function of \( z \) itself while \( M \) and indeed \( N \) uneven functions. These having been set, the abscissa will be equal to an even function, while the
appricate and indeed the curve itself to uneven functions. For \( dL \) will be an uneven function, \( ddL \) an even function, \( dM \) and \( dN \) even functions, also \( ddM \) and \( ddN \) uneven functions. From this, it will be seen that lines \( AP \), \( PM \) and \( AM \) will have the requisite property. Therefore, it will be

\[
AQ = M + \frac{(dL^2 + dM^2 - dN^2)(dMdN + dL\sqrt{(dL^2 + dM^2 - dN^2)})}{dLddNdL + dMddMdM - dL^2ddN - dM^2ddN + (dLddM - dMddL)\sqrt{(dL^2 + dM^2 - dN^2)}}
\]

And the applicate of the reciprocal trajectory \( AEN \) will be

\[
QN = N - L + \frac{(dL^2 + dM^2 - dN^2)(dL^2 + dM^2 - dN^2)}{dLddNdL + dMddMdM - dL^2ddN - dM^2ddN + (dLddM - dMddL)\sqrt{(dL^2 + dM^2 - dN^2)}}
\]

From this construction, all reciprocal trajectories will flow, if in the place of \( L \), \( M \) and \( N \) not only algebraic functions are substituted, but also the transcendent ones. Indeed all algebraic reciprocal trajectories will be obtained, if these quantities are algebraic functions, and indeed such that it is required that \( L \) is an even function, and \( M \) and \( N \) are uneven functions.

Furthermore, it should be noted that \( \sqrt{dL^2 + dM^2 - dN^2} \) must be an uneven function of \( z \) itself as is \( dL \). Therefore, should it be put that \( \sqrt{dL^2 + dM^2 - dN^2} = dL - dS \), where \( dS \) is an uneven function of \( z \) itself, so \( S \) will be even function \( \frac{dN^2 - dM^2 + ds^2}{2ds} = dL \), from which \( dL \) will be an uneven function as it is required, and \( \sqrt{(dL^2 + dM^2 - dN^2)} = \frac{dN^2 - dM^2 - ds^2}{2ds} \). Therefore it will be \( L = \frac{1}{2} \int \frac{dN^2 - dM^2}{ds} + \frac{1}{2} S \). In order that the trajectory becomes reciprocal, \( \frac{dN^2 - dM^2 + ds^2}{dS} \) needs to be integrable. The abscissa will be \( AQ = M + \frac{(dN^2 - dM^2 - ds^2)((dN + dM)^2 - ds^2)}{4dS(dSddM + dSddN - dNddS - dMddS)} \) and \( QN = N - \frac{1}{2} \int \frac{dN^2 - dM^2}{ds} - \frac{1}{2} S + \frac{(dN^2 - dM^2 - ds^2)(dMddN + (dN - dM - dS))}{4dS(dSddM + dSddN - dNddS - dMddS)} \).

As long as \( \sqrt{dL^2 + dM^2 - dN^2} \) remains a surd quantity, there is no need for peculiar determination, indeed the square root from a non-quadratic even function is both an even and uneven function.