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The Surface Area of a Scalene Cone as Solved by Varignon, Leibniz, and Euler

Daniel J. Curtin
Northern Kentucky University, curtin@nku.edu

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The Scalene Cone: Euler, Varignon, and Leibniz

Daniel J. Curtin, Northern Kentucky University (Emeritus)
148 Brentwood Place, Fort Thomas, KY 41075
curtin@nku.edu

Abstract
The work translated alongside this article is Euler’s contribution to the problem of the scalene cone, in which he discusses the earlier work of Varignon and Leibniz. He improves Varignon’s solution and corrects Leibniz’s more ambitious solution. Here I will discuss the problem, provide extensive notes on Euler’s paper, and lay out its relation to the other two. For ease of reference translations of the papers of Varignon and Leibniz are appended, with the Latin originals. Euler’s version in Latin is available at https://scholarlycommons.pacific.edu/euler-works/133

1 The Problem

1.1 The History of the Problem
Finding the surface area of a right cone, a cone with circular base and vertex directly over the center of the circle, was known from antiquity, certainly by Archimedes.

The problem of finding the surface area of a scalene (or oblique) cone, one with circular base but vertex not necessarily directly over the center of the circle, seems to have been solved first by Pierre Varignon. In 1727 he published his result in the Miscellanea Berolinensia [3]. He called his paper a schediasma, a Latin version of a word that in Greek means a caprice, or whimsy. Varignon probably meant a sketch or rough draft. In fact, the mathematics in his paper is fully formed, but the layout and referencing are not carefully done. He solved the problem by giving a curve whose arc length at certain points gives the surface area of the corresponding part of the cone.

In the same volume, in fact in the article immediately following Varignon’s, Gottfried Leibniz [4] gave his suggested improvements. Varignon’s curve was transcendental, implicitly using trigonometric functions. Leibniz gave an algebraic curve to solve the problem and outlined how to extend his solution to base curves other than the circle. The two papers share a single diagram, printed in between them.
In his introductory paragraph, Leibniz asserted that Gilles de Roberval told Leibniz he had a solution, but never published it, but Leibniz noted that he could not find such work among Roberval’s papers, after his death.

A bit later, Euler took up the problem, publishing his solution in 1750 [2]. In his introduction, he referred to the two earlier papers by Varignon and Leibniz. He remarked that Leibniz’s solution was excellent, but involved a slight error. Euler used a Greek word in Latin form, *sphalma* meaning a misstep or stumble, perhaps to soften the criticism of a mathematician he greatly admired. Euler gave his own version of Varignon’s solution, then extended it in several ways. He explained where Leibniz went wrong, then corrected the misstep.

### 1.2 The Differential Approach

A cone is defined by starting with a circle in the plane and a vertex $V$ that is above the plane but not necessarily located directly over the center of the circle. A point on the circle $A$ is selected and line segment $VA$ is connected. Then the cone is defined by following points $M$ around the circle as $VM$ sweeps out any section $AVM$ of the cone that is desired. This dynamic definition allows infinitesimal methods to be used. For all three authors, a solution to the problem is a curve whose arc length gives the areas of the sectors of the cone. For either the arc length problem or the surface area problem, the integrals that result are rarely integrable in simple terms, since they both involve the element of the arc length $ds = \sqrt{dx^2 + dy^2}$ or something similar.

### 2 The Right Cone

To illustrate the idea let’s do the surface area of the right cone. We don’t need infinitesimal arguments here. Let $C$ be the center of the circle and $r$ the radius [Figure 1.]. Consider the segment $AVM$ of the cone, with slant height $l$ and central angle $u = \angle MCA$. In the plane the circular arc $AM$ has length $ru$. Then $AVM$ can be unfolded to a plane figure $A'V'M'$ that is part of a circle of radius $l = A'V'$, with arc $A'M'$ of length $ru$. The central angle of $A'V'M'$ is $ru/l$, so the area of $A'V'M'$ is

$$\text{Area} = \frac{1}{2} lru = \frac{1}{2} ru$$

Now $l$ and $r$ are constants, so as a function of $u$ this is a curve, in fact a line, whose arc length for any given point that is parametrized by a particular value $u$ gives the areas of that circular sector.
3 The Scalene Cone with Circular Base

If the vertex is off center things get quite a bit more complicated. This was first solved by Varignon [3]. Then Leibniz [4], using a different approach, claimed to solve it with an algebraic curve. He also claimed his approach extends to the case of an arbitrary base curve. Euler points out that this is not quite correct and removes the error, as we will see later.

The fundamental idea in all cases is to look for another point $m$, close to $M$. The area of the region $MVm$ is then the building block for the surface area. Taking $m$ infinitesimally close to $M$ makes $MVm$ what they called the element of the surface area, for us the integrand of the integral that would give the surface area. For Euler and the others, solving the problem meant finding a curve whose arc length element is this element of surface area.

4 Notes on Euler’s Paper

In what follows, I will outline what Euler does in various paragraphs, while leaving the reader to see much of the details in the work itself. I also try to include things I wish I had seen the first time I read this paper, in hopes they may prove helpful.

In [1], I go into much more detail about paragraphs 1–16, but note that, as usual, Euler himself does a good job of explaining his own ideas.

4.1 Paragraphs 2–4

Euler first sets out to reproduce Varignon’s curve [Figure 2]. He uses the central
angle of the circle $u = \angle ACM$ as the parameter for his solution. He defines three constants: $a$, the radius of the circle; $b$, the height of the cone ($VD$); and $c$, the distance from the center of the circle to the foot of the perpendicular from $V$ to the plane of the base ($c = CD$). He then derives the area as

$$AVM = \frac{1}{2}a \int du \sqrt{b^2 + (c \cos u - a)^2}.$$  

Euler generally writes $bb$ for $b^2$, etc. In this commentary I will use the more modern notation, since they mean the same thing. I have not moved his $du$, etc., to the modern location, since I believe we understand the $du$ quite differently from Euler. I left the notation in my translation as close to Euler’s as possible.

In paragraph 4 he concludes the curve is given, in what we would call parametric form:

$$p = bu,$$
$$q = c \sin u - au.$$  

He then says, “It will be clear to the attentive reader that this curve is the same as Varignon discovered.” It took me a bit of work, but I was able to verify this equivalence; see section 5 below.

### 4.2 Paragraph 5

Euler does not stop here. As usual he works to get as much out of the idea as possible. He notes that this solution involves transcendental functions, while
Leibniz has found (almost correctly) an algebraic solution, though by a different method that Euler discusses later.

4.3 Paragraphs 6 & 7
Euler then proceeds to substitute \( z = \cos u \) and thus to generate not only one algebraic solution but in fact a whole family of algebraic solutions.

4.4 Paragraphs 8–15
Euler works out several examples, also detailed in [1]. He always finds a parametrized curve, never a numerical answer.

4.5 Paragraph 16
Euler solves the case in which the perpendicular from vertex \( V \) passes through the point \( A \) on the circle and where the angle of the axis \( VC \) of the cone to the base is \( 60^\circ \). He derives parametric equations for a curve that solves the problem, then, almost as an afterthought, notes that the actual area of the part of the cone swept out from \( E \) to \( M \) will be \( \frac{1}{2}az \), where \( a \) is the radius of the base circle and \( z = \arcsin \angle ECM \), the angle \( ECM \) being the complement of the angle \( u = \angle ACM \) used in determining the curve.

5 Euler’s Solution Is the Same as Varignon’s
This section is for those interested in Varignon’s paper [3], which is translated in Appendix 1.

At the end of Euler’s paragraph 4 he gives a parametrized curve whose arc length (multiplied by \( \frac{1}{2}a \)) is the surface area of the scalene cone. The curve is given by abscissa \( p = bu \) and ordinate \( q = c\sin u - au \). He then asserts, “It will be clear to the attentive reader this is the same curve as Varignon discovered.” Let’s see how.

I will use mostly Varignon’s notation. For the radius of the circle, Varignon uses \( r \), Euler \( a \). For the distance from the center of the circle to the foot of the perpendicular from the vertex, Varignon has \( r + a \), Euler \( c \). For the distance from the vertex to the plane of the circle both use \( b \). Euler uses \( u \) for the generating angle \( \angle DCM \). Varignon uses no letter for his angle \( \angle AOH \), but I will use Euler’s notation \( u \). Further, for the arc swept out on the circle through the angle \( u \), Varignon uses \( AH \), Euler \( AM \). Either of these is equal to \( ru \).
Thus in Varignon’s notation (plus \( u \)) Euler’s curve becomes
\[
p = bu, \\
q = (r + a) \sin u - ru.
\]

Varignon gives his curve, beginning in the part entitled Proof, with a free constant \( m \). For us the relevant parts are as follows, first in the form Varignon wrote them, using the letters from his diagram:
\[
AE = \frac{BC \cdot AH}{2m} = \frac{b}{2m} AH, \\
EP = \frac{AO \cdot AH + OC \cdot GH}{2m} = \frac{rAH + (r + a)y}{2m}.
\]

Here if we suppose \( p \) is \( AE \) and \( q = EP \), then try \( 2m = r \), we get
\[
p = AE = \frac{b}{a} AH, \\
q = EP = \frac{rAH + (r + a)y}{r}.
\]

Further, looking at Varignon’s diagram, we see
\[
\sin u = -\frac{y}{r}.
\]

With \( AH = ru \) this gives us
\[
EP = \frac{r^2u + (r + a)r(- \sin u)}{r} = ru - (r + a) \sin u.
\]

Thus we correct our guess to make \( q = -EP \), which makes no difference to the length.

Finally, as mentioned above, Euler had factored \( a/2 \) out of his integral for the surface area before deriving \( p \) and \( q \), but Varignon’s curve can account for this by adjusting the value of the constant \( m \) to be \( \sqrt{a} \).

6 Leibniz’s Solution Perfected

For the remainder of the paper, Euler takes up the general problem of a cone with any curve in the plane whatsoever as base [Figure 3]. After setting this up in paragraphs 17–20, he returns to Leibniz’s idea and in paragraphs 21–28 corrects his solution.
6.1 Paragraphs 17 & 18

Euler no longer has the central angle available, so he takes any point $M$ on the curve and the foot $D$ of the perpendicular from the vertex to the plane, and he sets $x = DM$. He then takes the tangent line to the curve at $M$ and identifies the point $Q$ on that line for which $DQ$ is perpendicular to the tangent. He sets $y = DQ$ and notes that “there will be an equation between $x$ and $y$.”

We’ll follow the general trend of the argument. The details are not hard to read in the text.

As before, he considers a point $m$ on the curve infinitely near $M$. The triangle $MVm$ is the element of the surface area of the cone. Recall that $b = VD$ is the height of the cone. Since $VQ$ is the height of this triangle,

$$MVm = \frac{1}{2} Mm \cdot VQ.$$  

He shows that

$$MVm = \frac{1}{2} \int x dx \sqrt{b^2 + y^2} \over \sqrt{x^2 - y^2},$$

so

$$AVM = \frac{1}{2} \int x dx \sqrt{b^2 + y^2} \over \sqrt{x^2 - y^2}.$$
The last equation gives an integral for the area desired.

6.2 Paragraph 19

Euler unfolds the cone onto the plane [Figure 4], as we do with the right cone, but here he approaches the triangle $MVm$ in a different way. He uses the perpendicular, $Mr$ from $M$ to $V m$, which would be the height of the triangle relative to $V m$, and derives the distances $Mr$ and $mr$ for later use.

![Figure 4: The section of the same cone spread out on the plane.](image)

6.3 Paragraph 20

Next Euler returns to the cone in space [Figure 3], and he assumes the existence of the solution curve $AFS$. He will derive properties of the point $S$ and eventually show how to find it. Leibniz had based his argument on selecting an angle, essentially $\angle SMN$ to lead to his solution curve $FS$. In fact, Euler’s Fig. 4 [Figure 5] resembles the figure for Leibniz’s paper [4].

To see how the area can be defined using an angle, Euler defines $v = \angle AVM$. He also defines $u = \angle ADM$. Then since $VM$ and $V r$ are radii of an infinitesimal circle centered at $V$ and $dx = Mr$ is an arc of the circle,

$$dv = \frac{Mr}{VM}.$$  

By a similar argument, using $D$ as the center,

$$du = \frac{Mn}{DM}.$$
Using previous calculations for $M_r$, $VM$, $Mn$, and $DM$ from paragraphs 17–19, Euler obtains

$$dv = \frac{x dx \sqrt{b^2 + y^2}}{(b^2 + x^2)\sqrt{x^2 - y^2}}$$

and

$$du = \frac{y dx}{x \sqrt{x^2 - y^2}}.$$

He solves the second for $y$, then substitutes into the first, which yields

$$dv = \frac{\sqrt{b^2} dx^2 + (b^2 + x^2)x^2 du^2}{\sqrt{b^2 + x^2}}.$$

Thus, since $u$ depends on $x$, the angle $v$ can be found by integrating, and it then can be used to find the surface area $AVM$.

### 6.4 Paragraph 21

Euler now returns to the basic integral needed to solve the problem,

$$\int \frac{x dx \sqrt{b^2 + y^2}}{\sqrt{x^2 - y^2}},$$

and he notes that it can be solved in many different ways. Since Euler’s goal is to correct Leibniz’s approach, which he describes as “the most elegant,” he sets out to follow Leibniz’s idea.
6.5  Paragraph 22

Euler begins to trace Leibniz’s procedure. As noted above, Euler assumes that the desired curve $FS$ is given, then deduces what is needed to show its existence and nature.

For each point $M$ on the base curve $AM$, there is the segment $MS$ to $S$ on the rectifying curve. For a point $m$ infinitely near $M$, the corresponding segment $ms$ should be such that $S$ is infinitely near $s$. The arc length of the base curve is given as $s = AM$. (Yes, Euler uses the same letter for two different things, but the context always makes it clear which is which.) Let $v = \angle SMm$, i.e., the angle of $MS$ to the tangent at $M$. Then $ds = Mm$ and $\angle smN = v + dv$. He defines the osculating circle to $AM$ at $M$ whose center is $R$ and whose radius is $r = MR$. Then after a bit of work he obtains

$$MS = \frac{rds \sin v}{ds + rdv},$$

from which he claims the construction of the curve $FS$ follows, as we shall see. Here $F$ is the point on the curve corresponding to $M = A$.

6.6  Paragraph 23

Euler sets $z = MS$, so $ms = z + dz$. Using a bit of right angle trigonometry he shows that the element of arc length of the resolving curve $Ss = ds \cos v + dz$. Integrating with respect to $z$ he obtains

$$FS = \int ds \cos v + z + C.$$

We have $z = MS$, and when $s = 0$, $MS = AF$. Assuming $\int ds \cos v$ is 0 for $s = 0$, we have

$$\int ds \cos v = FS - MS + AF.$$

We need to look more closely at this integral.

6.7  Paragraph 24

From previous work,

$$Mm = ds = \frac{xdx}{\sqrt{x^2 - y^2}},$$

and the surface area $AVM$ may be written in two ways:

$$\frac{1}{2} \int ds \cos v = \frac{1}{2} \int \frac{xdx \sqrt{b^2 + y^2}}{\sqrt{x^2 - y^2}}.$$
Thus we want the angle $v$ to be such that $\int ds \cos v$ leads to $\int ds \sqrt{b^2 + y^2}$.

### 6.8 Paragraph 25

Euler now obtains $v$ from $VQ = \sqrt{b^2 + y^2}$ by taking a constant $k$ larger than, or equal to, the largest possible value of $VQ$ along the base curve, then setting

$$\cos v = \frac{\sqrt{b^2 + y^2}}{k}.$$  

Thus the surface area is given by

$$\frac{1}{2} \int ds \sqrt{b^2 + y^2} = \frac{1}{2} k \int ds \cos v.$$  

Geometrically this is

$$\frac{1}{2} k (FS + AF - MS).$$

### 6.9 Paragraph 26

Recall that $r$ is the radius of the osculating circle to $AM$ at $M$. After a bit of work Euler obtains

$$MS = z = \frac{k^2 r \sin^2 v \cos v}{k^2 \sin v \cos v - y \sqrt{x^2 - y^2}}.$$  

### 6.10 Paragraph 27

Euler uses this last characterization of $MS$ to show how to derive $MS$ geometrically. With the angle and distance in hand, the point $S$ is found. Doing this for each $M$ gives the rectifying curve.

To this end [Figure 5] he takes $MK$ perpendicular to the tangent $MQ$ so that $MK = k$. Then he forms the semicircle $KPM$ with diameter $MK$, where $P$ is the point on the semicircle such that $KP = \sqrt{b^2 + y^2}$. (Euler does this geometrically.) Then he takes $R$ on $MK$ such that $MR = r$, the radius of the osculating circle. Next he takes $MT$ perpendicular to $MP$ at $T$. He marks $X$ on $KP$ so that

$$PX = \frac{DQ \cdot MQ}{MP}.$$  

Finally,

$$MS = \frac{KP \cdot MT}{KX}.$$  

Euler states that the distance $MS$ can be found easily for each $M$, so $S$ is found and the resolving curve is known.
6.11 Paragraph 28

To end, Euler returns to the case where $AM$ is a circle. The curve $FS$ constructed above is the same curve that Leibniz derived [4]. Euler notes however that the solution is not $FS$, as Leibniz said, but rather $FS - MS + AF$. He ends with modest triumph, “Thus in this way, we have not only emended the Leibnizian construction, which was suitable only for scalene cones, but we also extend it to the cones whose bases are arbitrary figures.”

References


Appendix 1

Pierre Varignon’s sketch, on the dimensions of the surface of a cone that is oblique to a circular base, by means of the length of a curve whose construction depends only on the quadrature of the circle.$^1$

Now for many previous centuries the surfaces of cones whose bases are circular and standing upright [i.e., a right cone] have been known to have a rectangular area that depends on the quadrature of the circle. Not so for the oblique case, where I knew no one who had determined their surface area. Here is the curve that provides their areas by its length and whose construction depends on nothing other than the quadrature of the circle.

I. [See the diagram at the end.] Let the cone of any obliquity you like be $AHDZAB$, with base the circle $AHDZA$, vertex $B$, and height $BC$ perpendicular at $C$ to the plane of the base $AHDZA$. From that point $C$ through the center $O$ of the base, let the diameter $DA$ be produced indefinitely in the direction of $S$. Then on the diameter $AD$ let there be the bases of two ordinates as close to each other as you like, $GH$ and $gh$, from whose extremities $G$, $H$, $h$ to the vertex $B$ are drawn the lines $GB$, $HB$, and $hB$. Finally from the point $h$ let there be drawn $hk$ perpendicular to $HB$ at $k$ so from the center $B$ through $h$ there will be the indefinitely small circular little arc $hk$.

II. Now since $BC$ is normal to the base plane $AHDZA$ (I.) it follows that planes $ABC$ and $AHDZA$ are perpendicular to one another. Thus, since $HG$ is indeed perpendicular to their common section $AD$ (I.), it will also be perpendicular to the plane $ABC$ and thus also to the line $GB$.

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$^1$Varignon, Pierre, Schediasma de Dimensione Superficiei Coni ad basim circularem obliqui, ope longitudinis Curve, cujus constructio `a sola Circuli quadratura pendet, Miscellanea Beroliniensis, ad incrementum scientiarum ex scriptis Societati Regiae Scientiarum exhibitis edita III, 1727, pp. 280–284.

Translator’s note: I have modernized the notation and formatting somewhat. Those interested in details of the history of mathematical notation should look at the original. All footnotes and bracketed comments are mine.

$^2$The point $k$ is not labelled, but lies directly over the point labeled $K'$ in the figure. The point $K'$ is not used in this paper, but it is in the Leibniz paper that follows it.
Therefore setting \(AO = r, AG = x, GH = y, CD = a, BC = b\)

\[
BH = \sqrt{HG^2 + GB^2} \\
= \sqrt{HG^2 + BC^2 + GC^2} \\
= \sqrt{y^2 + b^2 + (2r + a - x)^2} \\
= \sqrt{y^2 + b^2 + 4r^2 + 4ar + a^2 - 4rx - 2ax + x^2}.
\]

From the circle \(AHDZA\) comes \(y^2 = 2rx - x^2\) [so we have further:]

\[
[BH] = \sqrt{4r^2 + 4ar + a^2 + b^2 - 2rx - 2ax}.
\]  
**[Eq. 1]**

Let \(c\) be [such that]

\[
c^2 = 4r^2 + 4ar + a^2 + b^2 \\
= (2r + a)^2 + b^2 \\
= AC^2 + BC^2 \\
= AB^2
\]

and also

\[f = r + a = OC.\]

[From this and Eq. 1]

\[
[BH] = \sqrt{c^2 - 2fx}.
\]  
**[Eq. 2]**

[Since \(Hk = -d BH\),] this will make\(^3\)

\[
Hk = \frac{fx}{\sqrt{c^2 - 2fx}}
\]

positive because of the alternate increasing and decreasing of \(AG (x)\) and \(BH\).

Then

\[
hk = \sqrt{HH^2 - Hk^2} \\
= \sqrt{dy^2 + dx^2 - dx^2 \frac{f^2}{c^2 - 2fx}}.
\]

\(^3\)Varignon displays this as \(BH = V(HGq. + GBq.)\) using a large \(V\) for \(\sqrt\) and \(q.\) to abbreviate \(quadratus\). As was customary at the time, Varignon writes \(yy\) for \(y^2\), etc. I will use modern notation. Further, Varignon writes this string of equations inline. I display them in the modern style.

\(^4\)Varignon has \(HK\), but I believe \(K\) is a typo for \(k\) in the next two displayed equations.
Now with the equation of the base, \( y^2 = 2rx - x^2 \), this gives

\[ \pm dy = \frac{rdx - xdx}{\sqrt{2rx - x^2}}. \]

So

\[
[hk] = \sqrt{dx^2 \left( \frac{r^2}{2rx - x^2} + dx^2 - dx^2 \left( \frac{f^2}{c^2 - 2fx} \right) \right)}
\]

\[ = dx \sqrt{\frac{r^2}{2rx - x^2} - \frac{f^2}{c^2 - 2fx}} \]

\[ = dx \sqrt{\frac{c^2r^2 - 2fgrx + f^2x^2}{(2rx - x^2)(c^2 - 2fx)}}. \]

Now setting \( g = r + f = 2r + a = AC \),

\[ [hk] = dx \sqrt{\frac{c^2r^2 - 2fgrx + f^2x^2}{(2rx - x^2)(c^2 - 2fx)}}. \] \[ \text{[Eq. 3]} \]

Thus [from Eq. 2 and Eq. 3]

\[ HBh \left( \frac{HB \cdot hk}{2} \right) = dx \sqrt{\frac{c^2r^2 - 2fgrx + f^2x^2}{2\sqrt{2rx - x^2}}} \]

will be the sought element of the surface of the cone, from whose summation [integral] would come the surface at hand, with the help of the length of the following curve.

**III.** Let the semiperimeter of the circle \( AHD \) be rolled out in a line \( AQ \) equal to it and perpendicular to the diameter \( AD \). Of course the unrolling of the first point \( D \) describes a curve \( DQ \). Then taking any constant \( m \) you please, draw from \( Q \) a line \( RQ \), perpendicular to \( AQ \), such that

\[ \frac{RQ}{AQ} = \frac{BC}{2m}. \]

So

\[ RQ = \frac{AQ \cdot BC}{2m} = \frac{AHD \cdot BC}{2m}. \]

Then the rectangle \( QS \) is completed and its side \( RS \) produced to \( F \) so that

\[ \frac{SF}{AS} = \frac{AO}{BC}. \]
Then
\[ SF = \frac{AS \cdot AO}{BC} = \frac{RQ \cdot AO}{BC} = \frac{AQ \cdot AO}{2m} = \frac{AHD \cdot AO}{2m}. \]

Next after drawing the lines \( AR \) and \( AF \) and taking the indeterminate line (on \( AQ \)) \( AL = AH \), the arc rolled out on it by \( AL^5 \) describe \( LM \) parallel to \( QR \). From the point \( M \) (where the line [i.e., \( LM \)] meets \( AR \)) make the indefinite line \( MP \) parallel to \( AQ \) so that it meets the lines \( AS, AF \) at \( E, N \) [respectively]. Finally for \( MP \) make it that
\[ NP = \frac{OC \cdot GH}{2m}, \]
and so on, everywhere.

I say that this is the curve \( APF \) passing through each of the points \( P \) whose product \( m \cdot AP \) of any arc \( AP \) by the constant \( m \), whatever it may be, is equal to the corresponding part \( AHB \) of the surface of the cone that is sought.

**Proof**

1.\(^6\) Since (by construction) there is
\[ \frac{2m}{BC} = \frac{AQ}{RQ} = \frac{AL}{LM} = \frac{AH}{AE}, \]
then
\[ AE = \frac{BC \cdot AH}{2m}, \quad [\text{Eq. 4}] \]
and so the parallel element is
\[ \delta p = \frac{BC \cdot Hh}{2m} \]
by the notation in II. Since the circle \( AHDZA \) yields
\[ Hh = \frac{AO \cdot Gg}{HG}, \quad [\text{Eq. 5}] \]
then
\[ [\delta p] = \frac{br \, dx}{2m\sqrt{2rx - x^2}}. \]

---

\(^5\)My correction: Varignon has \( HL \) instead of \( AL \).

\(^6\)Varignon restarts the sequence of Roman numerals. For clarity, I have changed this second series of steps to Hindu-Arabic numerals.
2. Further, (by construction) not only is

\[
\frac{BC}{AO} = \frac{AS}{SF} = \frac{AE}{EN} = \frac{BC \cdot AH}{2m} \quad \text{[Eq. 4]}
\]

Now with \([EN] = \frac{AO \cdot AH}{2m}\),

and also \(NP = \frac{OC \cdot GH}{2m}\),

then

\[
EP = \frac{AO \cdot AH + OC \cdot GH}{2m}
\]

Next (taking \(HV\) perpendicular to \(gh\) at \(V\)) its element will be

\[
\delta P = \frac{AO \cdot Hh + OC \cdot Vh}{2m}
\]

(Then from the previous section [Eq. 5], using the equation \(y^2 = 2rx - x^2\), with \(f = r + a\),

\[
[\delta P] = \frac{r^2dx}{2m\sqrt{2rx - x^2}} + \frac{(r + a)(rdx - xdx)}{2m\sqrt{2rx - x^2}}
\]

\[
= \frac{dx(2r^2 + ar - rx - ax)}{2m\sqrt{2rx - x^2}}
\]

\[
= \frac{(2r^2 + ar - fx)dx}{2m\sqrt{2rx - x^2}}.
\]

3. Therefore there will be the element \(pP\) [of the arc length] of the curve \(APF\), namely

\[
\sqrt{\delta p^2 + \delta P^2} = \frac{dx \sqrt{b^2r^2 + 4r^4 + 4ar^3 - 4fr^2x + a^2r^2 - 2farx + f^2x^2}}{2m\sqrt{2rx - x^2}}
\]

\[
= \frac{dx \sqrt{c^2r^2 - 2frx + f^2x^2}}{2m\sqrt{2rx - x^2}} \quad \text{(since } b^2 + 4r^2 + 4ar + a^2 = c^2) \]

\[
= \frac{H Bh}{m}.
\]

\(^7\) I correct a printing error in the original.
Therefore \( HBh = m \cdot Pp \), whose integral, or such a summation, is the sought surface of the part of the cone, \( AHB = m \cdot AP \), and so on for other parts of this surface obtained in this way. Thus the integral extends so that the surface area of this oblique cone = \( 2m \cdot APF \). \textit{Quod erat Dem.}

**IV.** From this measure of the surface of an oblique-angled cone that of the common type, given by a right angle, also follows. Now if the cone \( AHYZAB \) be a right cone, then certainly its height \( BC \) coincides with \( BO \) – for this reason making both [segments] equal and normal to the base of the cone, which would nullify \( OC \). Then (from 2) the curve \( APF \) has ordinate

\[
EP = \frac{AO \cdot AH + OC \cdot CG}{2m},
\]

reducing to

\[
\frac{AO \cdot AH}{2m} = EN,
\]

while the abcissa (see 1) remains the same:

\[
AE = \frac{BC \cdot AH}{2m}.
\]

The curve \( APF \) would be changed into the right line \( ANF \), and its arc \( AP \) changed into \( AN \). From this in the right cone, it would follow that

\[
m \cdot AN = m \cdot AP = AHB. \quad \text{(from 3)}
\]

And then

\[
AN = \sqrt{AE^2 + EN^2} \quad \text{(from 3)}
\]

\[
= \sqrt{(BC \cdot AH)^2 + (AO \cdot AH)^2} \frac{2m}{2m} = \frac{AH \sqrt{BC^2 + AO^2}}{2m}.
\]

(Then because \( BO = BC \) and is perpendicular to the plane \( AHYZA \) and thus also to the radius \( OH = AO \),)

\[
[AN] = \frac{AH \cdot \sqrt{BO^2 + OH^2}}{2m} = \frac{AH \cdot BH}{2m}.
\]
Thus in this case
\[
\frac{AH \cdot BH}{2} = AHB,
\]
or the portion \( AHB \) of the right convex cone will be equal to the product
\[
\frac{AH \cdot BH}{2}
\]
from half of the side \( BH \) times the corresponding base arc \( AH \). Thus the entire surface of such a convex cone taken together will be equal to the product of half the side and the whole circuit of the base, which geometers, by other reasoning, had observed long before now.

The measure of the surface area of the same right-angle cone also follows immediately from the oblique-angle element
\[
HBh = \frac{dx\sqrt{c^2r^2 - 2fgx + f^2x^2}}{2\sqrt{2rx - x^2}}, \quad \text{(from 3)}
\]
in which \( c = AB \) and \( f = OC \). Now since the surface of the rectangular cone forces \( AB = BH \) and \( OC = 0 \), or \( c = BH \) and \( f = 0 \), the oblique-angle element becomes, in the right-angle case,
\[
HBh = \frac{dx\sqrt{r^2BH^2}}{2\sqrt{2rx - x^2}} \cdot rdx = \frac{2BH \cdot OH \cdot HV}{2} = \frac{BH \cdot Hh}{2},
\]
and after integrating, or summing, it will give (and quickly) the surface of the right-angle cone
\[
ABH = \frac{BH \cdot AH}{2},
\]
etc., as above.

I will not dwell upon this any longer, but only wanted it known how great is the affinity between the preceding case of the oblique-angle cone and the common right-angle cone.
V. This part looks at the curve \( APF \) (III.) that yields the measure of the surface of the oblique-angle cone.

1) It establishes the angle \( ADB \) at \( F \) across from \( A \) with the ordinate \( SF \) and the angle \( ABC \) at \( A \) across from \( S \) with the axis \( AS \). Then the side \( AB \) of the oblique cone \( AH'DZAB \) is normal to this curve at \( A \), which, if it is right, also has its side \( AB \) normal at the point \( A \) to the line \( ANF \) which thus has been added to the curve.

2) Then the curve will be such that the largest of its ordinates \( EP \) corresponds to

\[
AG = \frac{AC \cdot AO}{OC},
\]

which will always be larger than \( AO \) and less than \( AD \).

3) The total segment of that curve, \( ANFPA \), and any desired indeterminate part \( ANPpA \), can be completely rectified\(^8\) This is possible from the known construction of the quadrature of the circle \( AH'DZA \). Yet you would not say the areas \( FPAS, PpAE \) are both quadrable, since the size of the triangles \( FNAS, NAE \), though rectilinear, depend on the quadrature of this circle.

4) However, the length of \( APF \) and any part of it \( AP \) is measured by the quadrature of the curve placed between two parallel asymptotes with the distance between them the diameter \( AD \) of the base of the cone, and [the length] is expressed by the equation

\[
z = \frac{\sqrt{c^2r^2 - 2fgrx + f^2x^2}}{2\sqrt{2rx - x^2}}
\]

whose coordinates, normal to each other, are \( x, z \).

It may be noted that if the perpendicular \( BC \) to the base plane \( AH'DZA \) of the oblique cone \( AH'DZAB \), which here [in the diagram] falls outside of the base, were to fall inside, over \( OC \) and with the same point \( D \), nothing different from the preceding would occur, except the change of the sign of the quantity \( a \) or \( DC \), and with the removal of this term the same construction would remain and the use of the above curve \( APF \).

\(^8\)He says quadrari, to be found by making a square.
The figure from Varignon’s paper, which is also used by Leibniz’s addendum, immediately following in the same volume of the Miscellanea.
PETRI VARIGNONIS

Schediasma
de
Dimensione Superficiæ Coni ad basim
circularem obliqui, ope longitudinis Cur-
væ, cujus constructio à sola Circuli
quadriutra pendent.

Jam à multis retrò sæculis notum est conorum, basibus
circularibus rectè insistentium superficies, mensuram in rectangulis
habere, à quadratura circuli pendentem: At non item in obliquangulis,
quorum superficies qui definierit, neminem novi. Ecce
curva quæ horum mensuram sua longitudine praebet, cujusque constructio
non aliunde quam à quadratura circuli pendent.

I. Sit obliquitatis cujusvis Conus A H D Z A B, bæsos circularis
A H D Z A, verticis B, altitudinis B C perpendicularis in C ad planum
bæsos A H D Z A; ex quo puncto C per bæses centrum O sit diamete-
er ejus DA indefinite producâ versus S; Hanc ad diametrum A D int
bæsos ordinate duæ, invicem quam proximæ quævis G H, g h, ex quærum
extremitatis G, H, h per coni verticem B, ducantur rectæ GB, HB, h B. Tandem ex puncto h singatur h k perpend
cularis ad HB in k, sive ex centro B per h sit circularis arculus indefinite parvus h k.

II. Jam cum sit (art. 1.) B C normalis ad planum bæsos A H D Z A,
liquet, plana ABC & A H D Z A ad invicem esse perpendicularia; atque
que adeo cum HG fit quoque (art. i.) perpendicularis ad communem
orum sectionem AD, erit pariter ea perpendicularis ad planum ABC,
& consequenter etiam ad rectam GB.

Ergo positis AO = r, AG = x, GH = y, CD = a, BC = b, erit
BH = V(HGq. + GBq.) = V(HGq. + BCq. + GCq.) = V(yy + bb +
(2r + a - x) q.) = V(yy + bb + 4ar + aa - 4rx - 2ax
+ xx) (circulo AHDZA exigente yy = 2rx - xx) = V(4ar +
4ar + aa + bb - 2rx - 2ax) [sint cc = 4ar + 4ar + aa + bb =
(2r + a) qu. + bb = AC q. + BC q. = AB q. & f = r + a = OC]
= V(cc - 2fx). Unde sit HK = f dx: V(cc - 2fx) positiva, pro-
pter alternatim crescentes aut decrecentes AG (x) & BH. Ergo hk =
V(Hh qu. - HK qu.) = V(dydy + dx - dx - dx (ff.: cc - 2fx))
[haec eos aequatione yy = 2rx - xx dante + dy = r dx - xx dx:
V(2rx - xx)] = V(dx dx ((r - x) qu.; 2rx - xx) + dx dx -
dx dx (ff.: cc - 2fx)) = dx V(2rr; 2rx - xx) - (ff.; cc - 2fx))
= dx V(ccrr - 2frxx - 2frxx + ffxx, (2rx - xx) (cc - 2fx))
[sit g = r + f = 2r + a = AC] dx V(ccrr - 2frxx + ffxx,:
(2rx - xx, cc - 2fx)). Igitur superficiem conicæ quæ sit et
elementum erit HBh [HB. h.k.2] = dx V(ccrr - 2frxx + ffxx): 2V(2rx - xx)
quælium summationem superficieris, tædem habetur, ope longitudinis Cur-
væ sequentis.

III. Fingatur semiperipheria circularis AHD evolvi in rectam ipsi
æqualem AQ perpendiculararem ad diametrum ejus AD, primo scilicet
evolutionis ejus puncto D describente curvam DQ. Dein ad perpen-
dicularum ipsi AQ (summa quavis constante m) ponatur in Q recta RQ ita,
ut sit RQ: AQ = BC: 2m. Sive RQ = AQ. BC: 2m = AHD. BC:
2m; perfected rectangulo QS, latus ejus RS producatur versus F,
donc sic SF: AS = AO: BC. Sive SF = AS. AO: BC = RQ. AO:
BC = AQ. AO: 2m = AHD. AO: 2m. Tum duæs rectæ AR, AF,
indeterminateque sumpta (in A) recta AL = AH arcu in ipsam evo-
luto, per HL, fiat LM parallela ad QR; & ex puncto M (occurrere ejus
cum AR) item fiat indefinita MP parallela ad AQ, quaèe occurrat
in E, N, rectas AS, AF. Tandem in illa MP summatur NP = OC. GH:
2m; & sic ubique.

N n

Dico
Dico eam esse Curvam APF per omnia ejusmodi puncta P transfixas, quæ producunt m. AP ex arcæ quovis ejus AP per constatatem m., ubique reddat, æquale respondenti portioni AHB conica superficiis quaestae.

Demonstr. I. Si quidem (constr.) ubique est 2m : BC = AQ : RQ = AL : LM = AH : AE est etiam ubique AE = BC. AH = 2m, & ipsi parallellum elementum ejus gp = BC. HH = 2m (juxta nominis art. 2, & circulum AHDZA exigentem HH = AO. GG = HG) = b drx : 2m v(2r x — xx)

II. Pariter cum sit (constr.) non modo BC = AO = AS = SF = AE: EN (num. 1.) = (BC : AH : 2m) : EN = AO, AH = 2m. Sed etiam NP = OC. GH = 2m; erit EP = AO, AH + OC. GH = 2m, & elementum ejus (ducès HV perpendiculari in V ad gh) 2p = AO. HH = OC. vb = 2m (ex premisso nominibus & ex aequatione yy = 2r x — xx art. 2.) = (r. rdrx : 2m V(2rx — xx)) + (r + a) (r drx — x dx) : 2m V(2rx — xx) = dx (2rx + ax — rx — ax) : 2m V(2rx — xx) [art. 2. dante f = r + a] = 2r + ar — f x, dx : 2m V(2rx — xx)

III. Ergo (num. 1, 2.) erit Curvæ APF elementum PP vel V(dP qu. dP qu.) = dx V(bbr + 4rr + 4ar + 4ar + 2r + c) = dx V(crr — 2fr + 2fr + fxx) = 2m V(2rx — xx) = 2m V(2rx — xx) [art. 2.] = HH = m. Igitur erit HBH = m. PP, cujus integrale, seu qualium summatio est conicae superficiis quaestae portione AHB = m. AP; & sic de ceteris ejusdem superficii portionibus hac ratione sumptis: Unde pater integrum hanc Coni obliqui superficie esse = 2m AP. Quod erat Dem.

IV. Ex hac obliquangula conicae superficie mensura fluent ulterius vulgaris ea, quæ rectangula tribuitur. Nam si Conus AHDZAB fieret rectius, is nimimum, cujus altitudo BC, conversa in BO, hac ratione sìam ipsi æqualem & normalem ad Coni basim, abolenter OC; tunc Curvæ APF ordinatis EP (art. 3. num. 2.) = AO. AH + OC. CG = 2m evadentibus = AO. AH : 2m = EN interim dum abscessa manerent exdem AE (art. 3. num. 1.) = BC. AH = 2m, Curva hac APF mutaretur in rectam ANF, & ejus arcus AP in AN: Unde hoc in Cono rectanguulo foret m. AN = m. AP (art. 3. num. 3.) = AHB. Atque AN = V(AEq.
$V(\text{AEqu} + \text{ENqu}) (\text{art. 3. num. 1. 2.}) = \frac{V(\text{BC AH}) q. \times (\text{AO AH}) q.)}{2m = \text{AH} V(\text{BCq. AOq.}) : 2m} \text{ (Siquidem hic est BO = BC & perpendicularis ad planum AHDZA, indeae ad radium OOH = AOH, V(BOq. OHq.)) : 2m = AHB BH : 2m. Ergo hoc in eodem cafu erit AHB BH : 2 = AHB, sive Coni recli convexæ superficie portio AHB æqualis erit producéto AHB BH : 2 ex dimidio latere BH per respondentem baseos arcum AHB : Unde tota hujusmodi Coni superficies convexa pariter erit æqualis producéto ex dimidio laterre per totum baseos ambitum; quod Geometris alia ratione jam pridem notum est.}

Eadem Conicæ superficie rectangulæ mensura statim fluít etiam ex obliquangulæ elemento H Bh (\text{art. 2.}) = dx V(cx r - zsfgr x + ffx xx) : 2V(2r x - xx) in quo sunt (\text{art. 2.}) c = AB, & f = OC. Nam cum rectangula superficies conica exigat AB = BH & OC = o, seu c = BH, & f = o; itid obliquangulæ elemento in rectangula siet H Bh = dx V(rr x x) : 2V(2r x - xx) = (BH:2) (OH HV HG) = BH Bh : 2; atque integrando, sive summamundo erit (ut mox) rectangulæ superficie Conicæ portio AHB = BH AH : 2, &c. ut superius.

Porro his tamdiu non immoror, nisi ut magis pateat quanta sit affinitas inter precedentem (\text{art. 3.}) Conicæ superficie obliquangulæ mensurae & rectangulæ vulgarem.

V. Quod spectat Curvam APF quæ (\text{art. 3.}) obliquangulæ illius superficie Conicæ mensuram præbet.

1. Ea constituit in F verius A cum ordinata SF angulum = ADB & in A cum axe AS verius S. angulum = ABC; atque ita hanc ad Curvam normalis est in A Latus AB Coni obliqui AHDZA; qui, si rectus effet, pariter haberet Latus suum AB normale in eodem puncto A ad re- dem ANF in quam ulla curva mutaretur.

2. Dum ea curva est, ordinatarum ejus EP maxima respondet AG = AC AO: OC semper major quam AO, & minor quam AD.

3. Ejusdem Curvæ segmentum totum ANFP A, ac etiam indeter-minata quævis ejus portio ANPpA, possunt absolute quadrari, Licee sint constructionis exiguæ quadraturas circuli AHDZA. Ne tamen dixeris areas FPAS, PpAE, pariter esse quadrables; cum triangulatium FNAS, NAE, et s æquidistantum, dimensio ab ista circuli quadratura pendet. 

N n 2 4 Lon-
4. Longitudo autem APF, & pars ejus quævis AP mensuram habent à Quadratura Curvæ posita duas inter asymptotos parallelas distantes ab invicem quantitate diametri AD baseos Coni, & exprimetur per æquationem \( z = V(ecrr - 2fgrx + ffxx) + 2V(2rx - xx) \), cujus co-ordinatae ad invicem normales sunt \( x, z \).

Notandum si perpendicularis BC ad planum baseos AHDZA Coni obliqui AHDZAB, qua hic cadit extra basim istam, caderet intra, super OC, aut in ipsum punctum D; nihil inde varietatis praecedentibus eventurum esse, prater mutationem signi quantitatis \( o \) vel DC, aut hujus quantitatis extinctionem, manentibus semper isdem constructione & usu curvæ superioris APF.
Appendix 2

An Addition by G.G.L. Setting forth an explanation of the surface area of any conoidal surface whatsoever; and a step-by-step explanation of the surface area of a scalene cone, so as this or that portion whatsoever is presented equal to a rectangle, by means of the length of a right curve constructed by ordinary geometry.\footnote{Leibniz, Gottfried Wilhelm, \textit{Additio G. G. L. Ostendens Explanationem superficiei conoidalis cujuscunque; & speciatim explantionem superficiei Coni scaleni, ita ut ipsi vel ejus portioni cuicunque exhibeatur rectangulum æquale, interventu extensionis in rectam curvæ, per Geometriam ordinariam construendae, Miscellanea Berolinensia ad incrementum scientiarum ex scriptis Societati Regiae Scientiarum exhibitis edita III, 1727, pp. 285–287.}

The surface area of a right cone was already laid out by Archimedes and it is equal to a given circle. It is certain that the ancients did not arrive at the surface area of a scalene cone. Ægidius [Gilles] de Roberval told me when I was young that he had found a solution for it, but he did not tell me how he had managed it, nor was I able to find anything about it among his papers. In fact now with the same method that appeared in the previous sketch [Appendix 1], we will easily be able to give plane figures, constructed by ordinary geometry, that are equal to the surface area not only of scalene cones, but also of conoids generated by lines $BH$ (see the Figure above) through a fixed point $B$ taken above and passed along any curve $Hh$ whatsoever in the plane $AHC$ below.

Now let $BC$ be normal to the plane below, and therefore to the line $AC$, whose directrix $HG$ is applied normal to the curve $Hh$ and meeting $AC$ at $G$. It is clear that\footnote{The text has $H(H)$, but clearly $Hh$ is meant}$

$$BG = \sqrt{BC^2 + CG^2},$$

which is normal to $HG$, and thus

$$BH = \sqrt{BG^2 + GH^2}.$$

Now $GH$ is given from $AG$ by the nature of the curve, and $BC$ and $AC$ are given. Then $CG$ is the sum or difference of $CA$ and $AG$. Further, $BH$ is also given from $AG$. Therefore also is given the ratio of the momentary increment,
or element, of the same $BH$ and certainly of $HK$ itself to $HV$, the element of $AG$. (Where $[K]$ is placed down [i.e., directly below $k$] from the normal $hk$ from $h$ to $BH$.)

Now if the square of $HK$ is subtracted from the square of $Hh$ there remains the square of $hK$, the element of the arc length of the curve. This line is set in ratio to $HV$, the increment of $AG$, and this ratio is called $r/a$, where $a$ is any constant whatsoever assumed once arbitrarily, and $r$ is determined from $AG$. Then it will be that

$$hK = \frac{r}{a} \, dAG.$$

But half of the rectangle $BH$ by $hK$ is equal to the triangle $BHh$, which is the element of the surface area of the conoid. So if the line $$(r/2a)BH$$ (which will clearly be to $BH$ as $r$ to $2a$) is applied to $dAG$, that is to $Gg$, or if $GH$ is always summed from $G$ in the direction of $H$, then a curvilinear figure will be made whose element is $$(r/2a)BH \, dAG$$, the same as the element of the surface of the conoid.

Thus the appropriate part of this figure summed will be equal to the corresponding part of the surface of the conoid, both lines being derived from the point above and the base curve. Hence the surface of any conoid can be laid out by ordinary geometry giving a plane figure equal to it, if the base of the conoid $AHh$ is a figure from ordinary geometry.

Now Archimedes presented the surface of a right cone, or a part of it, as equal to a circle, or a part of a circle. Then the area of a circle, or a part of it, may easily be turned into a rectilinear figure through the length of a particular curve (an arc of a circle, in fact). From this the desired [curve] was found in a most elegant way by that most renowned man, Pierre Varignon, as laid out immediately above [Appendix 1], whereby the plane figure equal to the surface area of the scalene cone can be measured by the length of a certain curve.

Now the curve that he sets forth is not ordinary, but transcendental, which it might not have been if it did not require the quadrature of the circle or the rectification of the arc of the circle for its construction. I prefer to seek an ordinary curve that would provide the same [result]. Here is how I proceeded. In the same way as before, let $GW$ be taken from $G$ towards $D$, which will be to $AG$ as $CD$ is to the radius $DO$ and let $BW$ be connected, which will yield

$$BH = \sqrt{c^2r^2 - 2rfgx + f^2x^2} \frac{r}{r}.$$

---

4The $r$ and $a$ here are not those of Varignon’s paper. Leibniz uses $(r : a)$ where I have $r/a$, and similarly elsewhere.

5I use the word "length"; Leibniz says "by extension in a line."
[From Varignon’s paper: \( OA = r \) is the radius of the circle, \( CD = a \), \( AB = c \), \( OC = f = r + a \), \( AC = g = r + f \). The expression follows by a relatively straightforward calculation from \( BW = \sqrt{CW^2 + BC^2} = \sqrt{(OW + OC)^2 + BC^2} \), using \( BC^2 = c^2 - g^2 \) and \( OW = r - AG - GW = r - (1 + a/r)x \).

Then I place \( A \) to be the farthest point of the diameter \( AD \) from \( B \). \( HI \) is led from the point \( H \) to the line \( AC \) so that, when \( O\psi \) is led from \( O \) normal to \( HI \), \( [O\psi] \) is to the radius \( HO \) as half of \( BW \) is to the constant line, which is larger than half of \( BA \). From the same line \( HI \), \( O\phi \) is taken from the center \( O \) perpendicular to \( CO \), and continued on the same side as \( H \) to make \( \phi \) [on \( HI \)]. Likewise all this is done to points \( h \) and \( g \), as \( BW \) was drawn, so on to \( hi \), and \( O\phi \) is made to \( (\phi) \) [on \( hi \)].

Now \( O\phi \), \( O(\phi) \), etc., are equal to \( I\Omega \), \( i\omega \), etc., the ordinates applied normal to \( OI \), \( Oi \), etc., that make the curve \( \Omega\omega \), whose tangent \( \omega T \) meets \( AC \) at \( T \). Finally along \( AC \) from \( I \) is taken \( IX \) to the side where \( IH \) [and] \( ih \) approach each other. Then \( IX \) is to \( IO \) as \( TI \) is to \( TO \).

From any such point \( X \) found in this way the normal \( XY \) is taken, running to the corresponding position \( Y \) on the given line \( HI \). This will be the applied ordinate of a new curve \( Y(\Omega) \). Then with the elementary arc between point \( Y \) and \( (Y) \), and with the intercept defined by the process given above, and with the above-mentioned constant that is no less than \( \frac{1}{2}BA \), the included rectangle will be equal to the triangle \( BIIh \), which is, of course, the element of the surface of the scalene cone. Thus by the evolution or extension of the appropriate curve \( Y(\Omega) \) in a line, if the thread rolled out is led along using the same constant, a rectangle will be produced equal to \( ABDHA \), the surface of half the scalene cone, or any portion of it (made up, of course, of half of the line from \( B \) and the arc of the base).
Additio G. G. L.

Ostendens Explanationem superficiem conoïdalis cujuscunque; & speciatim explanationem superficiei Coni scaleni, ita ut ipsi vel ejus portioni cuicunque exhibeatur rectangulum aequale, interventu extensionis in rectam curvæ, per Geometriam ordinariam consequendæ.

Superficies Coni recti jam ab Archimede explanata est; & aequalis ei circulus assignatus. Coni Scaleni superficiem veteres, quod confer, non attigere. Aegidius Robervaliius, me juvente ajebræ, ejus explanationem sibi esse notam, sed qualem habuerit non dixit, nihilque ea de re inter ejus schedas repertum accepi. Equidem hodie ad eum modum quo processum est in Scholiasmate precedenti, postulam facile dare figuras planas easque per Geometriam ordinariam consequendas, non tantum quæ aequantur superficiebus coni Scaleni, sed etiam cujuscunque Conoidis generatis rectâ BH (vide eandem quæ supra Figuram) per punctum B, constans, in sublimi transiente, & circumdata per curvam quamcunque H (H) in plano subjeco AHC positam. Nam cum BC sit normalis ad planum subjectum, adeoque & ad rectam AC, que sit directrix ipsarum HG normaliter applicarum ex curva HH, & ipsi AC occurrentium in G; patet BG = V(BC. BC + CG. CG) fore normalis ad H; & atque adeo BH fore V(BG. BG + GH. GH). Datur autem GH ex AG per naturam curvæ; & BC, AC sunt data, & CG est summa vel differentia ipsarum CA, AG; habetur ergo & BH ex AG. Ergo & ratio datur incrementi momentanei seu elementi ipsarum BH; nempe ipsius HK (si ponatur ex h demissa in BH normalis HK) ad HV elementum ipsarum AG. Quod si jam quadratum ipsius HK detrahatur à quadratro ipsius HH, elementi arcus curvæ, supererit quadraturum ipsius HK, cujus rectâ adeo etiam debitur ratio ad HV, incrementum ipsius AG; que ratio vocetur r: a: constan-
constante quacunque pro arbitrio semel assumt., & \( r \) determinat\( e \) ex AG. Erit ergo \( hK = (r:a) \cdot dAG \). Sed dimidium re\( \delta \)anguli ex B H in \( hK \) est \( \alpha \)quale triangulo B H h, id est elemento superficiei Conoidis, ergo si re\( \delta \a (r:2a) \cdot BH \) (nempe qu\( \alpha \) sit ad BH ut \( r \) ad \( 2a \)) accommodetur ad \( dAG \), id est, ad GG, seu sumatur semper in ipsa GH ex G versus H; si\( \dot{\iota} \)t figura curvilinea, cujus elementum erit \( (r:2a) \cdot BH \cdot dAG \) idem cum elemento superficiei conoidis. Itaque ipsius hujus Figure portio rit\( \varepsilon \)rum \( \alpha \)quabivtur respondenti superficiei Conocidis portioni, binis re\( \delta \is ex\) pun\( \dot{c} \)to in sublimi, \& arcu\( \cdot \) basis comprehens\( \alpha \); \& proinde qu\( \alpha \)vis superficies Conoidalis explanati potest, adhibita figura plana ei \( \equiv \)quali, per Geometriam ordinariam construenda, si ipsa AH h basis Cono\( \cdot \)idis, fit figura Geometri\( \alpha \) ordinar\( \alpha \).

Sed quia Archimedes Circulum, vel Circuli portiones, superficiei co\( n \)i re\( \delta \)i vel ejus portionibus \( \equiv \)quales exhibuit; Circuli autem vel ejus portionum area vel conversio in figuram rectilineam facili\( \alpha \)me habuer\( \beta \)t per extensionem curv\( \alpha \) (nempe ipsius arcus Circularis) in re\( \delta \)am: hinc \( \dot{\iota} \)nus Celeberrimo Petro Varignone qu\( \alpha \)stitus est inventusque modus elegans paulo ant\( \cdot \)e postitus, quo figura plana \( \equiv \)qualis superficiei coni scale\( \cdot \)ni etiam men\( \dot{f} \)rurati potest, per extensionem curv\( \alpha \) alicius in re\( \delta \)am. Cum vero, curva illa quam exhibet non \( \equiv \)t ordinaria, sed transcendens, quamvis non nisi tetragonismo Circuli \( \equiv \)su\( \cdot \) rectificatione arcus Circularis ad constru\( \cdot \)ctionem indigeat; placuit mihi querere curvam ordinariam qu\( \alpha \)m idem \( \equiv \)\( \alpha \)st, feu cujus rectificatione exhibetur superficiei Coni Scaleni vel ejus portioni cuvis \( \equiv \)qualis figura rectilinea. Hoc ita sum con\( \dot{f} \)curatus:

Lisdem qu\( \alpha \) an\( \iota \)ea posti\( \alpha \)tis, \( \dot{\iota} \) pun\( \dot{c} \)d G versus D sumatur CGW, qu\( \alpha \) sit ad AG ut CD ad radium DO & jungatur BW, qu\( \alpha \) erit \( V(cca rrr\dot{\iota}rffggxhffxxx):r \). Pomo autem A esse extremum diametri AD remotius \( \equiv \)B. Ex pun\( \dot{c} \)d H ad re\( \delta \)tam AC ducatur HI ita ut ex O d\( \alpha \)da ad HI normalis O \( \dot{\iota} \)sit ad HO radium ut dimidia BW ad re\( \delta \)tam constantem, qu\( \alpha \)m major dimidia BA; Et re\( \delta \a HA ip\( \cdot \)sam \( \alpha \) \( \alpha \) ex centro O perpendiculariter ad CO ed\( \alpha \)tam ad partes H, secet in \( \Phi \). Eadem omnia fiant ad pun\( \dot{c} \)d h \& g, ut ducatur BW, ducatur \& hi, qu\( \alpha \)m ip\( \cdot \)sam \( \Phi \) secet in \( \Phi \). Jam ipsis \( \Phi \), \( \Omega \), \( \Phi \), \&c. \( \equiv \)quales I, \( \iota \), \( \omega \), \&c. ordinam.
tim applicentur normaliter ad OI, Oi, &c. ut fiat curva Ωω, cujus tangens Ων occurs ad ipsi AC in T. Tandem in AC ex I sumatur IX ad partes ad quas congruent IH, ih, quae xIX fit ad IO ut TI ad TO; & ex punto X quovis, hoc modo reperto, normaliter edudā XY, occurs respondeint recta positione datā HI in Y, erit ordinam applicata curvæ novæ Y(Y) cujus sub arcu elementari inter puncta Y & (Y) praescripta ratione determinanda intercepto, & sub constante supradicta, non minore quām BBA, comprehensum rectangulum æquabur triangu(lo BHH), nempe elemento superficie coni scalenī ; atque adeo evolutiōne seu extensione debita curvæ Y(Y) in rectam, filo evoluto dūdo in ipsam constanter supradictam, exhibebitur rectangulum æquale ipsī ABDHA superficie dimidī coni Scaleni vel ejus portioni (rectis sic-licet binis ex B & arcu basis comprehensa) cuicumque.

XX.

Joh. Wilh. Wagneri
Annotationes ad Eclipsin Solis,

Maxima in obscuratione aēr colorem simulabat cinctum, & lux diei superfetes crepera erat, pallida vel glauca, legendæ tamen adhuc scripturæ, & rebus procul mediecrēt tum discernendis sufficiens.

Stella, præter Venerem seu helioperum, nulī apparebat, liceet Spectatores, vidu quorum plurimi pollebant, in coelum hinc inde oculorum aciem direxerint. Aēr & erat magis, ac ante & post, figidiusculus. ros non cadebat. hirundines in volatu suo, ad terram plus tamen demidē, perseverabant.

Circa