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Euler's three-body problem

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Abstract

In physics and astronomy, Euler's three-body problem is to solve for the motion of a body that is acted upon by the gravitational field of two other bodies. This problem is named after Leonhard Euler (1707-1783), who discussed it in memoirs published in the 1760s. In these publications, Euler found that the parameter that controls the relative distances among three collinear bodies is given by a quintic equation. Later on, in 1772, Lagrange dealt with the same problem, and demonstrated that for any three masses with circular orbits, there are two special constant-pattern solutions, one where the three bodies remain collinear, and the other where the bodies occupy the vertices of two equilateral triangles. Because of their importance, these five points became known as Lagrange points. The quintic equation found by Euler for the relative distances among the collinear bodies was also found later by Lagrange, and because of that, Euler has also been given credit for the discovery of the three collinear Lagrange points. A practical application of the collinear points for satellite location is also presented.

1. Introduction

In celestial mechanics, the Lagrange points (also Lagrangian points, L-points, or libration points) are points near two large orbiting bodies. Normally, the two objects exert an unbalanced gravitational force at a point, altering the orbit of whatever is at that point. At the Lagrange points, the gravitational forces of the two large bodies and the centrifugal force balance each other. This can make Lagrange points an excellent location for satellites, as few orbit corrections are needed to maintain the desired orbit. Small objects placed in orbit at Lagrange points remain in equilibrium relative to the large bodies.

Of the five Lagrange points, three are unstable and two are stable. The unstable Lagrange points - labeled L1, L2 and L3 - lie along the line connecting the two large masses. The stable Lagrange points - labeled L4 and L5 - form the apex of two equilateral triangles that have the large masses at their vertices. L4 leads the orbit of the Earth and L5 follows (Fig. 1).

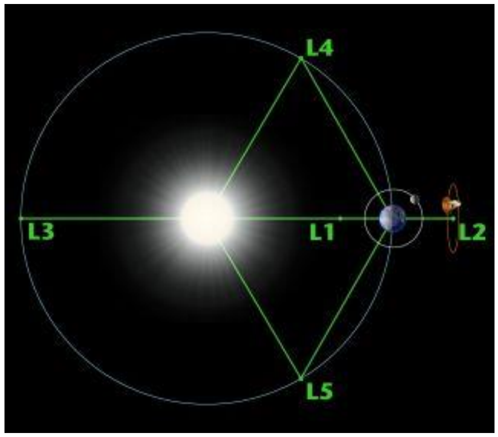

Figure 1: Diagram of the Lagrange points associated with the Sun-Earth system. (Image: © NASA / WMAP Science Team)

Lagrange points are named in honor of Italian-French mathematician Joseph-Louis Lagrange (1736-1813). This mathematical problem, known as the 'General Three-Body Problem' was considered by Lagrange in his prize winning paper *Essai sur le Problème des Trois Corps, 1772* [1]. In the first chapter he considered the general three-body problem. From that, in the second chapter, he demonstrated two special constant-pattern solutions, the collinear and the equilateral, for any three masses, with circular orbits. Lagrange states in the *Avertissement* of his essay that: "…The purpose of the second Chapter is to examine how and in what case the three Bodies could move so that their distances were always constant, or at least kept relations between them constant. I find that these conditions can only take place in two cases: one, when the three Bodies are arranged in the same straight line, and the other, when they form an equilateral triangle…"

Despite the fact that Euler never really advertised the existence of these points, the three collinear Lagrange points (L1, L2, L3) are generally also attributed to Leonhard Euler (1707-1783), because, as we shall see below, the developments in two of his publications, allow the determination of the positions of the three masses in a collinear configuration.

In physics and classical mechanics, the three-body problem is the problem of taking the initial positions and velocities (or momenta) of three point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation. The three-body problem is a special case of the n-body problem. Unlike two-body problems, no general closed-form solution exists, as the resulting dynamical

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system is chaotic for most initial conditions, and numerical methods are generally required.

Historically, the first specific three-body problem to receive extended study was the one involving the Moon, the Earth, and the Sun. In an extended modern sense, a three-body problem is any problem in classical mechanics or quantum mechanics that models the motion of three particles.

2. Euler's Quintic Equation to Find the Three Collinear Lagrange Points

In E327 -- *De motu rectilineo trium corporum se mutuo attrahentium,* 1767 [2] (On the rectilinear motion of three bodies mutually attracting each other), Euler considers the problem of three bodies lying on a straight line, and which are attracted to each other by central forces inversely proportional to the square of their separation distance (inverse-square law).

Perhaps the most important contribution of this publication was to show that there is a parameter, here called $\propto$, that controls the relative distances among the three bodies, and that can be found from the solutions of a fifth degree equation (quintic equation) given by

$$(A+B)\propto^5 + (3A+2B)\propto^4 + (3A+B)\propto^3 - (B+3C)\propto^2 - (2B+3C)\propto - B - C = 0,$$

where A, B, C are the masses of three bodies (Fig. 2). In this publication, Euler shows that there is a unique positive root of this equation, which, as will be shown later, allows the determination of the relative distances among the three bodies.

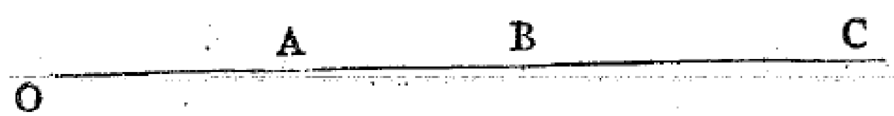

Figure 2: Three bodies aligned in a straight line. (From Euler, 1767 [2])

In E400 -- *Considérations sur le problème des trois corps*, 1770 [3] (Considerations on the three-body problem), with a similar content, Euler not only obtains the same quintic equation for the case of three bodies, but also expands the formulation considering the case of four aligned bodies.

Nonetheless, these developments do not consider cases where the bodies are in orbital motion. It is only in E304 -- *Considerationes de motu corporum coelestium*, 1766 [4] (Considerations on the motion of celestial bodies) that Euler deals with three celestial bodies (Sun, Earth and Moon), in which he also found a similar type of quintic equation controlling the relative distances among these planets. An important conclusion of this publication, is that if the Moon were four times more distant from the Earth (either in conjunction or in opposition), a motion of this kind would have been possible to exist, such that the Moon would always appear connected to the Sun, with these three celestial bodies in perpetual alignment in syzygy — three bodies on a straight line, executing orbital motions. Since then, other proofs of the validity of Euler's quintic equation for bodies in orbital motion have appeared in the specialized literature (see for instance Ref. 5).

3. Practical Application – Satellite Location

The next paragraphs reproduce the very interesting characteristics of the Lagrange points for satellite location provided by NASA at https://wmap.gsfc.nasa.gov/mission/observatory_l2.html, Retrieved on March 26 2021.

"The L1 point of the Earth-Sun system affords an uninterrupted view of the sun and is currently home to the Solar and Heliospheric Observatory Satellite SOHO. The L2 point of the Earth-Sun system was the home to the WMAP spacecraft, current home of Planck, and future home of the James Webb Space Telescope. L2 is ideal for astronomy because a spacecraft is close enough to readily communicate with Earth, can keep Sun, Earth and Moon behind the spacecraft for solar power and (with appropriate shielding) provides a clear view of deep space for our telescopes. The L1 and L2 points are unstable on a time scale of approximately 23 days, which requires satellites orbiting these positions to undergo regular course and attitude corrections."

"The L4 and L5 points are home to stable orbits so long as the mass ratio between the two large masses exceeds 24.96. This condition is satisfied for both the Earth-Sun and Earth-Moon systems, and for many other pairs of bodies in the solar system. Objects found orbiting at the L4 and L5 points are often called Trojans… There are hundreds of Trojan Asteroids in the solar system. Most orbit with Jupiter, but others orbit with Mars. In addition, several of Saturn's moons have Trojan companions."

"In 1956 the Polish astronomer Kordylewski discovered large concentrations of dust at the Trojan points of the Earth-Moon system. The DIRBE instrument on the COBE satellite confirmed earlier IRAS observations of a dust ring following the Earth's orbit around the Sun. The existence of this ring is closely related to the Trojan points, but the story is complicated by the effects of radiation pressure on the dust grains. In 2010 NASA's WISE telescope finally confirmed the first Trojan asteroid (2010 TK7) around Earth's leading Lagrange point."

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A practical application of these results is to find the collinear Lagrange points associated with the Earth-Moon system for satellite location. Figure 3 provides a graphical interpretation of the collinear Lagrange points, for the application of the quintic equation. Here, L1, L2, L3 are distances measured from the barycenter of the Earth-Moon system which is located inside the Earth at about 4,700 km from the Earth's center.

This exercise considers that the Earth-Moon distance equals $384{,}400\ km$, the mass of the Earth equals $5.98 \times 10^{24} kg$, the mass of the Moon equals $7.34 \times 10^{22} kg$, and that the mass of the satellite is null in comparison to the two other masses. Then, the three collinear Lagrange points can be found from the quintic equation as follows:

- In the first configuration, the satellite is placed between the Earth and the Moon, for which, $A = m_1 = m_{Earth}$, $B = m_2 = 0$, and $C = m_3 = m_{Moon}$, giving $\propto_1 = 0{,}17737$, and $L_1 = \rho = \frac{d_{Earth-Moon}}{1+\propto_1} = 326{,}490\ km$.
- In the second configuration, the Moon is between the Earth and the satellite, for which, $A = m_1 = m_{Earth}$, $B = m_2 = m_{Moon}$, and $C = 0$, giving $\propto_2 =$ 0,167417, and $L_2 = \rho(1 + \propto_2) = d_{Earth-Moon}(1 + \propto_2) = 448{,}755\ km$.
- In the third configuration, the Earth is between the satellite and the Moon, for which, $A = m_1 = 0$, $B = m_2 = m_{Earth}$, and $C = m_3 = m_{Moon}$, giving $\propto_3 =$ 1,00711, and $L_3 = \rho = \frac{d_{Earth-Moon}}{\propto_3} = 381{,}686\ km$.

These collinear Lagrange points can also be obtained by the following approximate expressions that have been proposed (see for instance https://www.mat.univie.ac.at/~westra/lagrangepoints.pdf, Retrieved on July 31 2021):

$$L_1 = d_{Earth-Moon}\left(1 - \sqrt[3]{\frac{m_{Moon}}{3m_{Earth}}}\right) = 323{,}043\ km.$$

$$L_2 = d_{Earth-Moon}\left(1 + \sqrt[3]{\frac{m_{Moon}}{3m_{Earth}}}\right) = 445{,}757\ km.$$

$$L_3 = d_{Earth-Moon}\left(1 + \frac{5m_{Moon}}{12m_{Earth}}\right) = 386{,}366\ km.$$

These approximate results to find the collinear Lagrange points, favorably agree with the more exact results given above by the quintic equation.

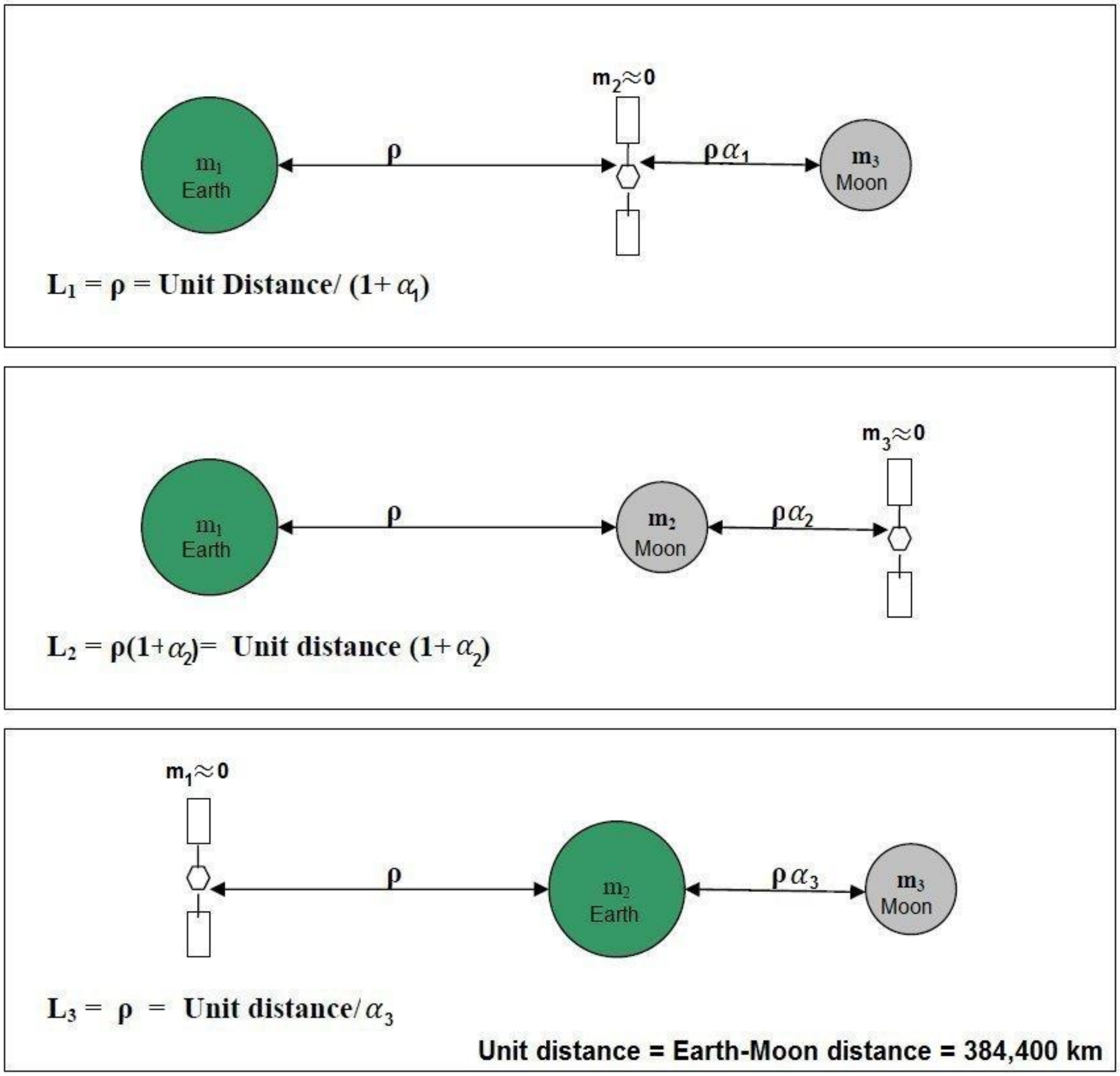

Figure 3: Graphical interpretation of the collinear Lagrange points, for the application of the quintic equation (Adapted from: Problem Set 1 Solution, Satellite Engineering, David W. Miller and John E. Keesee. Available at https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-851-satellite-engineering-fall-2003/assignments/ps1_nn_solution.pdf, Retrieved on July 31 2021)

4. Conclusion

The present work highlights Euler's contribution in E327 to the discovery of the three collinear Lagrange points, showing their importance in practical applications such as the location of satellites. While a deeper analysis would be required to determine Euler's ultimate influence on Lagrange's later work in this area, we see here that Euler laid the groundwork for an application of mechanics to the three-body problem, at least in part.

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[5] Geiges, H. (2016). *The Geometry of Celestial Mechanics*. Cambridge University Press, Cambridge, UK.