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Euler's Theories of Musical Tuning With an English Translation of Du Véritable Caractère de la Musique Moderne

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Du Véritable Caractère de la Musique Moderne

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Introduction

Euler and Music

Here is a passage from the oration delivered at Euler's funeral by his grandson-in-law Nicholas Fuss. It is worth quoting at length:

> *Euler's chief relaxation was music, but even here his mathematical spirit was active. Yielding to the pleasant sensation of consonance, he immersed himself in the search for its cause and during musical performances would calculate the proportions of tones. One may say that his Attempt at a New Theory of Music was the fruit of his relaxation. This deeply pondered work, full of new or newly presented ideas, did not meet with the success anticipated, perhaps because it contains too much mathematics for a musician or too much music for a mathematician.* [7]

It is surprising, then, that nothing seems to be known about Euler's musical education or tastes. Did he sing? Play an instrument? Were any particular composers, contemporary or not, favorites of his? Any particular forms or styles? It is rather a mystery. Be this as it may, Euler left a substantial body of work on the theory of music, including the book-length work *Tentamen Novae Theoriae Musicae* [6], *Du Véritable Charactère de la Musique Moderne* [3], *De Harmoniae Veris Principiis* [4], a number of shorter works, including [5], and a large body of unpublished material in his notebooks.[1] [12]

None of this can be said to have been very influential in the course of development of Western art music, though the *Tentamen* has aroused a certain amount of controversy over the years. In particular, the papers *Du Véritable Caractère* and *De Harmoniae* have been neglected by scholars. Taken together, though, they shed an extraordinary light not only on Euler's evolving thought, but also on some perennial themes in the theory of

music. In *Du Véritable Caractère*, he takes certain ideas about the nature of music and pushes them, almost recklessly, to extremes. In *De Harmoniae*, published only seven years later, he seems to have returned to severest orthodoxy. It should be emphasized that, with the partial exception of the *Tentamen*, the published works are specifically about *tuning theory*, that is, about what combinations of sound frequencies are suitable for use in music.

Notes

1. In a recent magisterial biography of Euler (Calinger [2], p.158) we find   ...*Euler considered music theory a dry and tiresome subject, one in which it was difficult to maintain interest.* One might reasonably take exception to this statement.

Cultural Background

The first thing that strikes the reader of Euler's musical works is the constant reference to "principles of harmony" or "true harmony." Indeed, in *Du Véritable Caractère* almost every paragraph contains such a term. Euler assumed, probably correctly, that to his readers it would be obvious what he meant.

He was in fact expressing a view of the cosmos that was widespread among savants from the time of Pythagoras up to his own day, and beyond. This view assumes a highly structured cosmos in which events unfold according to rational laws. Music is a reflection of and an embodiment of these laws, and harmony in music is embodied in ratios of small whole numbers. This view perhaps found its most complete expression in the works of the philosopher Gottfried Leibniz, to whom Euler often refers. (For an exposition of Leibniz' views, see [10].)

Three specific principles of music Euler is assuming may be summarized as follows; details are given in the appendix. A *tone*[2] is a vibration in the air with a certain fixed frequency.

Relativity Principle: The relationships among tones (specifically, the ratios of their frequencies) are what are significant, rather than individual tones.

Octave Principle: Any two tones whose frequencies have ratio 1:2 are to be strongly identified with each other. They are said to form an *octave*.

Twelve-Tone Principle: If we construct a set of tones for use in music, starting with some specific tone and arranging them in order of increasing frequency, the thirteenth tone will have twice the frequency of the first. Another way of saying this is that there are twelve notes per octave.

It may be argued that the first two of these are based in physics and physiology. The third is the result of cultural evolution, and although it has been standard in Western music for a long time- think of the arrangement of notes on an ordinary keyboard instrument- it is by no means universal. Indeed, by the end of this work, Euler has developed a 24-note octave.

Notes

2. It is important to keep in mind that the French term *ton*, like its English cognate *tone*, has two distinct meanings. The first is the one given here. The other refers to a certain relationship between *two* such vibrations. In this sense, a *ton*, or *ton entire*, is in English a *whole tone*, and a *demi-ton* is a *half-tone*. The context always makes clear what Euler means by *ton*; we have tried to avoid ambiguities.

Some Notes on the Translation

Euler freely uses the terms *ton* and *son*; subject to the caveat above, we have translated these as *tone*, *note*, or *pitch*, depending on context. A *clavecin* might refer specifically a harpsichord, but for Euler's examples we may just as well think of a generic keyboard, which in his day would have been found on harpsichords, clavichords, fortepianos, and most organs. Standard modern instruments are arranged in the same way. Euler uses the German *H* and *B* for the notes we call in English *B* and *Bb*. The time values of some notes in the examples given have been altered for convenience. This does not affect the meaning, since Euler is concerned only with pitches, that is, frequencies.

We have translated Euler's *la musique ancienne* and *les anciens* as *ancient music* and *the ancients*, but it should be kept in mind that in the context of this paper that the term *ancient* simply means previous, or former. This brings up a vexing question. Euler insists that there is a sharp distinction between ancient and modern music, but does not specify where the one ends and the other begins. The first two paragraphs alone might suggest to the reader that modern music is that of his own time, the middle of the eighteenth century, and ancient music that which came before. This is almost certainly not the case, however. Based on the evidence of the text, we venture the hypothesis that the ancient/modern divide is what we now refer to as the divide between late renaissance and early baroque music, occurring roughly towards the end of the sixteenth century. (See notes to paragraph 6.)

Now it is time for Mr. Euler to speak for himself.

1. Everyone agrees that there is a very essential difference between modern music and that which was practiced formerly, but opinions about exactly what characterizes this difference are strongly divergent, and it seems that no one has yet perceived the real difference that prevails between ancient music and modern. Those who imagine that the entire difference consists of nothing but certain artifices which musicians are putting into practice today, but which were unknown formerly, do not sufficiently distinguish between the two types of music. On the other hand, those who place the distinction of modern music as a free use of all sorts of dissonances, which would have seemed insupportable formerly, push the difference too far, even beyond the limits of true harmony, of which the principles must always serve to govern modern music just as much as older.

2. It is thus an incontestable truth that, however great the difference between modern and ancient music may be, the one and the other ought absolutely to be in accord with the principles of harmony, and anything that is contrary to them can never be put into practice with any success. The judgement of the ear, to which everything must be

referred, however strange it may often seem, is however nothing less than arbitrary[3], but it is reliant on certain principles which are those of true harmony; and if we employ nowadays many dissonances which to the ancients would have seemed absolutely incompatible with the principles of harmony, it is necessary that they should not be at all contrary to them, and that by the same criterion, that they do not at all offend the ear.

Notes

3. Euler has *arbitraire*.

3. On this occasion, it is important to remark that the term *dissonances* is hardly fit to express the idea which one attaches to it. This idea is nothing less than the opposite of that which we attach to the word *consonance*, as the etymology would seem to indicate; and consequently, since consonances are agreeable to the ear, it must not be imagined that dissonances should be disagreeable to it, or repulsive; for on that basis, dissonances should be without doubt completely banned from all music. Dissonances thus differ from consonances, properly so called, only in that they are less simple or more complex, and it is just as necessary that this greater complexity be agreeable to the ear as is for the simplicity of consonances.[4]

Notes

4. Euler himself is not entirely consistent about the terms *consonance* and *dissonance*. His meaning, however, is clear- what were often called dissonances did not sound disagreeable at all but rather displayed a more complex sort of consonance. He briefly discusses what he considers to be true (unusable) dissonances in Paragraph 41.

4. After this remark I maintain, and I will prove, that the distinctive character of modern music consists of a certain class of consonances, understood in the sense that I have just explained, which were unknown to the ancients, or which they had not the boldness or skill to employ. This sentiment in itself does not need to be proven, since no musician will deny that modern works are absolutely full of such dissonances, which we do not find at all in the ancients; but it is principally a matter of explaining the nature of these novel dissonances, and to make evident how they may accord with the principles of harmony; or rather, as it is a thing involved with the judgement of the ear, that these new dissonances are in agreement with the principles of harmony, it is a matter of giving a clear and complete explanation of this agreement.

5. In order to bring this material into the light, I will begin by proving that ancient music was confined within such bounds as entirely excluded these new dissonances, and I will then demonstrate that the bounds of modern music are extended much farther, and that the new dissonances conform to them perfectly well, so that the true character of modern music is to be found in a very considerable extension of the bounds of ancient music, which doubtless makes a very essential difference between these two types of music. However, it is certain that the one and the other equally conform to the principles of true harmony, and if the thing should still seem doubtful, that would a be sign that one has not understood the foundation of true harmony well enough.

6. To prove that the new dissonances do not have a place in ancient music, I first remark that only three fundamental consonances were admitted, which are: 1st the octave, 2nd the fifth, and 3rd, the major third; and that all other consonances and dissonances which one can employ there are always composed of these three. Now, we know that the octave contains two tones which are in the ratio of 1 to 2; the fifth, two in the ratio of 2 to 3; and the major third, two in the ratio of 4 to 5. Thus, this music admits only those tones for which the relations may be expressed by only these three prime numbers 2, 3, and 5, that is, all numbers which cannot be decomposed with only these three numbers as factors are excluded from this music. Thus the numbers fit to represent these tones are: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, 50, 54, 60, 64, 72, 75, 80, 81, 90, 96, etc, and the others, which include prime numbers greater than 5, are excluded; concerning which the great Leibniz once said *In music, we need not count beyond 5.*

Notes

Here we have an indication of what Euler means by ancient music. The scheme here is similar to that of the eminent theorist Zarlino, who was active in the mid sixteenth century [13]. Zarlino propounded a system based on frequencies in proportions 1, 2, 3, 4, 5, 6, and 8, the so-called *senario*. To speak in very general terms, we may say that the basic building blocks of harmonic structure were at that time *triads*, that is, groups of three notes. The most common of these were the *major triad*, with proportions 4:5:6, and the *minor triad*, with proportions 10:12:15. Euler has not yet mentioned the *minor third*, which is the relation of two notes in ratio 5:6. We may regard a major triad as a minor third on top of a major third, and a minor triad as a major third on top of a minor third.

Around 1600, at the beginning of what we call the baroque era, certain *seventh chords* became common. Typical of these is the so-called dominant seventh, which we may think of as of a minor third atop a minor third atop a major third- for example, C E G Bb. The frequency ratios in such a chord are 20:25:30:36. The first and last have ratio 20:36, or 5:9. Now, 5 and 9 are "numbers fit to represent tones," according to Euler, but the *ratio* 5:9 is not considered in any way fundamental before the baroque era.

7. It is in regards to such numbers, by which we represent the tones of ancient music, and which make up the *diatonic genre*[5], that it is necessary to remark that the smaller they are, the simpler the resulting music will be. And so the tones expressed by 2 and its powers are called *F, f, f', f''*, etc.[6] The octaves will become more and more filled with tones, as will be seen in the following arrangements:

```
   I      |    II      |      III         |      IV
          |            |                  |
1     2   |  2,  3,  4 |  4,  5,  6,  8   |  8,  9, 10, 12, 15, 16
          |            |                  |
F     f   |  f   C   f |  F   A   C   f   |  F   G   A   C   E   f


              V                              VI
                               |
16  18  20  24  25  27  30  32 | 32  36  40  45  48  50  54  60  64
                               |
F   G   A   C   C#  D   E   f  | F   G   A   B   C   C#  D   E   f
```

VII

64	72	75	80	81	90	96	100	108	120	125	128
F	G	G#	A	A★	B	C	C#	D	E	f★	f

VIII

128	135	144	150	160	162	180	192	200	216	225	240	250	256
F	F#	G	G#	A	A★	B	C	C#	D	D#	E	f★	f

Notes

5. We may think of this as a *major scale*.

6. He in fact just calls them F and f throughout.

There is no Bb in any of these octaves. It appears later. Also, Euler does not include 3^5 = 243, which would be an alternative E.

We note that Euler is not yet extending the twelve-tone principle (See introduction.) He is postulating two slightly different versions of the notes A and f.

8. About these various octaves, which we may relate to the diatonic genre, I make the following remarks.

1. The first, which contains only two tones in the ratio 1:2, or an octave, is too simple to be able to be used in music, except in the lower octaves.

2. The second contains a fifth in addition to an octave, and provides a very agreeable chord, but is still too simple to be susceptible to much variety.

3. The third adds to the octave and the fifth a major third, and provides what is called a perfect chord[7] in music; also, the majority of chords which were used in ancient music may be reduced to this one.

4. The fourth octave receives, in addition to the preceding, the two new tones G and e, that is, a major second G and a major seventh e from the fundamental tone F: these tones together already form a chord too complicated for music, and which disturbs the ear. However, taking from these only the tones(10) A, (12) c and (15) e, we have a perfect minor third chord for the mode called *minor* [8].

5. The fifth octave receives additionally two new tones c# and d, which already make this octave sufficiently complete and susceptible to great variety, in light of the fact that it includes three[9] perfect chords which may succeed one another: for it is no longer a question of sounding them all at once.

6. The sixth furnishes the complete scale of the diatonic genre, which represents the principal tones[10] of keyboard instruments, besides the tone c#, which seems superfluous, although it may be very essential.

7. The subsequent octaves are even more laden with tones, and one finds among them tones like A***** and f***** which are not found at all on keyboards; but in their place one uses the tones A and f, which do not differ much from them.

Notes

7, 8. These are the major and minor triads; see notes to paragraph 6.

9. They are F:A:C, A:C#:E, and C:E:G, all of which reduce to 4:5:6.

10. We may think of these as the notes given by the white keys on a modern keyboard; sequentially, starting on C, they produce the *C major scale***.**

9. The consideration of these *foreign tones* A***** and f***** leads me to a reflection which will furnish us with a complete clarification of the question at hand. Although these tones do not at all have a place in the scale represented on harpsichords, musicians do not fail to employ them in practice, or rather, the ear imagines itself to perceive them, even though it in effect hears other tones, which differ from them only very little; this is doubtless a paradox, which deserves to be developed more carefully. We know by experience that when fifths or thirds are not tuned exactly, the ear is nevertheless accommodating enough to hear them as if they were perfect consonances, provided that the difference is not too great. Thus two tones tuned in the ratio 27 to 40 are taken by the ear for a perfect fifth, since the ratio 27 to 40 differs from the true ratio of the fifth, 2 to 3, by only the *comma*[11] contained in the ratio 80 to 81: for the ratio 27 to 40 being the same as 54 to 80, it differs almost not at all from 54 to 81, which reduces to 2 to 3. For the same reason, two tones represented by the numbers 25 and 32 are taken for a major third, since these numbers form almost the same ratio as 4 and 5.

Notes

11. Generally, a *comma* is a discrepancy found when a note is arrived at by two different methods. The one referred to here is the *syntonic comma*, or *comma of Didymus*. This is *the* amount by which Didymus corrected the Pythagorean major third to a just major third. It results from comparing four pure fifths with one pure major third For example, ascending by a pure (just) major third from C to E, taking C = 1, we have E = 5/4 = 80/64. Proceeding by fifths, we have C = 1, G = 3/2, D = 9/8, A = 27/16, and E = 81/64; this is the Pythagorean third.

The material in this paragraph deserves the closest attention. It is undeniable that even the most discerning ears are willing to forgive some degree of imprecision in frequencies, whether the result of some comma, as above, or of faulty adjusting of an instrument, or of unskilled performance. Music, as Euler later notes, would be utterly impossible were this not the case. The question is, how far can this tolerance be stretched? Very far indeed, according to this work.

10. The explanation of this paradox is not difficult, when we reflect that the measure of each tone is nothing but a certain number of vibrations by which the organ of hearing is

struck within a certain fixed period of time, and that a tone is considered higher or deeper according to whether the number of these vibrations produced within the same time is correspondingly larger or smaller. Now, the sensation of a consonance is aroused when the ear, being struck by two tones at the same time, notices the relation which prevails between the two numbers of vibrations given in the same time; from which we easily comprehend that the relation must be sufficiently simple to be able to be perceived by the ear. But when two tones differ very little from such a simple relation, the ear will be almost equally affected, and will sense the same agreement, as if the two tones were holding between themselves that simple agreement, the perception of which is pleasant to the ear.

11. And in effect, if the mind only enjoyed this sweet and agreeable sensation when the tones were perfectly tuned according to simple ratios, which make up the essence of consonances, that would be the end of all music, since it almost never happens that the tones of instruments are so exactly tuned. There are even musicians who claim that in order to fulfill all the requirements of music, it is necessary to make all of the twelve *semitones* in each octave equal.[12] But in this case there would be neither perfect fifth nor perfect third, without which harmony would be destroyed; which is what would happen nevertheless if the ear always sensed exactly the ratios which truly prevail among tones. From this it is necessary to conclude that as soon as the ratio between two tones closely approaches the ratio 1 to 2, or 2 to 3, or 4 to 5, the ear is equally affected by them as if these consonances were perfect: it is also sufficiently confirmed by experience that a piece of music need not lack success at all, even though the instruments may not be perfectly tuned.

Notes

12. This is the *equal-tempered system*, anathema to Euler and many others. Its construction is described in the appendix.   Here is a sample of what he had to say about it elsewhere. The quote from De Harmoniae [4] is especially relevant.

... *unacceptable because of a lack of a rational relation between sounds, excepting the difference of an octave*. (Euler [6], quoted in Gertsman [8])

... *we are seen to assume all consonances discussed here [thirds, fifths, and octaves] to be displayed in musical instruments so exactly that it should not be possible for even the least aberration to be sensed. Therefore, those musicians have greatly withdrawn from this rule, which is so greatly necessary for the production of harmony, who have thought that an interval of one octave should be divided into twelve equal parts, because in such a way musical ensemble may be transposed into all keys. Because, however, in this way no pure fifth may be given in the entire musical scale, and all major thirds deviate considerably from the true ratio, this opinion has now been rejected by the majority, to be sure, of musicians, who have easily recognized that to be removed from the true principles of harmony for the sake of transposition is not at all fitting*. (Euler [4])

There is a stark paradox here. We have just been reminded, and will continue to be reminded, that the ear is able and willing to "fill in" correct frequency ratios when confronted with small imperfections- but this does not seem to apply to imperfections introduced by equal temperament. In paragraph 9, for example, we have that the ratio 40/27 ≈ 1.481 is acceptable as a fifth, which has pure ratio 3/2 = 1.500. The impure fifth produced by the equal-tempered system is $2^{7/12}$ ≈ 1.498. Likewise, again referring to paragraph 9, 32/25 = 1.280 is to be sensed as a major

third, which has pure ratio 5/4 = 1.250 The equal tempered system gives $2^{4/12} \approx 1.260$. In both these cases, the result of equal temperament is *much* better than the other approximation. It seems that Euler was willing to forgive all sorts of deviations, as long as there were, if only in principle, perfect intervals *somewhere*. The idea that all perfection was to be abandoned entirely, except for octaves, he could not accept.

Now, to be fair, we have found one passing mention of the equal tempered system that is not entirely dismissive. (Euler [5]) For a very engaging cultural history of these ideas, see Isacoff [9].

12. It is thus an incontestable truth, that the ear does not very severely judge the tones it hears; but, provided they do not differ at all too sensibly from the exact proportions, which constitute the essence of consonance, it substitutes as if without thought these exact proportions, in order to recover from them the agreeable sensations which suit it. It is not that the ear does not sense these small deviations at all. Rather, it suppresses them in order not to be troubled in the enjoyment of harmony. However, there is no doubt that if the instruments were perfectly well tuned, and all the tones did not deviate at all from their true proportions, the pleasure the ear would derive from them will be much greater.

13. From this, we know how the same tone on a musical instrument is able to stand in place for two somewhat different tones according to its various combinations with other tones- thus in the arrangements of an octave marked VII and VIII above, the tone designated A* is much the same as A on instruments; and when we hear it in combination with either the note e, which is the fifth with it, or with F, with which it is the major third, the ear attaches the number 80 to it, which forms with the numbers 120 and 64 the ratios of 2 to 3 and 5 to 4, although this tone may be a little sharper on instruments, and expressed by the number 81. But when one combines the same tone A with the tone d = 108, or the tone an octave lower D = 54, then the ear imagines to hear the tone A*, which corresponds to the number 81, to enjoy the sensation of a fifth, even though it may be that instruments sound a tone slightly deeper, corresponding to the number 80. But if instruments contained both tones A = 80 and A* = 81, and we used the first in combination with the tones F and E, and the other with the tone D, we could not doubt that it would result in a much more perfect harmony[13]

Notes:

13. Here Euler is hinting at a *split keyboard*, containing more than twelve notes per octave. The history of such instruments is long and rich; indeed, there are some organs with split keyboards in use today. There is a technical discussion in [1].

14. This circumstance leads me to a reflection, which must be of great importance in the practice of music: it is that when the same note on instruments can hold the place of two different tones, these divergent uses should not be employed at the same time, or immediately one after another: but it is necessary that a few moments elapse, as if to make the ear forget the use which it had made of it previously. It even seems that musicians in effect follow this rule. For, in the example reported, as long as we combine the note A with F as its major third, we carefully avoid combining it with the note D as its fifth: experience has without doubt taught musicians that it disturbs the harmony; but, as

soon as one begins to use the note A as the fifth of the note D, it is regarded as having changed the key, and for example to have passed from the C major to G major: and then we should no longer join the same tone A with the tone F, because it would no longer be its major third, being now the fifth to the note D.

15. The same rule also applies to the other tones of which the numbers do not hold a sufficiently simple ratio. Let us consider the sixth arrangement of the octave related above in paragraph **7**, which, ignoring the note c# = 50, corresponds to the key of C major: here the interval of the notes d and f, expressed by the ratio 54 to 64, or 27 to 32, differs a little from the ratio 5 to 6, which is the true measure of a minor third: and if one employs this interval, the ear substitutes for it the ratio 5 to6, or indeed attributes to the note *d* the number 53 1/3: but that itself disturbs the harmony since this same note d is already used as the fifth to the note G, so that as soon as we should want to treat this tone d as a minor third from f, it would be necessary to renounce the previous use, which would produce a change of key.

16. After these reflections on ancient, or rather, *common* music, for distinguishing it from modern, or rather, *sublime*, music, since its character consists of a higher degree of harmony, as I will show; I am going to demonstrate, that the chords which distinguish modern music are absolutely incompatible with the nature of the consonances which I have just developed. For this effect, I need only consider certain chords which one uses in the key of C major, to compare them with the scales given above in paragraph **7** for this key. Here (below) the chords numbered 1, 3, 5, and 7 conform perfectly with the common principles of harmony, since they contain perfect agreement of the note C with the major third. But the second chord, being reduced to numbers, is

D	d	f	b
13½	27	32	45

where the first interval D:d is truly an octave, but the second d:f is not a perfect minor third, and the third f:b is a false fifth, which has the effect of making this chord incompatible with the principles of harmony. It is the same with the fourth and sixth of these chords, which reduce to the following numbers.

4						**6**			
F	d	f	a	c	G	d	f	g	b
16	27	32	40	48	18	27	32	36	45

where the ratios 27:32 and 32:45 must again disturb all harmony, not to speak of the defective fifth d and a in the fourth chord. If we wanted to say that the ear substitutes in place of the interval 27:32 a perfect minor third, as I remarked above, then, either the note f would no longer remain the fourth from the fundamental tone c, or the tone d would no longer be the fifth of the principal tone G, of which however the one and the other are absolutely necessary according to the principles of harmony.

Notes:

Here are Euler's chords in modern notation:

Here, for comparison, is the original:

17. Musicians agree that such chords cannot be reconciled with the principles of harmony, and they try to maintain them under the term dissonance, which they impose on them, but, if they mean by this term a chord in which the ear is not able to find any relation, we should be able to use with just as much success any other collection of tones, however absurd it may be: musicians are far from admitting this. We will also be little satisfied with the explanation that Mr. Rameau[14] gives of this phenomenon, in saying that in chord no. 6 (above), the tone *f* is only added to warn listeners that they should attribute this relation to the key of C, and not to the key of G. In chord no. 4, the tone *d* serves, according to the same author, to warn that this chord should not be regarded as belonging to the key of F, so that according to him, this addition is only used to characterize the key C major. I do not think that this explanation stands in need of being refuted.

Notes:

14. Jean-Philippe Rameau (1683-1764), French composer and theorist. See [11].

18. If these collections of tones did not present to the ear any proportion to perceive, they would doubtless be contrary to the principles of harmony, and would be banned from music. But musicians, far from maintaining that, find on the contrary something rather pleasant in these chords; and without the addition of these tones, which seem to

disturb all harmony, these chords would seem to them too simple and too scanty; in the same way that if one wanted to remove the third from perfect chords in common music, they would become too empty, and little suited to satisfy the, ear. It is in order to make music fuller that one adds to the reported chords these tones which seem to us contrary to harmony, and it is necessary that they produce an effect similar to that produced when one began to add the third to chords that previously contained only the octave and the fifth; and as it was not randomly that a tone was added to them, but that the very principles of harmony demanded the third, we also must be persuaded that the chords described above are equally founded on the principles of harmony.

19. Here are, then, two facts that we must take into consideration. The first is that the chords 2, 4, and 6 described in paragraph 16 above trigger in the ear a certain sensation of pleasure. The other is that these same chords represented by the numbers that have been attached to them, should be unbearable to the ear, since they contain impure intervals, expressed by numbers too complicated to be perceived by the ear. It is thus absolutely essential that the ear in hearing these chords should substitute in the place of one or two tones others which differ from them only very little, being expressed by numbers holding among themselves proportions simple enough to be perceived by the ear. There is also no doubt that these chords make a very particular type of consonance, which cannot be represented purely by the numbers 2, 3, and 5; for, in whatever manner one changes ever so slightly the numerical relations expressed above, admitting only the three stated prime numbers, one always arrives at numbers which are larger, and consequently more contrary to harmony.

20. All these reasons oblige us to recognize that it is necessary to resort to the prime number 7 to explain the success of these chords, so that among the ratios that make up the nature of these new chords, there enters beyond the prime numbers 2, 3, and 5, the next one, 7, and consequently we may say with the late Mr. Leibnitz that music has now learned to count up to seven. In effect, we have only to change by a little bit a single tone in the chords reported to lead them back to the principles of harmony. To begin with, let us consider the second one, expressed by the whole numbers

D	d	f	b
27	54	64	90

and change only the number 64 of the tone f into 63, in order to have the numbers 27, 54, 63, 90, which, being all divisible by 9, the tones will be in the same relation among themselves as the numbers 3, 6, 7, 10, which are certainly small enough to produce an agreeable sensation in the ear, and there is now no further doubt that the ear, in hearing this chord, substitutes in the place of the tone f another slightly lower, by the ratio 64 to 63, and that it then senses a very good relation among these tones, which must be much more agreeable than that which would result from the first numbers 27, 54, 64, 90, even supposing that the ear should be capable of perceiving them.

21. The sixth chord of the preceding passage reduces in the same manner to the principles of harmony. For, the tones being represented in this way by

G	d	f	g	b
36	54	64	72	90,

we only have to substitute 63 in place of the number 64, & those numbers being divisible by 9, they reduce again to these very simple proportions 4, 6, 7, 8, 10. This chord is precisely of the same nature as the preceding, since the tone 8 is nothing but the octave of the bass 4. Nevertheless, the preceding is a little simpler because the cube of 2 is not found, and in taking the tone G yet one octave deeper in order to have the numbers 2, 6, 7, 8, 10, the ear will find there yet more agreement. But, since here the tone f undergoes an alteration in the judgment of the ear, we see very well that this chord must neither precede nor follow any chord in which the tone f is found in its true significance. Also, musicians carefully observe this rule, to which simple experience has doubtless led them.

22. The fourth chord in the passage related above is a little more difficult to explain: for, in doubling the numbers which I have attached to it, to have

F	d	f	a	c
32	54	64	80	96,

we would mar everything if we wanted to substitute the number 63 in place of 64, and we would not arrive at a common divisor at all, for making the proportions simple enough. But, returning to the first numbers, which were

F	d	f	a	c
16	27	32	40	48,

it is apparent that if we give the tone d the number 28 in place of 27, all the numbers will be divisible by 4, and will reduce to the following very simple proportions:

F	d	f	a	c
4	7	8	10	12,

which chord is once again of the same nature as the preceding ones. However, as it is the tone d which is altered, I comment once again that this chord must neither precede nor follow any other which contains the tone d in its natural value.

23. It will doubtless seem very hard that, in order to make this last consonance harmonious, the ear should be obligated to change the tone d almost by an interval of a **h**alf-tone. I agree that this change is very considerable, and that if a fifth should differ so much from the true proportion of 2 to 3, it would be insufferable, and that the ear would try in vain to remedy it. But I remark that, although octaves and fifths hardly admit any deviation from their true proportions, thirds admit much more considerable deviations, which can even surpass the interval named *diese*,[15] consisting of the ratio 125 to 128,

without destroying harmony. Thus if consonances, the less they are simple, allow a greater deviation, it is very natural that our new consonance, the proportion of which contains the number 7, should not be too troubled by a tone which deviates from accuracy by a factor of 27 to 28.[16] But there is no doubt that this chord would be much more agreeable if in place of the tone d we would use another a little sharper, and if we should put in its place the tone d#, the proportion would be almost completely true. Also we see that the chord F A c d# is very much in use among musicians, from which we must conclude that the preceding should produce in the ear the same effect as that very chord.

Notes:

15. The *diese* is derived from a comparison of the true octave 64:128 with the result of raising the tone 64 by three major thirds, which results in 64:125.

16. In truth, it *is very* hard for the ear to tolerate a deviation as large as 28/27 in any musical context whatsoever.

What Euler has proposed here and in the previous few paragraphs is astonishingly bold. He says that our ears spontaneously make large adjustments in order to perceive harmonious tone relations based on the prime 7.

24. We have here arrived at our goal, which is to characterize the true character of modern music, and we can no longer doubt that this character consists of the use of a new class of consonances, which had been completely unknown in the music of times past. These new consonances, being expressed by numbers, contain the prime factor 7, while formerly only numbers resolvable into the three prime numbers 2, 3, and 5 were admitted. It is thus in effect a higher degree of perfection to which we have brought music, in having introduced this new type of consonance. But also for this very reason, modern music demands more delicate and clever ears, for perceiving and distinguishing these new consonances, and so we should not be surprised if many people find no enjoyment in these new pieces of music, for as soon as these new consonances exceed the range of their ears, they must seem like veritable dissonances to them.

25. In order for us to form an idea of the skill of the ear required to grasp these new consonances, I begin by observing that there is probably no ear that is not capable of distinguishing an octave well. As soon as someone begins to study music, it is necessary to have an accurate idea of an octave, and to be capable of adjusting two tones exactly in an octave on musical instruments, that is, it is necessary for the ear to have a correct idea of the ratio 1 to 2. And then we assume an equal skill in recognizing fifths, and adjusting two tones to them exactly on instruments, that is, the ear should be put in a state of perceiving the ratio 2 to 3, which is already more difficult. To get better accustomed to this, it is good to begin with intervals composed of one octave and one fifth. The ratio being 1 to 3, the ear grasps this more easily and will sense the agreement. A little practice will suffice for this effect, and will put the ear in a state of well distinguishing, not only the fifth itself comprised of the ratio 2 to 3, but also the fourth, that is, the ratio of 3 to 4, and to notice the least aberration, if there is any.

Notes:

Euler says nothing about this, but perhaps the reason that ratio 1:3 is easier for the ear to grasp than ratio 2:3 is that 3 is in the overtone series of 1 but not of 2.

26. When the ear has acquired this double capacity of distinguishing octaves and fifths, together with fourths, it is necessary to accustom it to the major third, expressed by the ratio 4 to 5, which already requires a greater exercise of delicacy on the part of the ear. To provide some help, one may begin by making the ear recognize intervals composed of an octave and a major third, made up of ratio 2 to 5, or even those composed of two octaves and a major third, which correspond to ratio 1 to 5. As soon as the ear senses a certain agreement there, it will easily arrive at the discernment of the ratio 4 to 5, or the simple major third. Also at the beginning, we will be able to add the fifth, in order to be able to grasp well the perfect chord made up of the three numbers 4, 5, and 6. For, if the fifth is well- adjusted, we will perceive easily whether the third is exact or not, and in that case, the ear will acquire a true idea of the minor third, made up of the ratio of the numbers 5 and 6, and then also of the intervals which are derived from these, like the major sixth, made up of the ratio 3 to 5, and the minor sixth, in ratio 5 to 8.

Notes:

It may be easier to start with 1:5 than with 4:5 for the reason mentioned in the last section.

27. I believe that such an exercise, sustained for a sufficiently long time and varied through all tones, would be infinitely more useful to those who wish to study music than learning it by singing and forming tones according to some given scale, where most of the tones are arbitrary, while the consonances of which I have just spoken are founded on nature itself and are accompanied by a certain type of agreement, which it is above all essential to make well-sensed by the ear. Truly, how do we suppose that a student intones C, D, E, or do, re, mi, at a time when there is no correct idea of fundamental consonances? When the exact measure is caught upon occasion, it is only by pure chance. And then, since there are two types of interval called a whole tone, one made up by the ratio 8 to 9, and the other by that of 9 to 10, which of these two do we want the student to follow in rising from *do* to *re*? Do we wish to be content with something approximate? That would be to upset all principles of harmony, since it is assumed that by ascending through the scale *do*, *re*, *mi*, etc, one arrives at a true octave from the first *do*.

28. It would not be out of place to explain here how we can succeed much better, not only to teach how to sing well, but also to form from the beginning a just and precise taste for music, which is doubtless the main end towards which we should strive. I will thus begin with the exercise of which I have just spoken, to impress upon the ears a very exact sense of the three fundamental consonances of the octave, the fifth, and the major third; none of these will fail to make a particular impression and be accompanied by a certain approval by the ear; in this manner we will soon acquire the habit, any tone being put forth, of intoning exactly the octave, the fifth, or the major third, as we wish, equally ascending or descending from it. Such an exercise will, in the end, make these consonances so familiar to the ear that it will distinguish them in the midst of many other tones, and if these intervals should not be exact, that it will easily notice the aberration.

29. After this preparation, it will not be difficult to familiarize the ear with all the other intervals. We have only to consider how each interval results from the three principal consonances: so the major tone 8:9 decomposes into ratios 2 to 3 and 4:3. To rise by this interval from the tone C to D, we only have to represent the fifth above C, which is G, and then descend from there by an interval of a fourth, which will lead to the tone D. At the beginning it is necessary to allow the singing of this auxiliary tone G, but soon one will become accustomed to singing it only in thought, or, when the tone C is accompanied on an instrument by the fifth G, it will help in jumping from the start to the true tone D. But the leap from D to E should be a *minor tone* comprised of the ratio 9 to 10. It is necessary to decompose it into the three ratios 3:4, 3:2, and 4:5; thus one will first rise from D to its fourth G, from which one will descend by a fifth to C, from which one will ascend once more by an interval of a major third to E. Or, if the preceding tone C is still present in the ear, its major third E will be immediately determined: these mental leaps may be displayed as follows.

Major Tone　　　　　　　　　**Minor tone**

Notes:

For comparison, here is Euler's notation:

With one exception, musical examples will be given in modern notation in what follows.

The *minor tone*, ratio 9:10, is one of the two slightly different versions of the whole tone; the *major tone* is the other, in ratio 8:9. For the term *tone* in the sense of a relation, see remarks in the Introduction.

30. Now, in rising from E to F, or singing the interval *mi*, *fa*, which, being a *major semitone*, is composed of the ratio 15:16, we decompose it into the two ratios 5:4 and 3:4, and thus we first descend from E to C by the interval of a major third, and from C we ascend in turn by a fourth to F, like this:

The following leaps of the ordinary scale are the same as I have just explained, since from F to G there is a major tone, from G to A a minor tone, from A to B once again a major tone, and from B to C a major semitone. So the singing of this scale with the auxiliary tones will be represented as follows:

One will easily discern that these auxiliary tones are not superfluous, but that they may serve to fill out the melody and to determine the following tones.

31. In similar fashion one may learn to produce all the other intervals which one uses in music, and such an exercise will certainly perfect the discernment of the ear, and to make it more suitable for appreciating good pieces of music. I will thus go through these intervals.

1. The *minor semitone*, made up of the ratio 24:25: one descends a fifth from G to C, and from there ascends by two major thirds C to E and then E to G#:

2. The *minor third*, made up of the ratio 5:6: One descends by a major third from E to C, and from there rises by a fifth to G. Or, one rises at the beginning from E to its fifth B, and from there descends by a major third to G. But a little exercise will soon allow the ear to recognize the minor third immediately.

3. The *augmented forth*, made up of the ratio 32:45, or from F to B.: One first descends by a fourth from F to C, from where one rises by a fifth to G, and from there by a major third to B. But this interval is not much in use.
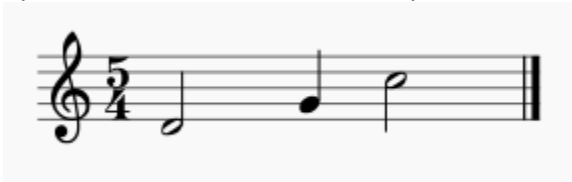
4. The *minor sixth*, made up of the ratio 5:8, as from E to c, is produced by descending by a major third to C, and from there rising by an octave to c.

5. The *major sixth*, made up of the ratio 3:5, as from C to A, is produced by rising by a fourth from C to F, and from there by a major third to A.

6. The *minor seventh*, made up of the ratio 9:16, as from D to c, is produced by rising by a fourth to G, and from there by another fourth to c.

7. The *other minor seventh*, made up of ratio 5:9, as from E to d, is produced by first descending by a major third to C, and ascending from there by two successive fifths to G and d.

8. The major seventh, made up of the ratio 8:15, as from C to B, is produced by first rising by a fifth to G, and from there by a major third to B.

32. Such an exercise sustained for some time will hone the ear in recognizing the nature of each of these intervals and to sense the agreements. We will also soon see that this method is much more suited for training up musical talents, than that which is ordinarily used, and which is not, for the most part, founded on any principle of harmony. There is also no doubt that by this means the sensitivity of the ear will become much more discerning, and that it will perceive the least disturbances of the exact proportions that should prevail among the consonances. But it still seems doubtful that such delicacy should be advantageous in music, since many very beautiful pieces would become unbearable to the ear. But it could also happen that the most excellent pieces would revolt such an over-delicate ear because of inexact performance. That is because, since the tones of instruments frequently deviate from the true proportions which consonances demand by a *comma* or more[17], the fear is that such a deviation would be unbearable to those ears.

Notes:

17. See notes to paragraph 9.

33. The intervals that I have just developed are those that modern music has in common with ancient. It will thus be very interesting to examine in the same way the consonances and intervals that are peculiar to modern music, and which make up its distinctive character. Now, all these new consonances derive from the number 7, and thus the principal one that serves as the foundation of all others will be that which is made up of the ratio 4:7, which is rather simple. It is thus at first a matter of accustoming the ear to this new consonance, so that it completely understands the nature and the pleasant feeling by which it is accompanied. But since the proportion 4:7 is not resolvable into anything simpler, it is essential that the ear grasp it immediately, in the same way that it perceives the octave, the fifth, and the major third; but there is no doubt that such skill demands much more diligence and longer training. Let us see then in what way we will be able to succeed in making the ear respond to this new consonance.

34. Let us take the tone C as that which corresponds to the number 4, and then the number 7 will correspond to a tone slightly deeper than the tone Bb, that is, included between A and Bb, since the interval 4:7 is a little smaller than the minor seventh 9:16, or a little greater than the major sixth, 3:5. So, while one sounds the tone C, we may produce successively several tones between A and Bb on a violin,[18] and we will find among them one which will have a very good agreement with C, and to be sure that this tone is neither the major sixth nor the minor seventh, we only have to add the tones E and G, the one as a major third and the other as a fifth with C, and we will see that the given tone between A and Bb produces with them a very beautiful harmony, endowed with a very particular type of agreement. By this means, one will obtain the true chord that constitutes the character of modern music. Let us, then, designate this new tone between A and Bb by the symbol B*, since it is closer to Bb than to A, and the four tones which make up this new chord C, E, G, B* will be expressed by the simple numbers 4, 5, 6, 7, and this same chord may be represented in musical notation:
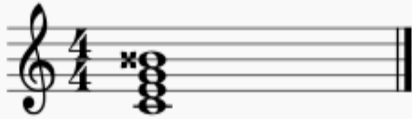
Notes:

This is the original; we have simplified the notation. The asterisk in our version should not be confused with the *double sharp*, which it sometimes signifies, here and in the following examples.


18. We mention in passing that if one wishes to produce the ratio 4:7 on a stringed instrument, the easiest way to do it is to use an open, that is, unfingered, string as the lower note.

There is some danger of confusion in the terminology here. Euler is expounding the idea of admitting the prime 7 as a factor in frequency ratios. This is not directly identified with *seventh chords* in music. (See notes to paragraph 6.)

35. This new chord thus contains, first, the ordinary perfect chord (major triad) on the fundamental tone C; that is to say, the fifth G and the major third E, but above these one adds the new tone Bb*, which forms with the fundamental C the ratio 7:4. These four tones thus form a chord more complete than the major triad, and the new tone B* which one adds to it gives it a very particular grace, which gives modern music its advantages. As this new tone Bb* holds ratio 7:4 with the fundamental C, let us also consider more closely its relationship with the other tones E and G. But the ratio of the tones E:Bb* being 5:7, this interval is almost an augmented fourth, or a false (diminished) fifth made up of ratio 32:45, the difference being merely the small interval 224:225, but, because of this small difference, this interval should be much more agreeable than the false fifth or the augmented fourth. Finally, the interval G:Bb* being expressed by the ratio 6:7, is a little smaller than a minor third, or the ratio 5:6, and if we would add to it the tone c above, the octave of the fundamental C, we would have the interval Bb*:c, expressed by the ratio 7:8, a little larger than a whole tone, 8:9.

36. The successive diminution of the intervals which make up this new chord C, E, G, Bb*, c is quite remarkable, the first C:E being a major third in ratio 4:5, the second E:G being a minor third in ratio 5:6, the third G:Bb* a little smaller than a minor third, and represented by ratio 6:7, and finally the fourth Bb*:c a little closer than a whole tone, of which the measure is the ratio 7:8. It is thus the simplicity and good order of these numbers 4:5:6:7:8 which makes this new chord agreeable, and lends it a very particular grace, unknown to ancient music. It is also clear that this chord may enter into music in many different guises, according to how one expresses these tones at one or two octaves higher or lower, and of these the simplest and most agreeable to the ear is doubtless that which is expressed by the simple numbers 1:3:5:7, which are represented thus in musical notation:
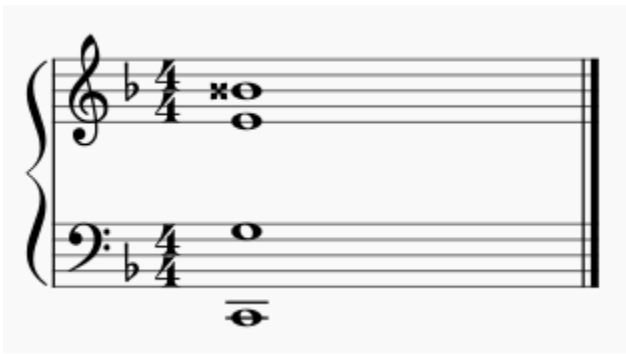
37. To better understand the reach of this new chord and its practical application, I will once again list all the variations which can occur in the space of an octave.
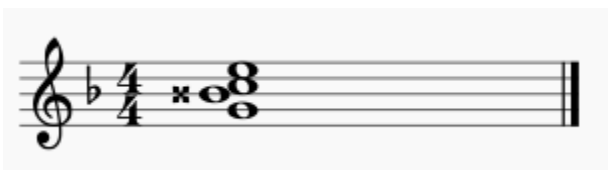
1. The first form is that which I have already described, and which corresponds to the numbers 4:5:6:7.
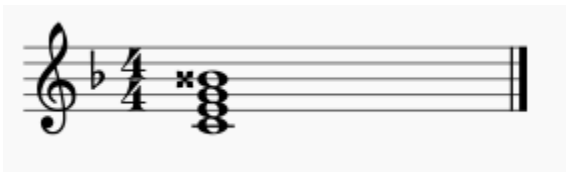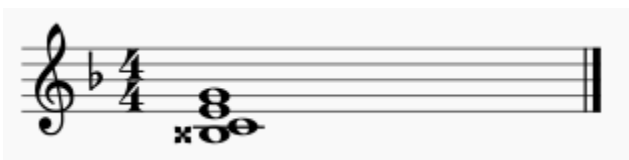

2. The second form is represented by the numbers 5:6:7:8, where the minor third occupies the lowest place.

3. the third form begins at the bottom with the tone 6, and contains numbers   in the order 6:7:8:10.

4. The fourth variation, finally, carries the new tone 7 to the deepest position according to the order 7:8:10:12.

We have only to cast our eyes over modern compositions of musicians, and we will see that all these variations are used there with the greatest success, with the only difference that instead of the new tone marked here with a star [*], they use whichever of the ordinary tones that is closest to it.

38. But music would doubtless have much more charm if one were capable of expressing all these tones exactly; this would be very possible to do with violins and similar instruments, where one is master of modifying tones to one's wishes. But, when one uses a harpsichord, or other instrument having only a certain number of fixed and determined tones, it is necessary to substitute, in the place of these new tones, others which differ from them by not much, and, as I have already remarked, the ear is not so scrupulous that it will not allow such a substitution, but rather will itself supply one, putting it in the place of those sounds which are not quite exactly the ones harmony demands. So, if the fifths and thirds of instruments are not exact, the ear is always ready to remedy this fault, as long as it is not too great, and even if instruments should be perfectly well-tuned, this accuracy would have place only in pieces composed in in the key of C major and a few others; for other keys there are always thirds or fifths, or both, which would not be exact, and this very deficiency seems to characterize each of the different keys, which without this difference would seem to resemble each other perfectly.

39. Since these new tones which enter into the chords characteristic of modern music are not found at all on instruments on which all the tones are fixed, on that account I will call them *foreign*, and it will be necessary to regard as a *musical license* that it be permissible to use, in place of these foreign tones, the tones on instruments which most closely approach them; also, we sometimes admit a greater deviation from exactness, as I have already remarked that in the place of the chord C:E:G:Bb which corresponds rather well to the numbers 4:5:6:7[19], we may use this one: C:E:G:A, since the true tone Bb*, which makes the chord exact, is found between the tones A and Bb, though it is much closer to Bb than to A. The reason one uses in this case the tone A rather than Bb is obvious: the tone Bb is not found in the diatonic scale, which characterizes the key of the piece. One prefers thus to use the tone A, which already enters into the chords, and by which the ear has already been struck, though it was in a completely different context.

Notes:

19. See notes to Paragraph 34.

40. From this I draw the rule, essential in musical composition, which is that these new chords should neither follow nor be followed by chords in which the foreign tones, or rather the ordinary tones which are used in their place, are used in their natural state. And so the chord described above C:E:G:B* properly has a place in the key of F, major or minor equally, where C is called the dominant tone, but it can be used neither before nor after the chord Bb:d:f when the tone Bb, which is the foreign one, has its natural value. It is the same with the chord F:d*:f:a:c, considered above in paragraph 21 where d* is the foreign tone represented by the number 7; this chord is characteristic of the key of C major, and thus must never immediately follow nor be followed by the chord G:B:d, where the tone d is found in its natural state. Now, I remark that this rule is usually well-observed by musicians, and if it should not be, we will readily agree that whatever

composers allow themselves is not, for that reason, founded in harmony. It often seems, rather, to be purely whimsical.

41. Finally, it is very necessary to be careful not to confuse these chords, which properly characterize modern music, with other complicated chords, known by the term dissonances, which, according to the rules of musicians, should be prepared and resolved in a certain way; whereas the chords of which I have spoken up to this point do not require any preparation, and can be used just like the perfect chords. But for those which we properly call dissonances, we must observe that these are only a meeting or complication of two perfect chords which must follow one another, but where the first gets applied for too long and so merges with the following one. But when these two chords, while following each other, produce a good effect, the ear should not be disturbed by hearing them both at the same time; and we will easily see that this is the true reason for the use of those chords called dissonances. But the one type of chord that I call here *new* is completely different, since its essence contains some foreign tones, which correspond to the number 7, unknown in ancient music.

42. This consideration may also lead us to discover, perhaps, some new chords of a kind that musicians have not yet seen fit to use. For this effect we have only to place before our eyes a scale of tones contained within an interval of an octave, which may be used in the same key; which is what happens when the numbers which are the measures of these tones are not too large. Here, then, is one such octave:

64: 72: 75: 80: 90: 96: 100: 108: 120: 128

F: G: G#: A: B: C: C#: D: E: f

which is as suitable for the key of C major as for A minor. Now we need only mark within this interval the numbers formed as the product of 7 and some other number whose only prime factors can be 2, 3, and 5; they are:

1st 70 = 10×7; 2nd 84 = 12×7; 3rd 105 = 15×7, 4th 112 = 16×7; and 5th 126 = 18×7.

These numbers thus give the foreign tones, in place of which one may use the tones of the clavichord which approach them most closely: thus

| For the Number	70	84	105	112	126
The Tone	G* or F#*	Bb*	D*	D#*	f*

Notes:

The first series of tones here is the same as VII from paragraph 7, with the exclusion of the tones A* and f*. Euler has used the term *foreign tone* in two different senses; neither A* (81) nor f* (125) has 7 as a factor. His scheme also excludes $2 \cdot 7^2 = 98$, which is perhaps not surprising.

43. If we now join to each of these multiples of the number 7 some similar multiples of the numbers 4, 5, 6, and 8, we will get the following new chords:

I.

40:	50:	60:	70:	80
A:	C#:	E:	G*	a
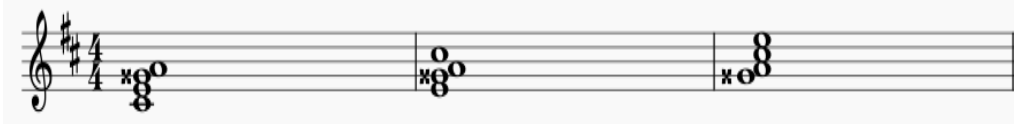
of which the various representations may be found as follows.

II.

48:	60:	72:	84:	96
C:	E:	G:	B*:	c

III.

60:	75:	90:	105:	120
E:	G#:	B:	D*:	e
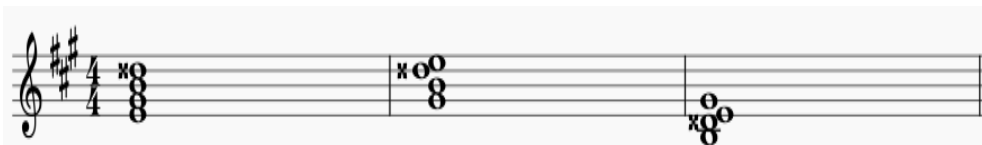
IV.

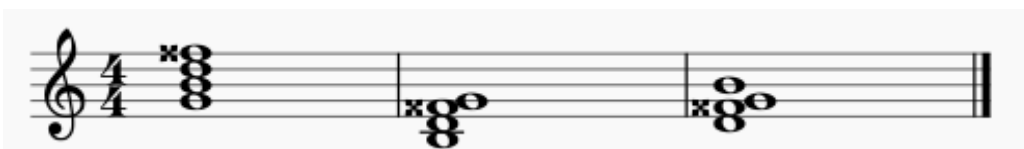| 64:	80:	96:	112:	128
F:	A:	C:	D#:	f

V.

72: 90: 108: 126: 144

G: B :   D: F*:   g

I have everywhere marked the foreign tones with a star (*).

44. But since there is no doubt that all these chords would produce a better effect if we were able to produce the foreign tones which enter into them exactly on instruments, music could be also in this regard be carried to a higher degree of perfection if we found a means of doubling the number of tones on harpsichords. Just as the twelve ordinary tones of an octave are included in the expression $2^n \cdot 3^3 \cdot 5^2$, of which all the divisors furnish these twelve tones: it would be necessary to use the expression $2^n \cdot 3^3 \cdot 5^2 \cdot 7$, which provides 24 tones in each octave, that is to say, twelve new ones, in order to represent exactly the tones which I have called foreign. Here are the ones and the others.

Principal Tones	Numerical Expression		Foreign Tone	Numerical Expression	
F	2^{12}	$= 4096$	F*	$2^6 \cdot 3^2 \cdot 7$	$= 4032$
F#	$2^5 \cdot 3^3 \cdot 5$	$= 4320$	F#*	$2^3 \cdot 3 \cdot 5^2 \cdot 7$	$= 4200$
G	$2^9 \cdot 3^2$	$= 4608$	G*	$2^7 \cdot 5 \cdot 7$	$= 4480$
G#	$2^6 \cdot 3 \cdot 5^2$	$= 4800$	G#*	$3^3 \cdot 5^2 \cdot 7$	$= 4725$
A	$2^{10} \cdot 5$	$= 5120$	A*	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$= 5040$
Bb	$2^3 \cdot 3^3 \cdot 5^2$	$= 5400$	Bb*	$2^8 \cdot 3 \cdot 7$	$= 5376$
B	$2^7 \cdot 3^2 \cdot 5$	$= 5760$	B*	$2^5 \cdot 5^2 \cdot 7$	$= 5600$
C	$2^{11} \cdot 3$	$= 6144$	C*	$2^5 \cdot 3^3 \cdot 7$	$= 6048$
C#	$2^8 \cdot 5^2$	$= 6400$	C#*	$2^2 \cdot 3^2 \cdot 5^2 \cdot 7$	$= 6300$

```
        |                        |        |
D       |  2⁸·3³      = 6912  |  D*    |  2⁶·3·5·7    = 6720
        |                        |        |
D#      |  2⁵·3²·5²   = 7200  |  D#*   |  2¹⁰·7       = 7168
        |                        |        |
E       |  2⁹·3·5     = 7680  |  E*    |  2³·3³·5·7   = 7560
        |                        |        |
f       |  2¹³        = 8192  |  f*    |  2⁷·3²·7     = 8064
```

It is evident that it is enough to have determined a single one of the foreign tones, since all the others may then be formed from it by simple fifths and major thirds.

Notes:

Once again, A* and f* don't mean here quite what they did in paragraph 7.

Euler is proposing in this paragraph a specific type of *split keyboard.* See notes to Paragraph 13.

Afterword

Here we point out that in Euler's next, and last, published work on music theory, De *Harmoniae* [4], he has abandoned the speculations in this one. Although the ratio 4:7 is mentioned in passing, the bulk of *De Harmoniae* is an exposition of his *Speculum Musicum*, which is a very strict system of *just tuning,* based purely on thirds and fifths. The scheme is as follows:

Starting with any fundamental tone, ascend by pure fifths to get three more tones. (At this point, it is as if the four strings of a violin have been tuned.)   Above each of those four tones, ascend by a pure major third. Above each of *those* four tones, ascend once again by a major third.

We have now a sort of commutative diagram, in which multiplying by 3/2 moves to the right, and multiplying by 5/4 moves upward. Division by 2 is done as needed to remain in [1, 2). This arrangement is now referred to as the Eulerian Monochord. (See Barbour [1].)

It is striking that in [5] this scheme of tuning is presented, without any elaboration at all, as *the ordinary system*; it is in fact one of many possible *just tunings*. (See appendix.)

Appendix

Musical tones consist of regular vibrations in the air, which when transmitted to our ears are converted to a sensation of tone. The greater the number of vibrations per second (hz), the higher the tone is, or the higher its *pitch*. Euler simply refers to the number of vibrations in some fixed period of time.

Western art music, and indeed many styles of music in many eras and many cultures, may be said to be built on three fundamental principles, which we may call the Principle of Relativity, the Octave Principle, and the Twelve-Tone Principle. (Here the term *tone* is used in its general sense, and does not refer to two notes related by a whole step. See notes to the Introduction.)

The Principle of Relativity says that in music, the relations *among* tones are more significant than individual tones themselves. This is almost obvious, psychologically. For example, if a piece of music were played and then replayed with all the pitches increased by, say, 20%, no one who is not deaf would have any hesitation about identifying it as the same piece. Nevertheless, the situation is peculiar. Imagine increasing all the *light* frequencies in a painting by 20%- the effect would be bizarre.

The Octave Principle says that two tones whose frequencies have ratio 1:2 are to be strongly identified with each other. Think for example of the note A = 440 hz on a modern keyboard. The notes with frequencies 55 hz, 110 hz, 220 hz; 880 hz, 1760 hz, etc, are also called A. There are physical reasons for this principle, which don't concern us here. Suffice it to say, as Euler does, that it requires little musical skill to identify octaves. The practical consequence of this principle is that if we want to construct a system of tones for use in music, we need only construct tones within some interval of form [α, 2α), that is, greater than or equal to α and less than 2α, and extend upward and downward by multiplying and dividing by powers of 2.

The Twelve-Tone Principle says that starting with any tone and moving through consecutive tones, upward or downward, the thirteenth tone and the first form an octave, that is, have frequency ratio 2:1. Another way of saying this is that there are exactly twelve tones within each octave. Twelve tones, rather than some other number, are used for convenience- there are enough for a vast variety of music, but not so many that the ear is bewildered. There have been other systems, however, and Euler himself has proposed a 24-note octave in this work.

Given these principles, how should a system of tones be constructed? Let us choose some tone at 1 hz. (It would be inaudible, but remember, it is the *ratios* that are significant.) How should we fill in eleven more tones between 1 and 2? This is a very long and tangled story.

The common starting point is the observation that two tones are heard to be *harmonious*, or *consonant*, if the ratio of their frequencies can be expressed in small whole numbers. The strongest of these ratios are 2:1, the *octave*; 3:2, the *fifth*; 5:4, the *major third*; and 6:5, the *minor third*. From these, others can be derived, and Euler does so. (A system based purely on such small whole number ratios is called a *just tuning*.) The trouble is that 2, 3, and 5 are pairwise relatively prime, and if we put in 7, as Euler does, or any other prime, that situation does not change. The practical outcome is that we generally cannot have harmonious relations among all tones in our family. One example follows; many more are to be found in the references.

Suppose we decide, as did the Pythagoreans several millennia ago, to base our tones on the fifth. That is, we start with 1, and repeatedly multiply by 3/2, dividing by 2 whenever necessary to remain in [1,2). Calling the first tone C, we get

$$1 = C, \quad 3/2 = G, \quad 9/8 = D, \quad 27/16 = A, \quad 81/64 = E, \quad \text{etc.}$$

It is evident that no pair of these tones can form a major or minor third. For instance, in order for E and C to form a pure Major third, the frequencies should have ratio $5/4 = 1.2500$; in this Pythagorean system we get ratio $81/64 \approx 1.2656$. This difference is quite conspicuous to the ear.

The Equal-Tempered System (ET)

This system was described as early as the sixteenth century by, among others, Vincenzo Galilei and Simon Stevin. There is evidence that it was known in China in the same era. It is now the default tuning system on keyboard instruments, but it came into general use only slowly and against bitter opposition. Euler in particular detested it.

In ET, the frequency ratios of all adjacent tones are identical. It is not difficult to see that this ratio must be $2^{1/12} \approx 1.0594$. The result of this is that octaves have the pure ratio 1:2, but other intervals are defective. For example, the fifth, say G and C, has ratio $2^{7/12} \approx 1.498$, extremely close to the pure ratio $3/2 = 1.500$. The major third, say E and C, has ratio $2^{4/12} \approx 1.260$, as opposed to the pure ratio $5/4 = 1.250$. The discrepancy is quite grating to a sensitive musical ear. Other intervals may be compared in a similar fashion.

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