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The Solution of a Problem of Searching for Three Numbers, of Which the Sum, Product, and the Sum of Their Products Taken Two at a Time, Are Square Numbers

Translated by Mark R. Snavely[(a)] and Phil Woodruff
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Abstract

This paper (E270) first appeared in *Novi Commentarii academiae scientiarum Petropolitanae*, Volume 8, pp. 64-73, and is reprinted in *Opera Omnia*: Series 1, Volume 2, pp.519-530. Euler seeks whole numbers x, y, and z, such that $x+y+z$, $xy+yz+xz$, and xyz are all perfect squares. The solutions he finds are quite large. Euler discovers significantly smaller solutions in "On Three Square Numbers, of Which the Sum and the Sum of Products Two Apiece will be a Square"(E523), in which he requires that x, y, and z themselves be perfect squares. In her senior thesis at Carthage College, Malorie Harder found smaller solutions to the problem posed in E523 using brute force methods. In her senior thesis at Carthage, Olivia Lutterman found much smaller solutions to the problem posed in this paper. She proved that while most of these smaller solutions do not satisfy Euler's assumption that $xy=u^2+v^2$ (a possibility that Euler mentions in Section 7), a few small solutions *do* satisfy this assumption, but it would have been exceedingly difficult for Euler to have guessed appropriate parameter values.

1.

Although problems of this sort, which are usually called Diophantine, seem to be of little use, it is still certain that Mathematical Analysis, and especially that area which concerns infinity, has received great benefits from the method of solving Diophantine equations. Problems of this sort, however, even if they are rather difficult, have generally expanded the boundaries of analysis, but also often sharpened wonderfully the force of the intellect so that even in other problems, one can quite easily ascertain how a solution should be established. For this reason, I am of the opinion that problems of this sort must by no means be disregarded, especially if the way of solving [them] has been quite hard to find. For while unique methods are required for their solution, from these same solutions one may also look for rich development for refining quite productively the entire field of Analysis.

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2. But on this point the proposed problem seems to be of this type, seeing that I spent such a long time in vain on it using various devices of the Diophantine method that I almost completely lost hope for its solution. But at last, almost unexpectedly, I came upon a solution which seems quite worthy to be recorded, because the very few numbers which I was able to elicit as still satisfactory are so thoroughly large that it is no surprise that a solution evolved by such great difficulties.[(b)] Therefore, since I arrived at this very solution by a remarkable method, I think that its fuller explanation would be of use in the examples below, since perhaps by this same method other questions much more difficult up to now can be overcome.

3. Therefore three numbers are desired for which the three following conditions apply.

I. That their sum be a squared number.[(c)]

II. That the sum of their products two at a time be a squared number.

III. That the product of all three be a squared number.

This problem can also be described in this way: that a cubic equation be sought

$$z^3 - pzz + qz - r = 0,$$

having all rational roots, whose coefficients p, q and r be squared numbers. This condition should be added as well, that they be whole numbers, for it is obvious that when three rational numbers have been found satisfying [the equation], whole numbers could easily be formed from these which also satisfy the equation. And so, whatever three numbers have been found to satisfy [the equation], then if these same numbers are each multiplied by some square number, they will satisfy the equation. In this way the fractions will easily be reduced.

4. Let three numbers of this sort be sought, nx, ny, nz, that satisfy these conditions:

I. That $n(x + y + z) =$ a squared number.

II. That $nn(xy + xz + yz)$ or $xy + xz + yz =$ a squared number.

III. That n^3xyz or $nxyz =$ a squared number.

But the first and third conditions will be satisfied if it were given that

$$xyz(x + y + z) = \text{ a squared number.}$$

Now let it be given:

$$xyz(x + y + z) = vv(x + y + z)^2,$$

[(b)]In paper E523 in the Euler Archive (http://eulerarchive.maa.org/docs/translations/E523en.pdf), Euler finds much smaller solutions to this problem.

[(c)]Here Euler meant squared rational numbers, and he indicates below that one can easily require that the solutions be whole numbers.

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which when divided by $x+y+z$ will yield $xyz = vv(x+y+z)$, and from this $z = \frac{vv(x+y)}{xy-vv}$. Therefore $xyz = \frac{vvxy(x+y)}{xy-vv}$, then in order that $nxyz$ appears, which should be taken to be a squared number, we set

$$n = m^2xy(x+y)(xy-vv)$$

and with these values for z and n assumed, the first and third conditions will be satisfied.

5. From here also our three numbers will be
first $nx = mmxxy(x+y)(xy-vv)$
second $ny = mmxyy(x+y)(xy-vv)$
third $nz = mmvvxy(x+y)^2$
where if any by chance fractions occur, they can be removed by the arbitrary number m. But let us now consider the second condition, which on account of $z = \frac{vv(x+y)}{xy-vv}$, requires that:

$$xy + \frac{vv(x+y)^2}{xy-vv} = \text{ a square number.}$$

Toward this end, let us set:

$$xy - vv = uu; \text{ so that } y = \frac{vv+uu}{x} \text{ and } z = \frac{vv(x+y)}{uu}.$$

It follows that $xy = vv + uu$ and $x + y = \frac{xx+vv+uu}{x}$ and it must work out that

$$vv + uu + \frac{vv(xx+vv+u^2)^2}{uuxx} = \text{ a squared number.}$$

6. Let $x = tv$; so that $y = \frac{vv+uu}{tv}$, and it ought to be that

$$vv + uu + \frac{(vv(tt+1)+uu)^2}{ttuu} = \text{ a square number,}$$

or by multiplying by $ttuu$

$$ttuuvv + ttu^4 + v^4(tt+1)^2 + 2uuvv(tt+1) + u^4(tt+1) = \text{ a square number,}$$

$$\text{or } v^4(tt+1)^2 + uuvv(3tt+2) + u^4(tt+1) = \text{ a square number.}$$

Let the square root of this number be set as $= vv(tt+1) + suu$, then

$$vv(3tt+2) + uu(tt+1) = 2svv(tt+1) + ssuu;$$

from which it arises that

$$\frac{vv}{uu} = \frac{tt+1-ss}{2s(tt+1)-3tt-2} = \text{ a square number.}$$

Therefore let $s = t - r$, and thus:

$$\frac{vv}{uu} = \frac{2rt - rr + 1}{2t^3 - (2r+3)tt + 2t - 2(r+1)}.$$

Let the numerator and the denominator be multiplied by $2rt - rr + 1$, so that it becomes

$$\frac{vv}{uu} = \frac{(2rt - rr + 1)^2}{4rt^4 - 2(3rr + 3r - 1)t^3 + (2r^3 + 3rr + 2r - 3)tt - 2(3r - 1)(r + 1)t + 2(r - 1)(r + 1)^2}.$$

7. Therefore the entire search for a solution to this point has been worked out, so that the denominator of this fraction be given as a square: for setting

$$4rt^4 - 2(3rr+3r-1)t^3 + (2r^3+3rr+2r-3)tt - 2(3r-1)(r+1)t + 2(r-1)(r+1)^2 = QQ,$$

there will arise for some t and r

$$\frac{v}{u} = \frac{2rt - rr + 1}{Q}, \text{ then indeed } x = tv \text{ and } y = \frac{vv + uu}{tv}$$

from which the numbers sought will be determined. But, before we reach this equation, we have already limited the solution by assuming $xy - vv = uu$, a restriction which must be duly noted, since there is no doubt that solutions of this sort may exist in which $xy - vv$ is not a squared number, and accordingly we will not find [solutions] among these. But I have been forced to make this limitation, so that one might arrive at such a formula equaling a square, obviously this has been set up in such a way that it be resolved by known methods. And thus the total strength of the solution is found in the reductions of the preceding paragraph.

8. This formula can be worked out as a square in more cases and, indeed, by countless methods, of which the foremost and the ones which offer themselves immediately are: 1) if the coefficient of t^4, namely $4r$, or r, will be a square; 2) if the last term $2(r - 1)(r + 1)^2$ or $2(r - 1)$ will be a square. For in each case by known principles suitable values for t can be elicited, then even further from any [solution] other new values can be found. But if both r and $2(r - 1)$ will be squares, one may obtain more appropriate values for t through a single operation, and indeed, as generally happens, no simpler solution offers itself here; although if $2(r - 1) =$ a square, it leads to the value $t = 0$, but from there it yields $x = 0$ and $y = \infty$, and these values are clearly incongruous for the nature of the inquiry. Solutions in which one of the three desired numbers vanishes are excluded, since then the question would be very easy and would depend on two numbers, of which just as the sum, the product too would be a square.

Case 1. in which $r = 1$

9. This case seems very simple since the last term of our equation disappears, and the first would be a square. Therefore, we have

$$4t^4 - 10t^3 + 4tt - 8t = QQ \text{ and } \frac{v}{u} = \frac{2t}{Q}.$$

To solve this equation, let us set $Q = 2tt - \frac{5}{2}t$[d] and thus

$$4tt - 8t = \tfrac{25}{4}tt;\ \tfrac{9}{4}t = -8;\ \text{and } t = -\tfrac{32}{9}.$$

[d]Euler chooses $Q = 2t^2 - \frac{5}{2}t$ so that the first two terms of Q^2 are $4t^4 - 10t^3$.

But from here, $\frac{v}{u} = \frac{4}{4t-5} = \frac{-36}{173}$, from which we will have

$$v = -36; u = 173; t = \frac{-32}{9} \text{ and } x = tv = 128$$

and then in turn $y = \frac{36^2+173^2}{128} = \frac{31225}{128} = \frac{25\cdot1249}{128}$. Therefore, $x + y = \frac{47609}{128}$ and $z = \frac{36^2\cdot47609}{173^2\cdot128}$ and the three desired numbers, because $xy - vv = uu$, will be

First $= \frac{128^2\cdot25\cdot1249\cdot47609\cdot173^2}{128\cdot128\cdot}mm$

Second $= \frac{128\cdot25^2\cdot1249^2\cdot47609\cdot173^2}{128^2\cdot128\cdot}mm$

Third $= \frac{36^2\cdot128\cdot25\cdot1249\cdot47609^2}{128\cdot128^2\cdot}mm.$

10. To reduce the fractions let us set $m = \frac{128}{5}$ and our numbers three at a time will be

$$\left.\begin{aligned} \text{First} &= 128^2 \cdot 173^2 \cdot 1249 \cdot 47609 = 128^2 \cdot 173^2 \\ \text{Second} &= 5^2 \cdot 173^2 \cdot 1249^2 \cdot 47609 = 5^2 \cdot 173^2 \cdot 1249 \\ \text{Third} &= 36^2 \cdot 1249 \cdot 47609^2 = 36^2 \cdot 47609 \end{aligned}\right| \text{ multiplied by } 1249 \cdot 47609$$

when these numbers are worked out, thus

First $= 490356736 \cdot 59463641$

Second $= 934533025 \cdot 59463641$

Third $= 617011264 \cdot 59463641$

of which the product is clearly a square

$$5^2 \cdot 36^2 \cdot 128^2 \cdot 173^4 \cdot 1249^4 \cdot 47609^4.$$

The sum is given

$$25 \cdot 59463641^2$$

and the sum of the products from two at a time:

$$173^2 \cdot 59463641^2 \cdot 18248924559376$$

whose square root is $173 \cdot 59463641 \cdot 4271876$.

11. To solve this same equation, set $Q = 2tt - \frac{5}{2}t - \frac{9}{16}$, so that the first three terms cancel and it will yield

$$-8t = +\frac{45}{16}t + \frac{81}{256}, \text{ or } 0 = 173t + \frac{81}{16}, \text{ and therefore } t = \frac{-81}{16\cdot173}.$$

From here $Q = \frac{81^2}{128\cdot173^2} + \frac{405}{32\cdot173} - \frac{9}{10} = \frac{9\cdot207563}{128\cdot173^2}$ and $\frac{v}{u} = \pm\frac{144\cdot173}{207563}$. For the value of Q itself can be taken as negative just as positive. Therefore, let

$$v = -144 \cdot 173; u = 207563; \text{ and thus } x = 9 \cdot 81 = 729$$

and $y = \frac{uu+vv}{729}$; from which it is now clear that the resulting numbers are so enormous that the previous solution would be judged much simpler than them. It would be unnecessary to produce over-complicated solutions of this sort any further, since in inquiries of this type we are generally accustomed to be satisfied by the simplest solution.

Case 2. in which $r = \frac{3}{2}$.

12. Given this, the final term of our equation should be a square, and thus $\frac{v}{u} = \frac{12t-5}{4Q}$, with

$$QQ = 6t^4 - \tfrac{41}{2}t^3 + \tfrac{27}{2}tt - \tfrac{35}{2}t + \tfrac{25}{4}.$$

Now, in order to remove the final three terms, let

$$Q = \tfrac{5}{2} - \tfrac{7}{2}t + \tfrac{1}{4}tt, \text{ and therefore}$$

$$6t^4 - \tfrac{41}{2}t^3 = \tfrac{1}{16}t^4 - \tfrac{7}{4}t^3 \text{ and } t = \tfrac{60}{19}$$

and from here $Q = \frac{4379}{722}$ and $\frac{v}{u} = \frac{19}{14}$, from which $v = 19$ and $u = 14$. At this point then $x = tv = 60$; and $y = \frac{vv+uu}{x} = \frac{557}{60}$ and for that reason $x+y = \frac{4157}{60}$ and the three desired numbers:

First $= \frac{60^2 \cdot 557 \cdot 4157 \cdot 196}{60 \cdot 60} mm = 14^2 \cdot 60^2 \cdot 557 \cdot 4157$

Second $= \frac{60 \cdot 557^2 \cdot 4157 \cdot 196}{60 \cdot 60 \cdot 60} mm = 14^2 \cdot 557^2 \cdot 4157$

Third $= \frac{361 \cdot 60 \cdot 557 \cdot 4157^2}{60 \cdot 60 \cdot 60} mm = 19^2 \cdot 557 \cdot 4157^2$

with $m = 60$: and these numbers are now clearly less than those which were found in the first case.

13. Therefore, since these numbers seem worthy of attention due to their small size, these numbers should be written as:

First $= 705600 \cdot 2315449$

Second $= 109172 \cdot 2315449$

Third $= 1500677 \cdot 2315449$

The sum of these numbers is $= 2315449^2$, and the product $= 14^4\, 19^2\, 60^2\, 557^4\, 4157^4$, and thus each number is a square.

But the sum of the products two-at-a-time will be

$$(14^2 \cdot 60^2 \cdot 14^2 \cdot 557 + 14^2 \cdot 60^2 \cdot 19^2 \cdot 4157 + 14^2 \cdot 557 \cdot 19^2 \cdot 4157)2315449^2$$

which is reduced to this form:

$$14^2 \cdot 2315449^2 \cdot 6631333489$$

the square root of which is

$$14 \cdot 2315449 \cdot 81433.$$

These numbers, however, are in turn about 15000 times smaller than those first found.

Case 3. in which $r = 3$.

14. With $r = 3$, it turns out that $\frac{v}{u} = \frac{6t-8}{Q}$, and this equation will be resolved as:

$$QQ = 12t^4 - 70t^3 + 84tt - 64t + 64.$$

Now, in order to remove the final three terms, let

$$Q = 8 - 4t + \tfrac{17}{4}tt, \text{ and thus}$$

$$12t^4 - 70t^3 = \frac{289}{16}t^4 - 34t^3$$

from which it follows that $t = -\frac{576}{97}$ and $Q = \pm\frac{8\cdot 213601}{97\cdot 97}$. Therefore $\frac{v}{u} = -\frac{97\cdot 259}{213601} = -\frac{97\cdot 23}{9287} = -\frac{23\cdot 97}{37\cdot 251}$. And for that reason $v = -23\cdot 97$ and $u = 37\cdot 251$: then $x = tv = 23\cdot 24^2$ and $y = \frac{91225730}{23\cdot 24^2}$. But it is easily seen that these numbers increase to a vast size; accordingly we refrain from working these out. Therefore, let us examine one case in addition in which, just as the first, the final term of the formula QQ becomes a square.

Case 4. in which $r = 9$.

15. With $r = 9$, it turns out $\frac{v}{u} = \frac{18t-80}{Q}$, resulting in

$$QQ = 36t^4 - 538t^3 + 1716tt - 520t + 1600.$$

Let us remove the first term and the last two terms by setting

$$Q = 40 - \frac{13}{2}t \pm 6tt, \text{ and thus}$$

$$-538t^3 + 1716tt = \mp 78t^3 \pm 480tt + \frac{169}{4}tt$$

from which we elicit for each sign

upper $t = \frac{5\cdot 191}{16\cdot 23}$	but the numbers in both cases
lower $t = \frac{5\cdot 1723}{32\cdot 77}$	become excessively large.

Let us then remove the three final terms by setting

$$Q = 40 - \frac{13}{2}t + \frac{1339}{64}tt;$$

from here, however, the numbers still become much too large. Further, to remove the first two terms we could set $Q = 6tt - \frac{269}{6}t \pm 40$. Indeed from here we will arrive at numbers much less simple.

16. From these examples it seems it can safely be concluded that the smallest numbers which satisfy this problem, which we produced in §13, which if they are completely worked out by multiplication, will be:

First $= 1633780814400$
Second $= 252782198228$
Third $= 3474741058973$.

But if very simple solutions were desired to be fractions, those can be derived from the same source by dividing these by 2315449^2; The resulting numbers would be:

First $= \frac{705600}{2351449}$
Second $= \frac{196}{4157}$
Third $= \frac{361}{557}$,

of which just as the sum is a squared number, so too the sum of the products taken two-at-a-time and the product of all three are squared numbers.

Solutio Problematis

de Investigatione Trium Numerorum,
Quorum tam Summa, Quam Productorum,
nec non Summa Productorum ex
Binis, Sint Numeri Quadrati

Auctore
L. EULERO

1.

Etsi problemata huius generis, quae Diophantea appellari solent, parum utilitatis afferre videnter: tamen certum est, Analysin Mathematicam, atque adeo etiam eam partem, quae circa infinita versatur, ex methodo problemata Diophantea solvendi, maxima incrementa cepisse. Non solum autem huiusmodi problemata, si sint difficiliora, fines Analyseos plurimum amplificaverunt: sed etiam vim ingenii mirifice acuere solent, ut etiam in aliis problematibus, quomodo solutionem institui oporteat, facilius perspicere valeat. Quam ob rem huius generis problemata, praecipue si modus solvendi magis fuerit reconditus, minime contemnenda esse arbitror. Dum enim singularia artificia ad eorum solutionem requiruntur, ab iisdem quoque egregia subsidia ad universam Analysin uberius escolendam expectare licebit.

2. Ad hoc autem genus potissimum referendum videtur problema propositum, quandoquidem id diu et multum per varia Methodi Diophanteae artificia frustra tractavi, ut fere etiam de eius solutione penitus desperaverim. Tandem vero, quasi inopinato, solutionem sum consecutus, quae eo magis notatu digna videbatur, quod minimi numeri, quos quidem adhuc satisfacientes elicere potui, sunt ita praegrandes, ut mirum non sit, solutionem tantis difficultatibus fuisse involutam. Quare cum methodo singulari ad istam solutionem pertigerim, eius ampliorem explicationem usu non esse cariturам arbitror, cum simili fortasse modo aliae quaestiones multo adhuc difficiliores superari queant.

3. Quaeruntur ergo tres numeri, quibus tres sequentes conditiones conveniant:

I. Ut eorum summa sit numerus quadratus.

II. Ut summa productorum ex binis sit numerus quadratus.

III. Ut productum omnium trium sit numerus quadratus.

Quod problema etiam hoc modo enunciari potest, ut quaeratur aequatio cubica $z^3 - pzz + qz - r = 0$, omnes suas radices habens rationales, cuius singuli coeficientes p, q et r sint numeri quadrati. Posset adhuc adiici haec conditio, ut isti numeri sint integri, verum per se est perspicuum, quomodo inventis ternis numeris fractis satisfacientibus, ex iis facile integri, qui etiam satisfaciant, formari queant. Quicumque enim terni numeri satisfacere fuerint inventi, iidem per numerum quadratum quemcunque multiplicati aeque satisfacient, quo pacto fractiones facillime tollentur.

4. Sint igitur nx, ny, nz tres huiusmodi numeri quaesiti, ac satisfieri oportebit his conditionibus:

I. Ut sit $n(x+y+z) =$ Quadrato

II. Ut sit $nn(xy+xz+yz)$ seu $xy+xz+yz =$ Quadrato

III. Ut sit n^3xyz seu $nxyz =$ Quadrato.

At primae et tertiae conditioni satisfiet, si reddatur,

$$xyz(x+y+z) = \text{ Quadrato.}$$

Ponatur ergo:

$$xyz(x+y+z) = vv(x+y+z)^2$$

unde per $x+y+z$ dividendo erit $xyz = vv(x+y+z)$, hincque $z = \frac{vv(x+y)}{xy-vv}$. Cum igitur hinc fiat $xyz = \frac{vvxy(x+y)}{xy-vv}$, ut $nxyz$ prodeat, quadratum capi debet,

$$n = m^2xy(x+y)(xy-vv)$$

Hisque valoribus pro z et n assumtis, satisfactum erit primae et tertiae conditioni.

5. Hinc itaque nostri tres numeri erunt
primus $nx = mmxxy(x+y)(xy-vv)$
secundus $ny = mmxyy(x+y)(xy-vv)$
tertius $nz = mmvvxy(x+y)^2$
ubi per numerum arbtrarium[(e)] m fractiones, si quae forte occurrent, tolli poterunt. Verum contemplemur iam secundam conditionem, quae ob $z = \frac{vv(x+y)}{xy-vv}$ requirit, ut sit:

$$xy + \frac{vv(x+y)^2}{xy-vv} = \text{ Quadrato.}$$

Ponamus in hunc finem:

$$xy - vv = uu; \text{ ut sit } y = \frac{vv+uu}{x} \text{ et } z = \frac{vv(x+y)}{uu}$$

erit $xy = vv+uu$ et $x+y = \frac{xx+vv+uu}{x}$ efficiendumque est, ut sit

$$vv + uu + \frac{vv(xx+vv+u^2)^2}{uuxx} = \text{ Quadrato.}$$

6. Ponatur $x = tv$; ut sit $y = \frac{vv+uu}{tv}$, esseque debet

$$vv + uu + \frac{(vv(tt+1)+uu)^2}{ttuu} = \text{ Quadrato,}$$

seu multiplicando per $ttuu$

$$ttuuvv + ttu^4 + v^4(tt+1)^2 + 2uuvv(tt+1) + u^4(tt+1)^{(\mathsf{f})} = \text{ Quadrato,}$$

$$\text{sive } v^4(tt+1)^2 + uuvv(3tt+2) + u^4(tt+1) = \text{ Quadrato.}$$

(e) Ed: Should be arbitrarium.

(f) $tt+1$ is missing in the original manuscript.

Statuatur huius quadrati radix $= vv(tt+1) + suu$, erit

$$vv(3tt+2) + uu(tt+1) = 2svv(tt+1) + ssuu;$$

unde elicitur

$$\frac{vv}{uu} = \frac{tt+1-ss}{2s(tt+1)-3tt-2} = \text{ Quadrato.}$$

Sit porro $s = t - r$, et habebitur:

$$\frac{vv}{uu} = \frac{2rt - rr + 1}{2t^3 - (2r+3)tt + 2t - 2(r+1)}$$

Multiplicetur numerator et denominator per $2rt - rr + 1$, ut fiat

$$\frac{vv}{uu} = \frac{(2rt-rr+1)^2}{4rt^4 - 2(3rr+3r-1)t^3 + (2r^3+3rr+2r-3)tt - 2(3r-1)(r+1)t + 2(r-1)(r+1)^2}.$$

7. Tota ergo quaestio huc est perducta, ut huius fractionis denominator reddatur quadratum: posito enim[(g)]

$$4rt^4 - 2(3rr+3r-1)t^3 + (2r^3+3rr+2r-3)tt \\ - 2(3r-1)(r+1)t + 2(r-1)(r+1)^2 = QQ$$

erit definitis hinc t et r

$$\frac{v}{u} = \frac{2rt-rr+1}{Q}, \text{ tum vero } x = tv \text{ et } y = \frac{vv+uu}{tv}$$

unde numeri quaesiti definientur. Ante autem, quam ad istam aequationem pertigimus, solutionem iam limitavimus positione $xy - vv = uu$, quae restrictio probe est notanda, quoniam nullum est dubium, quin eiusmodi extent solutiones, in quibus $xy - vv$ non sit numerus quadratus, easque propterea hinc non reperiemus. Verum hanc limitationem ideo facere sum coactus, ut ad istam formulam quadrato aequandam pervenire licuerit, quippe quae ita est comparata, ut per cognita artificia resolvi possit. Sicque tota solutionis vis in reductionibus § praeced. est sita.

8. Pluribus autem casibus haec formula et quidem infinitis modis quadratum effici potest, quorum praecipui, et qui statim se offerunt sunt: $1°$). Si coefficiens ipsius t^4, scilicet $4r$, seu r, fuerit numerus quadratus. $2°$) Si terminus ultimus $2(r-1)(r+1)^2$ seu $2(r-1)$ fuerit numerus quadratus: utroque enim casu per regulas cognitas valores idonei pro t elici, tum vero porro ex quolibet alii novi inveniri possunt. Sin autem simul et r et $2(r-1)$ fuerint quadrata, una operatione plures valores idoneos pro t eruere licet, neque vero hic, ut plerumque fieri solet, solutio simplicior se offert; etsi enim si $2(r-1) =$ quadrato, satisfacit valor $t = 0$, tamen inde prodit $x = 0$ et $y = \infty$, qui valores pro natura quaestionis plane sunt incongrui. Excluduntur enim solutiones, quibus unus trium numerorum quaesitorum evanesceret, quia tum quaestio esset facillima et circa duos numeros versaretur, quorum tam summa, quam productum, esset quadratum.

(g) missing parenthesis inserted after $r - 1$ in expression below

Casus 1. quo ponitur $r = 1$

9. Hic casus simplicissimus videtur, quia ultimus terminus nostrae formae evanescit, primusque sit quadratus. Habemus ergo

$$4t^4 - 10t^3 + 4tt - 8t = QQ \text{ et } \frac{v}{u} = \frac{2t}{Q}.$$

Ad hanc aequationem solvendam statuamus $Q = 2tt - \frac{5}{2}t$ eritque

$$4tt - 8t = \tfrac{25}{4}tt; \tfrac{9}{4}t = -8; \text{ et } t = -\tfrac{32}{9}.$$

At hinc fiet $\frac{v}{u} = \frac{4}{4t-5} = \frac{-36}{173}$; unde habebimus

$$v = -36; u = 173; t = \tfrac{-32}{9} \text{ et } x = tv = 128$$

indeque porro $y = \frac{36^2+173^2}{128} = \frac{31225}{128} = \frac{25\cdot1249}{128}$. Erit ergo $x + y = \frac{47609}{128}$ et $z = \frac{36^2\cdot47609}{173^2\cdot128}$ ac tres numeri quaesiti erunt, ob $xy - vv = uu$,

Primus $= \frac{128^2\cdot25\cdot1249\cdot47609\cdot173^2}{128\cdot128\cdot}mm$

Secundus $= \frac{128\cdot25^2\cdot1249^2\cdot47609\cdot173^2}{128^2\cdot128\cdot}mm$

Tertius $= \frac{36^2\cdot128\cdot25\cdot1249\cdot47609^2}{128\cdot128^2\cdot}mm.$

10. Ad fractiones tollendas ponamus $m = \frac{128}{5}$ eruntque terni nostri numeri

$$\left.\begin{aligned} \text{Primus} &= 128^2\cdot173^2\cdot1249\cdot47609 = 128^2\cdot173^2 \\ \text{Secundus} &= 5^2\cdot173^2\cdot1249^2\cdot47609 = 5^2\cdot173^2\cdot1249 \\ \text{Tertius} &= 36^2\cdot1249\cdot47609^2 = 36^2\cdot47609 \end{aligned}\right| \text{ in } 1249\cdot47609$$

quibus numeris evolutis erit

Primus $= 490356736\cdot59463641$

Secundus $= 934533025\cdot59463641$

Tertius $= 617011264\cdot59463641$

quorum productum manifesto est quadratum 2 quippe

$$5^2\cdot36^2\cdot128^2\cdot173^4\cdot1249^4\cdot47609^4.$$

Summa autem reperitur

$$25\cdot59463641^2$$

et summa productorum ex binis:

$$173^2\cdot59463641^2\cdot18248924559376$$

cuius radix quadrata est $173\cdot59463641\cdot4271876$

11. Pro eadem aequatione resolvenda poni potest $Q = 2tt - \frac{5}{2}t - \frac{9}{16}$, ut tres primores termini tollantur, ac prodibit

$$-8t = +\tfrac{45}{16}t + \tfrac{81}{256}, \text{ seu } 0 = 173t + \tfrac{81}{16}, \text{ ergo } t = \tfrac{-81}{16\cdot173}.$$

Hinc $Q = \frac{81^2}{128\cdot173^2} + \frac{405}{32\cdot173} - \frac{9}{10} = \frac{9\cdot207563}{128\cdot173^2}$ et $\frac{v}{u} = \pm\frac{144\cdot173}{207563}$. Sumi enim potest valor ipsius Q tam negative quam positive. Statuatur ergo

$$v = -144\cdot173; u = 207563; \text{ erit } x = 9\cdot81 = 729$$

et $y = \frac{uu+vv}{729}$; unde iam manifestum est, ad tam enormes perveniri numeros, ut solutio praecedens prae hac multo simplicior sit aestimanda. Superfluum autem foret, huiusmodi solutiones nimis complicatas ulterius evolvere, quia in huius generis quaestionibus solutione simplicissima plerumque contenti esse solemus.

Casus 2. quo ponitur $r = \frac{3}{2}$.

12. Hac positione ultimus formulae nostrae terminus sit quadratum, eritque $\frac{v}{u} = \frac{12t-5}{4Q}$, existente

$$QQ = 6t^4 - \frac{41}{2}t^3 + \frac{27}{2}tt - \frac{35}{2}t + \frac{25}{4}.$$

Iam, ad tres terminos ultimos tollendos, statuatur

$$Q = \frac{5}{2} - \frac{7}{2}t + \frac{1}{4}tt, \text{ eritque}$$

$$6t^4 - \frac{41}{2}t^3 = \frac{1}{16}t^4 - \frac{7}{4}t^3 \text{ et } t = \frac{60}{19}$$

hincque $Q = \frac{4379}{722}$ et $\frac{v}{u} = \frac{19}{14}$, unde $v = 19$ et $u = 14$. Nunc igitur erit $x = tv = 60$; et $y = \frac{vv+uu}{x} = \frac{557}{60}$ ideoque $x + y = \frac{4157}{60}$ et tres numeri quaesiti:

Primus $= \frac{60^2 \cdot 557 \cdot 4157 \cdot 196}{60 \cdot 60} mm = 14^2 \cdot 60^2 \cdot 557 \cdot 4157$

Secundus $= \frac{60 \cdot 557^2 \cdot 4157 \cdot 196}{60 \cdot 60 \cdot 60} mm = 14^2 \cdot 557^2 \cdot 4157$

Tertius $= \frac{361 \cdot 60 \cdot 557 \cdot 4157^2}{60 \cdot 60 \cdot 60} mm = 19^2 \cdot 557 \cdot 4157^2$

posito $m = 60$: hique numeri iam notabiliter sunt minores quam ii, qui casu primo sunt inventi.

13. Quoniam ergo hi numeri ob parvitatem attentione digni videntur, ii ita exhibeantur:

Primus $= 705600 \cdot 2315449$

Secundus $= 109172 \cdot 2315449$

Tertius $= 1500677 \cdot 2315449$

Quorum numerorum summa est $= 2315449^2$, et productum $= 14^4 19^2 60^2 557^4 4157^4$, sicque uterque numerus quadratus.

Ac summa productorum ex binis erit

$$(14^2 \cdot 60^2 \cdot 14^2 \cdot 557 + 14^2 \cdot 60^2 \cdot 19^2 \cdot 4157 + 14^2 \cdot 557 \cdot 19^2 \cdot 4157)2315449^2$$

quae reducitur ad hanc formam:

$$14^2 \cdot 2315449^2 \cdot 6631333489$$

cuius radix quadrata est

$$14 \cdot 2315449 \cdot 81433.$$

Sunt autem hi numeri circiter 15000 vicibus minores, quam primum inventi.

Casus 3. quo ponitur $r = 3$.

14. Posito $r = 3$, fit $\frac{v}{u} = \frac{6t-8}{Q}$, et habebitur haec aequatio resolvenda:

$$QQ = 12t^4 - 70t^3 + 84tt - 64t + 64.$$

Iam ad ternos ultimos terminos tollendos statuatur

$$Q = 8 - 4t + \frac{17}{4}tt, \text{ eritque}$$

$$12t^4 - 70t^3 = \frac{289}{16}t^4 - 34t^3$$

unde elicitur $t = -\frac{576}{97}$. et $Q = \pm\frac{8 \cdot 213601}{97 \cdot 97}$ Ergo $\frac{v}{u} = -\frac{97 \cdot 259}{213601} = -\frac{97 \cdot 23}{9287} = -\frac{23 \cdot 97}{37 \cdot 251}$. ideoque $v = -23 \cdot 97$ et $u = 37 \cdot 251$: tum $x = tv = 23 \cdot 24^2$ et $y = \frac{91225730}{23 \cdot 24^2}$. Verum facile perspicitur, hos numeros in immensum excrescere, unde iis evolvendis supersedemus. Contemplemur ergo adhuc unum casum, quo tam primus, quam ultimus terminus formulae QQ fiunt quadrati.

Casus 4. quo ponitur $r = 9$.

15. Posito $r = 9$, fit $\frac{v}{u} = \frac{18t-80}{Q}$, existente

$$QQ = 36t^4 - 538t^3 + 1716tt - 520t + 1600$$

Tollamus terminos primum et duos ultimos, ponendo

$$Q = 40 - \tfrac{13}{2}t \pm 6tt, \text{ et habebimus}$$

$$-538t^3 + 1716tt = \mp 78t^3 \pm 480tt + \tfrac{169}{4}tt$$

unde elicimus pro utroque signo
superiori $t = \frac{5\cdot 191}{16\cdot 23}$ | utrinque autem prodeunt
inferiori $t = \frac{5\cdot 1723}{32\cdot 77}$ | numeri nimis magni.
Tollamus ergo tres terminos ultimos, ponendo

$$Q = 40 - \tfrac{13}{2}t + \tfrac{1339}{64}tt;$$

hinc autem numeri multo adhuc maiores resultant. Posset porro pro binis terminis primis cum ultimo tollendis poni $Q = 6tt - \frac{269}{6}t \pm 40$, verum hinc multo minus ad numeros simpliciores perveniemus.

16. Ex his satis tuto concludi posse videtur, minimos numeros problemati satisfacientes esse eos, quos §13 elicuimus, qui ergo, si penitus per multiplicationem evolvantur, erunt:

Primus $= 1633780814400$
Secundus $= 252782198228$
Tertius $= 3474741058973$.

Sin autem in fractionibus numeri satisfacientes simplicissimi desiderentur, ii indidem assignari poterunt, his per 2315449^2 dividendis: ita ut hi numeri futuri sint:

Primus $= \frac{705600}{2351449}$
Secundus $= \frac{196}{4157}$
Tertius $= \frac{361}{557}$.

quorum tam summa, quam summa productorum ex binis, et omnium trium productum, sunt numeri quadrati.