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De summatione innumerabilium progressionum

Leonhard Euler

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DE SUMMATIONE INNUMERABILIVM PROGRESSIONVM.

Auct. L. Eulero.

I.

Quae in praecedente differtatione de progressionibus transcendentibus earumque terminis generalibus tradidi, multo latius patent, quam videri possent; et inter alia quam plurima, ad quae accommodari possunt, eximius earum potest esse vsus in inueniendis summis innumerabilium progressionum. Quemadmodum enim in superiore differtatione innumerae progressionis ad terminos generales sunt reuocatae, quae communem algebra transcendent; ita hic eandem methodum accommodabo ad terminos summatorios inueniendos progressionum, ad quas indefinite summandas communis algebra non sufficit.

§. 2. Progressio quaequam summari dicitur indefinite, si detur formula numerum indefinitum n continens, quae exponat summam tot terminorum illius progressionis, quot n comprehendit unitates, ita vt si ponatur v. gr. $n=10$ ea formula exhibeat summam decem terminorum a primo numeratorum. Formula haec vocatur terminus summatorius illius progressionis, atque est simul

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ter-

terminus generalis progressionis, cuius terminus quicumque aequatur summae tot terminorum illius progressionis, quot eius exponens in se continet unitates.

§. 3. Cum progressionibus quaeque exponantur terminis generalibus, quaestio de summendis progressionibus est haec, ut ex termino generali terminus summatorius inveniatur. Et quidem iam eo est peruentum, ut, quoties terminus generalis est functio rationalis ipsius indicis n exponentes sunt numeri integri affirmativi, semper terminus summatorius inueniri queat. Quando autem exponentes ipsius n sunt negativi, nisi excipiantur pauci casus, nemo adhuc terminos summatorios dedit. Ratio huius difficultatis est, quod tum termini summatorii plerumque algebraice exprimi nequeant, sed tales requirant formas, quae quadraturas in se contineant.

§. 4. Assumatur haec forma $\int \frac{1-x^n}{1-x} dx$, tanquam terminus generalis cuiusdam progressionis; quae scilicet integrata, ita ut fiat $=0$ si $x=0$ positoque $x=1$ daret terminum ordine n . Progressio quae hoc modo ex ea formatur erit haec x , $x + \frac{1}{2}$, $x + \frac{1}{2} + \frac{1}{3}$, $x + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, etc. cuius ergo terminus generalis est formula assumpta $\int \frac{1-x^n}{1-x} dx$

Series

Series vero haec inuenta summatoria est progressionis harmonicae $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ etc. cuius terminus generalis est $\frac{1}{n}$. Quamobrem huius progressionis terminus summatorius erit $\int \frac{1-x^n}{1-x} dx$, qui illius est terminus generalis.

§. 5. Cum terminus generalis progressionis $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3},$ etc. sit $\int \frac{1-x^n}{1-x} dx$. Poterit ex hoc ea progressio interpolari, seu quilibet terminus medius inueniri: vt si requiratur terminus, cuius index est $\frac{1}{2}$, oportebit integrari $\frac{1-\sqrt{x}}{1-x} dx$ vel $\frac{dx}{1+\sqrt{x}}$, cuius integrale est $2\sqrt{x} - 2l(1+\sqrt{x})$ quod cum fiat $= 0$ si $x=0$, ponatur $x=1$, erit terminus ordine $\frac{1}{2} = 2 - 2l2$. Deinde, quia generaliter terminus ordine $n+1$, terminum ordine n superat fractione $\frac{1}{n+1}$, erit terminus ordine $1\frac{1}{2} = 2\frac{2}{3} - 2\sqrt{2}$, et terminus ordine $2\frac{1}{2}$ hic $2 + \frac{2}{3} + \frac{2}{5} - 2l2$, etc. Series igitur interpolata erit

$$2 - 2l2, 1, 2 + \frac{2}{3} - 2l2, 1 + \frac{1}{2}, 2 + \frac{2}{3} + \frac{2}{5} - 2l2, \text{etc.}$$

§. 6. Ad hunc modum rem generalius complexus sum, et assumi formulam $\int \frac{1-P^n}{1-P} dx$, vbi P denotat functionem quamcunque ipsius x . Integrale hoc vt semper ita debet accipi vtposito $x=0$; id totum fiat $= 0$. Deinde hoc facto

non vt ante pono $x=1$, sed vt latius pateat pono $x=k$. Forma hoc modo resultans erit terminus ordine n progressionis cuiusdam, cuius terminus generalis est forma assumpta $\int \frac{1-P^n}{1-P} dx$. Progressio vero ipsa haec erit $k, k+\int P dx, k+\int P dx + \int P^2 dx$, etc. vbi in integralibus $\int P dx, \int P^2 dx$, etc. loco x iam positum esse k pono.

§. 7. Progressio inuenta, si quibus terminus a sequeute subtrahatur, praebebit hanc $k, \int P dx, \int P^2 dx, \int P^3 dx$ etc. Huiusque terminus summatorius aequalis est termino generali praecedentis progressionis, cuius terminus generalis est $\int P^{n-1} dx$, haec formula $\int \frac{1-P^n}{1-P} dx$. Sit $P=x^\alpha: a^\alpha$ erit progressionis huius $k, \frac{k^{\alpha+1}}{(\alpha+1)a^\alpha}, \frac{k^{2\alpha+1}}{(3\alpha+1)a^{2\alpha}}$ etc. terminus generalis $\frac{k^{(n-1)\alpha+1}}{(\alpha+(n-1)\alpha)a^{(n-1)\alpha}}$ atque terminus summatorius hic $\int \frac{a^{n\alpha}-x^{n\alpha}}{(a^\alpha-x^\alpha)a^{n\alpha-\alpha}} dx$.

§. 8. Inuentus ergo est terminus summatorius pro omnibus progressionibus quorum termini sunt fractiones, harumque numeratores progressionem geometricam, denominatores vero arithmeticam constituunt. Vt vero facilius ad omnes casus accommodari possit, sumatur haec progressio,

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fit, $\frac{b}{c}, \frac{b^{i+1}}{c+e}, \frac{b^{2i+1}}{c+2e}, \frac{b^{3i+1}}{c+3e}$, cuius terminus generalis est $\frac{b^{(n-1)i+1}}{c+(n-1)e}$, comparet ut hic cum illo

$$\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}} \text{ vel } \frac{ck^{(n-1)\alpha+1}}{(c+(n-1)\alpha)a^{(n-1)\alpha}} \text{ erit } a = \frac{e}{c} \text{ et } \frac{ck^{\frac{(n-1)e}{c}+1}}{\frac{(n-1)e}{c}} = b^{(n-1)i+1} \text{ atque } a = \frac{b^{\frac{(n-1)e}{c}+1}}{ck^{\frac{(n-1)e}{c}+1}}$$

$\left(\frac{ck^{\frac{(n-1)e}{c}+1}}{b^{(n-1)i+1}}\right)^{\frac{c}{(n-1)e}} = \left(\frac{ck}{b}\right)^{\frac{c}{(n-1)e}} \frac{k}{b^{ci:e}}$. Hic ne a pendeat ab n , debet enim a esse constans quantitas, oportet ut $\frac{ck}{b}$ fit $= 1$, erit ergo $k = \frac{b}{c}$, atque $a = \frac{b^{\frac{e-ci}{c}}}{c}$. Quocirca terminus summatorius est

$$\int \frac{b^{\frac{ne-ni}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)(e-ci)}{c}} (b^{\frac{e-ci}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} dx. \text{ Quae ita debet integrari ut fiat } = 0 \text{ si } x = 0, \text{ tum vero ponere oportet } x = \frac{b}{c}.$$

§. 9. Cognita summa progressionis indefinita habebitur summa progressionis in infinitum, si ponatur $n = \infty$. Terminus quidem summatorius inuentus non magis ad hunc casum quam ad aliumquemque accommodatus videtur. Est mihi vero alia methodus summas serierum infinitarum investigandi, quae latissime patet. Sit series $\frac{b}{c} + \frac{b^{i+1}}{c+1} + \frac{b^{2i+1}}{c+1}$

b^{2i+1}
 $c + 2e$ etc. Ponatur numerus terminorum n , et summa eorum A . Augeatur numerus n unitate, augetur summa A termino ordine $n+1$, qui est b^{n+1}
 $c + ne$. Si nunc n et A tanquam quantitates fluentes considerentur, quia n est quasi infinites maior quam 1, erunt earum differentialia dn et dA inter se ut augmenta 1 et $\frac{b^{n+1}}{c + ne}$. Vnde pro-

dit aequatio $dA = \frac{b^{n+1} dn}{c + ne}$. Quae integrata dabit aequationem inter summam A et numerum terminorum n .

§. 10. Ponatur $l(c + ne) = z$, erit $\frac{e dn}{c + ne} = dz$ atque $e + ne = g^z$ denotante g numerum, cuius logarithmicus est 1. Est ergo $n = \frac{g^z - c}{e}$ et b^{n+1}

$$= b^{\frac{g^z - c}{e} + 1} = b^{\frac{e - ci}{e}} b^{\frac{g^z}{e}}$$

$$= \frac{b^{\frac{e - ci}{e}} b^{\frac{g^z}{e}} dz}{e}$$

Haec quidem aequatio ita generaliter instituta integrationem nisi per series non admittit. Si vero ponatur $i = 0$, ut prodeat series

$$\frac{b}{c} + \frac{b}{c+e} + \frac{b}{c+2e} + \text{etc.},$$

$$= \frac{b}{e} dz \text{ et } A = \frac{b}{e} (z + lC) = \frac{b}{e} lC(c + ne).$$

Constantis quidem C non determinatur, sed tamen aequa-

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aequatio ad definiendam differentiam inter duas summas inseruit; ut sit alius numerus terminorum m , et summa B erit, $B = \frac{b}{e} LC(c+me)$. Ergo $A - B = \frac{b}{e} \sqrt{\frac{c+me}{c+ne}} = \frac{b}{e} \sqrt{\frac{m}{n}}$ quia m et n sunt infinita.

§. 11. Maneat $i=0$, et progressio erit haec $\frac{b}{c}, \frac{b}{c+e}, \frac{b}{c+2e}, \frac{b}{c+3e}$ etc. cuius terminus generalis est $\frac{b}{c+(n-1)e}$. Terminus autem summatorius est $\int \frac{b \frac{ne}{c} - c \frac{ne}{c} x \frac{ne}{c}}{b \frac{(n-1)e}{c} (b \frac{e}{c} - c \frac{e}{c} x \frac{e}{c})} dx$. Sumatur alia progressio $\frac{b}{c}, \frac{b}{c+2f}, \frac{b}{c+2f}, \frac{b}{c+3f}$ etc. cuius terminus generalis est $\frac{b}{c+(n-1)f}$ et summatorius $\int \frac{b \frac{nf}{c} - c \frac{nf}{c} x \frac{nf}{c}}{b \frac{(n-1)f}{c} (b \frac{f}{c} - c \frac{f}{c} x \frac{f}{c})} dx$;

in quo integrato itidem ponere oportet $x = \frac{b}{c}$. Addantur haec duae progressionis, scilicet terminus primus primo, secundus secundo, et ita porro prodibit haec progressio $\frac{2b}{c}, \frac{2bc+b(e+f)}{(c+e)(c+f)}, \frac{2bc+2b(e+f)}{(c+2e)(c+2f)}$ etc., cuius terminus generalis est $\frac{2bc+(n-1)b(e+f)}{(c+(n-1)e)(c+(n-1)f)}$. Terminus vero summatorius erit $\int dx \left(\frac{b \frac{ne}{c} - c \frac{ne}{c} x \frac{ne}{c}}{b \frac{(n-1)e}{c} (b \frac{e}{c} - c \frac{e}{c} x \frac{e}{c})} + \frac{b \frac{nf}{c} - c \frac{nf}{c} x \frac{nf}{c}}{b \frac{(n-1)f}{c} (b \frac{f}{c} - c \frac{f}{c} x \frac{f}{c})} \right)$

§. 12. Simili modo, sed vniuersalius, pro termino generali in cuius denominatore n duas tenet dimensiones, inuenitur terminus summatorius, si illius progressionis p cuplum ad huius q cuplum addatur. Obtinebitur hoc modo progres-

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sio, cuius terminus generalis est $\frac{pb}{c+(n-1)e} + \frac{qb}{c+(n-1)f}$
 $= \frac{(p+q)c+(n-1)b(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$. Terminus autem summa-

torius huic termino generali respondens erit

$$\int \frac{p dx}{b^{\frac{(n-1)e}{c}}} \left(\frac{b^{\frac{ne}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}}} \right) + \int \frac{q dx}{b^{\frac{(n-1)f}{c}}} \left(\frac{b^{\frac{nf}{c}} - c^{\frac{nf}{c}} x^{\frac{nf}{c}}}{b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}}} \right)$$

$$= \int dx \left(\frac{pb^{\frac{n(e+f)}{c}} - pb^{\frac{n(e+f)-f}{c}} c^{\frac{f}{c}} x^{\frac{f}{c}} - pb^{\frac{nf}{c}} c^{\frac{ne}{c}} x^{\frac{ne}{c}} +}{b^{\frac{(n-1)(e+f)}{c}}}$$

$$\frac{pb^{\frac{(n-1)f}{c}} c^{\frac{ne+f}{c}} x^{\frac{ne+f}{c}} + qb^{\frac{n(e+f)}{c}} - qb^{\frac{n(e+f)-e}{c}} c^{\frac{e}{c}} x^{\frac{e}{c}}}{(b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})}$$

$$- \frac{qb^{\frac{ne}{c}} c^{\frac{nf}{c}} x^{\frac{nf}{c}} + qb^{\frac{(n-1)e}{c}} c^{\frac{nf+e}{c}} x^{\frac{nf+e}{c}}}{(b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}})} \right). \text{ Ponatur } b=1$$

hoc enim modo vniuersalitati nihil decedit, erit-
 que terminus generalis $\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$. Sit cx

$=y$, erit $dx = \frac{dy}{c}$. Atque terminus summatorius ha-

$$\text{betur } = \int \frac{dy}{c} \left(\frac{p+q - py^{\frac{f}{c}} - qy^{\frac{e}{c}} - py^{\frac{ne}{c}} - qy^{\frac{nf}{c}} + qy^{\frac{ne+f}{c}} + qy^{\frac{nf+e}{c}}}{(1-y^{\frac{e}{c}})(1-y^{\frac{f}{c}})} \right)$$

in qua formula integrata, ita vt. posito $y=0$ ea
 quoque fiat $=0$, oportet ponere $y=1$.

§. 13. Assumatur iam terminus generalis hic

$\frac{\alpha+\beta n}{\gamma+\delta n+\epsilon n^2}$. Qui comparatus cum $\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$

dabit $c = \sqrt{(\gamma+\delta+\epsilon)}$, $e = \frac{\delta+2\epsilon+\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}}$, $f =$

$\frac{\delta+2\epsilon-\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}}$ $p = \frac{\alpha\delta-\beta\delta+2\alpha\epsilon-2\beta\gamma+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}(\delta\delta-4\gamma\epsilon)}$ atque

$q = \frac{\beta\delta-\alpha\delta+2\beta\gamma-2\alpha\epsilon+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\epsilon)}}{2\sqrt{(\gamma+\delta+\epsilon)}(\delta\delta-4\gamma\epsilon)}$. His in termino sum-

matorio substitutis, prodibit terminus summatorius

huius

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huius progressionis $\frac{\alpha+\epsilon}{\gamma+\delta+\epsilon}$, $\frac{\alpha+2\epsilon}{\gamma+2\delta+4\epsilon}$, $\frac{\alpha+3\epsilon}{\gamma+3\delta+9\epsilon}$,
 etc. cuius terminus generalis est $\frac{\alpha+\epsilon n}{\gamma+\delta n+\epsilon n^2}$.

§. 13. Eodem modo si in termino generali n plures duabus dimensiones habuerit, eruetur terminus summatorius, combinandis tot progressionibus simplicibus, quot dimensiones n habere debet, quemadmodum idem in casu duarum dimensionum factum est. Attamen hac ratione non ad quasvis, quae in huiusmodi terminis generalibus contineri videntur, series perueniri potest. Nam quoties denominator $\gamma+\delta n+\epsilon n^2+\zeta n^3+\eta n^4$ etc. duos pluresue habet factores simplices aequales, tum progressio in tot simplices progressiones resolui nequit, neque igitur eius terminus summatorius inueniri.

§. 14. Hanc ob rem aliam tradam methodum, quae hos casus non excludat. Sit progressio quaedam simplex $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{1}{a+2b}$ etc. cuius terminus generalis est $\frac{1}{a+(n-1)b}$. Huius terminus

summatorius erit $\int \frac{1-a\frac{nb}{a}x\frac{nb}{a}}{1-a\frac{b}{a}x\frac{b}{a}} dx$, vel ponatur ax

$=y$, erit is $\int \frac{1-y\frac{nb}{a}}{1-y\frac{b}{a}} \frac{dy}{a}$, in quo integrato poni

oportet $y=1$. Multiplicetur hic in $y^a dy$ et summa huius facti $\int y^a dy \int \frac{dy}{a} \left(\frac{1-y\frac{nb}{a}}{1-y\frac{b}{a}} \right)$ erit secundum

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modum

modum descriptum tractata terminus summatorius huius progressionis $\frac{a}{a\delta a}$, $\frac{a}{(a+b)(\delta a+b)}$, $\frac{a}{(a+2b)(\delta a+2b)}$ etc. positō breuitatis ergo δ loco $\alpha+2$. Huius progressionis terminus generalis est $\frac{a}{(a+(n-1)b)(\delta a+(n-1)b)}$

vel
$$\frac{a}{b^2 n^2 + (ab + \delta ab - 2bb)n + (a-b)(\delta a - b)}$$

§. 15. Assumamus progressionem generalem huius generis, quae facilius ad casus quosuis adaptatur, sit eius terminus generalis

Hic cum illo termino generali comparatus dabit
$$a = \frac{(2b-c)^2 - 4ac + (2b-c)\sqrt{(2b-c)^2 - 8ac}}{2}$$

$$b = \frac{2b-c + \sqrt{(2b-c)^2 - 8ac}}{4}, \quad \delta = \frac{2b-c - \sqrt{(2b-c)^2 - 8ac}}{2b-c + \sqrt{(2b-c)^2 - 8ac}}$$
 Hi valores si substituantur loco $a, b,$ et $\alpha,$ (est ve-

ro $\alpha = \delta - 2$) in $\int y^\alpha dy \int \frac{dy}{1+a} \left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}} \right)$ prodibit terminus summatorius progressionis propositae, $\frac{1}{a},$ $\frac{1}{a+b},$ $\frac{1}{a+2b+c},$ $\frac{1}{a+3b+3c}$ etc.

§. 16 Hoc modo ulterius progredi licet; multiplicetur $\int \frac{dy}{a} \left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}} \right)$ in $y^{\alpha-2} dy,$ et facti integrale $\int y^{\alpha-2} dy \int \frac{dy}{a} \left(\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}} \right)$ denuo in $y^{\delta-\alpha-1}$ huiusque producti integrale $\int y^{\delta-\alpha-1} dy \int y^{\alpha-2} dy \int \frac{dy}{a}$

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$\left(\frac{1-y\frac{nb}{a}}{1-y\frac{b}{a}}\right)^{\alpha}$ erit terminus summatorius progressio-
 nis huius $\frac{a^{\alpha}}{a^{\alpha} - b^{\alpha}}$, $\frac{a^{\alpha}}{(a+b)(\alpha a+b)(3a+b)\dots}$ etc. cuius terminus generalis est $a^{\alpha} : (a+(n-1)b)$
 $(\alpha a+(n-1)b)(3a+(n-1)b)$ similiter $\int y^{\alpha-1} dy \int y^{\alpha-2} dy \dots$
 $\int y^{\alpha-2} dy \int \frac{dy}{a} \left(\frac{1-y\frac{nb}{a}}{1-y\frac{b}{a}}\right)^{\alpha}$ est terminus summatorius
 progressionis, cuius terminus generalis est
 $\frac{a^{\alpha}}{(a+(n-1)b)(\alpha a+(n-1)b)(3a+(n-1)b)\dots}$. Hoc igitur
 modo ad omnes progressionem pervenitur, quarum termini sunt fractiones, numeratoribus exi-
 stentibus numeris constantibus, denominatoribus autem constituentibus quamcunque progressionem
 algebraicam.

§. 17. Si summae huiusmodi progressionum
 in infinitum continuarum desiderentur, oportet
 ponere $n = \infty$. Hoc posito postremum cu-
 iusque termini summatorii membrum scilicet $\int \frac{dy}{a}$
 $\left(\frac{1-y\frac{nb}{a}}{1-y\frac{b}{a}}\right)^{\alpha}$ transmutabitur in hoc $\int \frac{dy}{a(1-y\frac{b}{a})^{\alpha}}$. Quia
 enim y semper est minus quam 1, praeter ca-
 sum ultimum, quo fit $y = 1$, evanescet $y^{\frac{nb}{a}}$ prae
 1: atque ideo $1-y\frac{nb}{a}$ abit in 1. Propterea hu-
 ius seriei $\frac{a^{\alpha}}{a^{\alpha}} + \frac{a^{\alpha}}{(a+b)(\alpha a+b)} + \frac{a^{\alpha}}{(a+2b)(\alpha a+2b)} + \dots$ etc.
 in infinitum summa erit $\int y^{\alpha-2} dy \int \frac{dy}{a(1-y\frac{b}{a})^{\alpha}}$, et

huius $\frac{a^2}{a \cdot a \cdot 6a} + \frac{a^2}{(a+b)(aa+c)(6a+c)} + \frac{a^2}{(a+2b)(aa+2b)(6a+2b)}$
 + etc., summa erit $\int y^{6-a-1} y^{\alpha-2} dy \int \frac{dy}{a(1-y\frac{b}{a})}$
 et ita de reliquis omnibus.

§. 18. Sit $b=a$, ut fiat $\frac{b}{a}=1$, erit $\int \frac{dy}{a(1-y)}$
 $= A - \frac{1}{a} l(1-y)$. Quia posito $y=0$ totum integrale
 fieri debet $=0$, erit $A=0$, adeoque $\int \frac{dy}{a(1-y)} =$
 $-\frac{1}{a} l(1-y)$. Multiplicetur hoc in $y^{\alpha-2} dy$ habebi-
 tur $-\frac{y^{\alpha-2} dy}{a} l(1-y)$. Huius integrale ut inuenia-
 tur ponatur $1-y=z$, erit $y=1-z$, habebitur igitur
 integrandum $\frac{(1-z)^{\alpha-2} dz}{a} l z = (1 - \frac{\alpha-2}{1} z +$
 $\frac{(\alpha-2)(\alpha-3)}{1 \cdot 2} z^2 - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1 \cdot 2 \cdot 3} z^3 + \text{etc.}) \frac{dz}{a} l z$. Quia
 vero $z^\eta dz l z = C - \frac{z^{\eta+1}}{(\eta+1)^2} + \frac{z^{\eta+1}}{\eta+1} l z$, erit illius
 integrale haec series $\frac{1}{a} (C - z + z l z + \frac{(\alpha-2)}{1 \cdot 4} z^2 -$
 $\frac{(\alpha-2)}{1 \cdot 2} z^2 l z - \frac{(\alpha-2)(\alpha-3)}{1 \cdot 2 \cdot 9} z^3 + \frac{(\alpha-2)(\alpha-3)}{1 \cdot 2 \cdot 3} z^3 l z + \text{etc.})$
 Hoc integrale si fiat $y=0$ seu $z=1$ debet fieri
 $=0$, hanc ob rem erit $C = 1 - \frac{(\alpha-2)}{1 \cdot 4} + \frac{(\alpha-2)(\alpha-3)}{1 \cdot 2 \cdot 9}$
 $- \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1 \cdot 2 \cdot 3 \cdot 16} \text{ etc.}$

§. 19. Perspicuum est ex hoc integrali, quo-
 ties a sit numerus integer unitate maior, tum
 semper integralis eius terminorum numerum fo-
 re finitum, atque ideo summam progressionis de-
 finiri. Attamen etiam si terminorum numerus sit
 infi-

ininitus, summa propositae seriei dabitur per aliam seriem infinitam quae vero plerumque magis conuergit quam proposita, atque ideo perquam est utilis ad summam determinandam.

§. 20. Sit summa progressionis in infinitum continuatae $\int \frac{-y^{\alpha-2}}{a} dy l(1-y)$, quia hic est posi-

tum $b=a$, erit progressio ipsa $\frac{1}{\alpha a} + \frac{1}{2(\alpha+1)a} + \frac{1}{3(\alpha+3)a} + \frac{1}{4(\alpha+4)a}$ etc. Huius summa habetur si in illo integrali ponitur $y=1$, sed facto $y=1-z$ est integrale illud $\frac{1}{\alpha} (1 - \frac{(\alpha-2)}{1.4} z^2 + \frac{(\alpha-2)(\alpha-3)}{1.2.9} z^3 - \frac{(\alpha-2)}{1.2} z^2/z + \frac{(\alpha-2)(\alpha-3)}{1.2.3} z^3/z - \text{etc.})$. Si iam fiat $y=1$ vel $z=0$ erit summa seriei $\frac{1}{\alpha a} + \frac{1}{2(\alpha+1)a} + \frac{1}{3(\alpha+2)a}$ etc. aequalis summae huius seriei $\frac{1}{\alpha} - \frac{(\alpha-2)}{1.4.a} + \frac{(\alpha-2)(\alpha-3)}{1.2.9.a}$ etc. vel summa huius $\frac{1}{\alpha} + \frac{1}{2(\alpha+1)} + \frac{1}{3(\alpha+2)}$ etc. aequalis summae huius $1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9} - \text{etc.}$

§. 21. Praeterea alium habeo modum series valde conuergentes inueniendi, quarum summa aequalis sit seriei propositae. $\int -y^{\alpha-2} dy l(1-y)$ aequatur ita integratum vt fiat $=0$ si $y=0$ huic seriei $1 - \frac{(\alpha-2)}{1.4} + \frac{(\alpha-2)(\alpha-3)}{1.2.9} - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.16} + \text{etc.}$
 $-z + \frac{(\alpha-2)}{1.4} z^2 - \frac{(\alpha-2)(\alpha-3)}{1.2.9} z^3 + \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.16} z^4 - \text{etc.}$
 $+ z/z - \frac{(\alpha-2)}{1.2} z^2/z + \frac{(\alpha-2)(\alpha-3)}{1.2.3} z^3/z - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.4} z^4/z + \text{etc.}$ existente $z=1-y$; sed cum sit $-l(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} \text{ etc.}$ erit $\int -y^{\alpha-2} dy l(1-y) =$

$\frac{y^a}{a} + \frac{y^{a+1}}{2(a+1)} + \frac{y^{a+2}}{3(a+2)}$ etc. Haec series si a est numerus affirmativus est aequalis illi quicquid sit y , et ita multis modis series eiusdem summae reperiuntur, quarum altera alterius ope facilius summatur.

§. 22. Exemplo rem illustrabo fit $a=1$, habebitur trium sequentium serierum summa aequalis huic

$$+1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \text{ etc.} \\ -z - \frac{1}{4}zz - \frac{1}{9}z^3 - \frac{1}{16}z^4 \text{ etc.} \\ +z/z + \frac{1}{2}z^2/z + \frac{1}{3}z^3/z \text{ etc.}$$

Quia vero est $z + \frac{1}{2}zz + \frac{1}{3}z^3 \text{ etc.} = \frac{1}{1-z} = \frac{1}{(1-z)} =$
 $-ly$ erit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{y+z}{1} + \frac{y^2+z^2}{4} +$
 $\frac{y^3+z^3}{9} + \frac{y^4+z^4}{16} + \text{etc.} + ly/z$, est hic $y+z=1$,
 et manifestum est tales loco y vel z numeros assumi posse, ut series maxime convergat. Id vero evenit quando $y=z$ vel utrumque $=\frac{1}{2}$, eritque hoc casu $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = 1 + \frac{1}{8} +$
 $\frac{1}{32} + \frac{1}{128} + \frac{1}{512} + \text{etc.} = \frac{1}{2}$. Hoc modo summa progressionis $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$ valde prope haberi potest, est enim $\frac{1}{2} = \frac{1}{2} + \frac{1}{4} +$
 $\frac{1}{8} + \frac{1}{16}$. Summa progressionis $1 + \frac{1}{8} + \frac{1}{36} \text{ etc.}$ est quam proxime $=1$, 164481 et $\frac{1}{2} = 0,$
 480453, ergo summa seriei $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$
 etc. est $=1$, 644934 *q.p.* Si quis autem huius seriei summam addendis aliquot terminis initialibus determinare voluerit, plusquam mille terminos

nos

nos addere deberet, quo nostrum inuentum numerum reperiret.

§. 23. Ex his igitur methodum percipere licet, quomodo cuiuslibet progressionis, cuius termini sunt fractiones, quarum denominatores constituunt progressionem quamcunque algebraicam, terminum summatorium inueniri oporteat. Equidem vt hic rem considerauimus, numeratores deberent esse quantitates constantes; sed non difficulter haec methodus extendetur ad eas quoque progressionem etiam quamcunque algebraicam faciunt. Propterea haec methodus ad omnes progressionem quarum termini generales algebraice possunt exponi, accommodari potest, eiusque opetermini summatorii inueniri. Excipiendi tamen sunt casus, quibus terminus generalis irrationalis est.