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# Euler, Father of Haemodynamics

Sylvio R. Bistafa
*University of Sao Paulo*, sbistafa@usp.br



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# Euler, Father of *Haemodynamics*

Sylvio R. Bistafa
University of São Paulo, Brazil
sbistafa@usp.br

In 1742, the Dijon Academy launched its first contest with the title "To determine the difference in velocities between a liquid that flows through elastic and rigid tubes" (*Déterminer la différence des vitesses d'un liquide qui passe par des tuyaux inflexibles et de celui qui passe par des tuyaux élastiques*), in which Leonhard Euler submitted a manuscript (presumably) with the title *Principia pro motu sanguinis per arterias determinando*. Truesdell [1] traces the mishaps of this publication, indicating that: "[…] While it was the academy's principle to retain all manuscripts, the box on whose label 1742 appears contains no memoir earlier than 1765." *Principia pro motu sanguinis per arterias determinando* appeared in print much later, in 1862, and in fact dated 1775.

As originally published in Euler's *Opera Postuma* [2], E855 *Principia pro motu sanguinis per arterias determinando* of 1775 is an incomplete manuscript, lacking paragraphs 1 to 14, whereas the complete version appeared in Euler's *Opera Omnia* [3] (its Preface contains details about the missing parts). In fact, although in the introduction to the problem (paragraphs 1 – 8), Euler focuses on the discussion on blood flow through arteries, to such a degree to even proposing *adhoc* models for the behavior of their elastic cross-sections, the main body of the manuscript (paragraphs 9 – 34) is devoted to the modeling of flows through rigid tubes, driven by a piston pump. It is only in the remaining sections of the manuscript (paragraphs 35 – 43) that Euler investigates the formulas for the motion of fluids in elastic tubes, ending the work with a very pessimistic statement that "[…] since there appears to be no way in which this can be completely resolved and the investigation can be considered to transcend human powers, the work will end here".

In E855, Euler applied the principles of mass conservation and momentum conservation to the one dimensional flow of an incompressible fluid through an elastic tube driven by a piston pump simulating the heart (Fig. 1) to obtain, in Euler's notation, the equation

$$\left(\frac{\partial s}{\partial t}\right) + \frac{\partial (vs)}{\partial z} = 0, \tag{1}$$

from the mass conservation principle, and

$$2g\left(\frac{\partial p}{\partial z}\right) + v\left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial t}\right) = 0, \tag{2}$$

from the momentum conservation principle, which it is recognized as the one-dimensional form of what is now known as the Euler equations of fluid dynamics, which are used in inviscid flows calculations.

In these equations $s = s(z,t)$ is the cross-sectional area of the artery, $v = v(z,t)$ is the velocity, and $p = p(z,t)$ is the pressure (in fact, the pressure head), $t$ is time, and $z$ is the coordinate along the tube. Eq. 2 has units of acceleration (force per unit of mass), hence $p$ has units of length, and $2g$ is actually the gravity, due to Euler's peculiar form of writing equations involving forces.

To solve for $s$, $v$ and $p$, Euler had to add an algebraic pressure-area relationship, for which he proposes two expressions

$$s = \frac{\Sigma p}{c+p}, \qquad \text{or} \qquad s = \Sigma\left(1 - e^{-\frac{p}{c}}\right), \tag{3a, b}$$

where $\Sigma$ is the cross-sectional area of the tube at infinite pressure, and $c$ is a constant quantity that depends on the degree of elasticity of the tube. By adopting Eq. 3b, it would result in a simpler expression for the pressure gradient term in Eq. 2. Nonetheless, he makes no further use of either in the analysis.

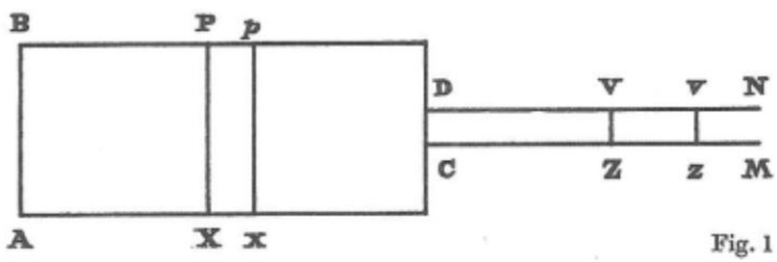


Figure 1: Euler's model for a one-dimensional flow of an incompressible fluid through an elastic tube driven by a piston pump simulating the heart [2].

Since Euler had succeeded in finding a solution for $p$ and $v$ for the rigid tube case, he then tried to do the same for the elastic tube case, but, with no avail.

An approximate solution to Euler's hemodynamic equations can be obtained from the linear theory, which is valid for small amplitude disturbances.

Let $s = s_0 + a$, such that $|a| \ll s_0$, $|v| \ll 1$, and $p \equiv p_e - p_0 = P(s)$, the transmural pressure. Then, $\frac{\partial p}{\partial t} = P'(s_0)\frac{\partial a}{\partial t} + O(a^2)$.

Under the assumption that the artery vessel has uniform cross-section ($s$ is a constant along $z$), and the blood has density $\rho$, then Equations 1 and 2 give

$$\frac{1}{s_0 P'(s_0)}\frac{\partial p}{\partial t} + \frac{\partial v}{\partial z} = 0, \tag{4}$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial t} = 0, \tag{5}$$

on neglecting small quantities. Here $p$ is actually the pressure in pascals.


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Eliminating $v$ gives

$$\frac{\partial^2 p}{\partial t^2} = c_0^2 \frac{\partial^2 p}{\partial z^2}, \tag{6}$$

where $c_0^2 = (\rho D_0)^{-1}$, in which $D_0$ is the distensibility of the artery vessel, defined as $\frac{d}{dp}\left(\frac{a}{s_0}\right) = {D_0}^{-1}$, or equivalently $D_0 = \frac{1}{s_0 P'(s_0)}$ .

Equation 6 is recognized as the wave equation, which admits the so-called D'Alembert solution[1] in the forms

$$\begin{aligned} p &= f(z \pm c_0 t\,), \\ v &= (\rho c_0)^{-1} f(z \pm c_0 t\,), \\ a &= (\rho {c_0}^2)^{-1} f(z \pm c_0 t\,), \end{aligned} \tag{7}$$

where $f(z \pm c_0 t\,)$ is the wave form, which propagates with speed $c_0$; the $-$ sign indicates waves travelling in the positive $z$-direction, and the $+$ sign indicates waves travelling in the negative $z$-direction. This linearization is valid provided $|v| \ll c_0$, with $c_0 = \sqrt{\frac{E}{\rho}\frac{h_0}{D_0}}$, , where $E$ is the modulus of elasticity of the material, $h_0$ and $D_0$ are, respectively, the wall thickness and the inside diameter of the artery vessel, and $\rho$ is the density of the blood.

The simulation presented in Figure 2, clearly shows the propagation of a pressure-pulse entering at the inflow, with a non-reflecting boundary condition at the outlet, as obtained in Ref. [6].

[1]In 1747, D'Alembert [4] derived the equation for the shape of a stretched string subject to vibrations in the form $y = \psi(t + s\,) + \Gamma(t - s)$, without explicitly deriving the wave equation. In 1759, Euler [5] instead, not only derived the wave equation, but also gave its solution in the form $y = \phi(x + ct\,) + \psi(x - ct)$, in an essay on the propagation of sound. Nonetheless, the general solution of the wave equation became associated with D'Alembert.

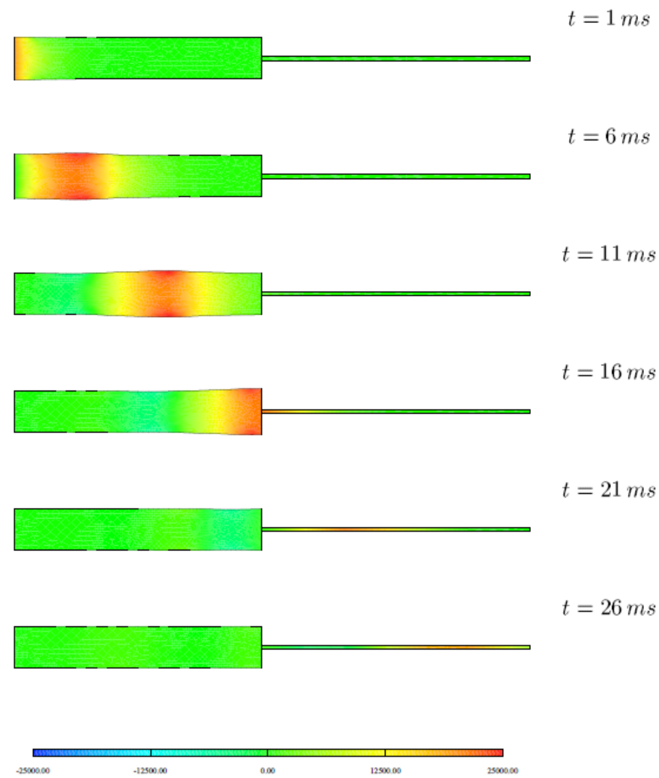


Figure 2: 2-D propagation of a pressure pulse in a compliant contraction. From Quarteroni and Formaggia [6].

For a long time, Euler's hemodynamic model was mainly considered an historical curiosity, because practical applications of his equations require rather advanced numerical computing. It was only in the last decades of the last century that we see a rapid growth in the blood flow analysis, mainly due to the developments in digital computer simulations. In fact, the major advances have been occurring since the beginning of the current century, and it is possible to say that this is an area of considerable research interest, with several publications each year.

The hemodynamic equations form a system of coupled hyperbolic equations, which can be solved using different numerical schemes. These include the method of characteristics, finite volumes, finite differences, and finite elements. A comparison of these methods has shown a good agreement in their ability to capture the main features of pressure, velocity and area waveforms in large arteries [7].

The hemodynamic equations developed by him in 1742, about 275 years ago, still undergird the most advanced numerical methods in use today for blood flow analysis in arterial networks. At the time Euler wrote his essay, the knowledge on flow in elastic tubes had not been yet subjected to mathematical analysis, Euler being the first to propose a plausible model to the problem. Nonetheless, by not recognizing the wave nature of the hemodynamic equations, led Euler to a dead end on trying to find a closed form solution to his equations. It was only in 1759 that Euler himself obtained the wave equation and its associated general solution in an essay on the propagation of sound. The propagation of waves in elastic tubes requires an adequate constitutive relation for the viscoelastic behavior of the walls, which Euler was unable to establish at that time. Then, for more 200 years, Euler's hemodynamic equations remain practically dormant, and it was only in the last decades of the 20th century that blood flow analysis became possible due to the advances in numerical computing. Today, because of the importance on better understanding cardiovascular diseases, blood flow in human arteries is a thriving area of research, and it is possible to say that all the particular features of arterial network can now be rather adequately modeled, which allow the simulation of pulse wave propagation in all the cardiac cycle phases in large arterial networks with very good accuracy. As Cerny and Walawender [8] had earlier pointed out "*Principia pro motu sanguinis per arterias determinando* perhaps is the first published mathematical treatment of circulatory physiology and haemodynamics, and Euler could very well be called the 'Father of Haemodynamics'".

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