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A history and translation of Lagrange's "Sur quelques problèmes de l'analyse de Diophante"

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Abstract

Among Lagrange's many achievements in number theory is a solution to the problem posed and solved by Fermat of finding a right triangle whose legs sum to a perfect square and whose hypotenuse is also a square. This article chronicles various appearances of the problem, including multiple solutions by Euler, all of which inadequately address completeness and minimality of solutions. Finally, we summarize and translate Lagrange's paper in which he solves the problem completely, thus successfully proving the minimality of Fermat's original solution.

In the work "Sur quelques problèmes de l'analyse de Diophante," Joseph-Louis Lagrange (1736-1813) tackled a problem that Pierre de Fermat (1601-1665) formulated over a hundred years prior and deemed intriguing enough to pose as a challenge to his own contemporaries. Not surprisingly, Leonhard Euler (1707-1783) took up the challenge along with Lagrange, and while the former gave a partial answer to the question, and even managed to make generalizations, it was the latter who ultimately explained and supported what Fermat had claimed without proof. In this article we first present a brief history of the problem and chronicle attempts to solve it, with a focus on how Lagrange's work fits in with Euler's endeavors. Then we present a synopsis of Lagrange's paper, pointing out the crucial technique that he employed. Finally, we end with a complete, annotated translation of Lagrange's work.

History of the Problem: From Fermat to Euler and Lagrange

The available evidence points to Fermat as the source of the problem. The problem in question is to find a right triangle the sum of the legs of which is a square and whose

hypotenuse is also a square; in other words, find two numbers whose sum is a square and the sum of squares of which is a fourth power. Fermat seems to have communicated this problem to several of his contemporaries including Jacques de Billy (1602-1679), Bernard Frénicle de Bessy (1604-1674), Brulart de St. Martin^a, and Marin Mersenne (1588-1648). It appears as one of three problems, included as a parting challenge, at the end of a letter that Fermat wrote to St. Martin on May 31, 1643.^b Three months later, in August 1643, Fermat wrote Mersenne, alluding to the fact that Fermat had previously communicated this problem to both Frénicle and St. Martin.^c In this letter, Fermat provided his answer to the problem, the lengths 4,565,486,027,761 and 1,061,652,293,520 for the two legs and 4,687,298,610,289 for the hypotenuse, without explaining his method. The problem also appears as one of Fermat's observations in Claude Gaspar Bachet's (1581-1638) translation of Diophantus' (c.200-c.284) *Arithmetica*, book VI, number 24 [1], where he included the same triple of solutions mentioned in the letter to Mersenne and where Fermat claimed (but did not prove) minimality. However, in the very same translation of *Arithmetica* by Bachet, the problem and solution appear in an annex of de Billy's *Doctrinae analyticae inventum novum* [3], which according to the included description is a collection of problems communicated to de Billy by Fermat in their correspondence. In the *Doctrinae* de Billy outlined a solution that leads to the values mentioned above.

Over a century later, Lagrange's work [10] emerged among several of Euler's solutions, the first of which appeared in Part II of *Elements of Algebra*, [E388]^d published in 1770[4]. This problem and Euler's algebraic solution of it conclude Chapter XIV. He did not cite a reference for the problem. In 1771, Lagrange published his French translation of Euler's *Algebra* [E388a] [5] which also includes several of his own commentaries, and is often referred to as Lagrange's "Additions to Euler's *Algebra*." Euler's next solution appeared in *Miscellanea Analytica* [E560] presented in 1773, which contains a wide variety of results [6].^e Problem IV is this problem of Fermat's, though Euler attributed it to Leibniz^f. The solution presented in E560 employs a Fermat-like reduction similar to the approach Lagrange used later on. His solution even passes through the same intermediate steps that Lagrange's does, though Euler made no mention if his list of answers was complete or contained a minimal solution.

A few years later, on March 20, 1777, Lagrange's solution, which includes his proof of minimality, was read at the Berlin Academy of Sciences, and was published in the *Academy Mémoires* in 1779 [10]. It is this very article whose translation we provide as part of this paper. In the following year, Euler revisited and generalized this problem in three subsequent works. His *De tribus pluribusve numeris inveniendis, quorum summa sit quadratum, quadratorum vero summa biquadratum* [E763] [7] was read to the St. Petersburg Academy on May 18. Next, his *Solutio problematis Fermatiani de duobus*

^aWe haven't found reliable dates for Brulart de St. Martin.

^bSee letter LVIII in the second volume of [12, p. 258].

^cSee letter LIX in the second volume of [12, p. 260].

^dSee Ch. XIV, Q. 17, §240.

^eOne such result is a proof of Wilson's Theorem.

^fWe have not found evidence of this problem in Leibniz' works.

Year	Author	Work
1643 (writ.)	Fermat	Letters to correspondents
1670 (publ.)	Fermat	Observations in <i>Arithmetica</i>
1770 (publ.)	Euler	<i>Algebra</i> , ch. XIV (E388)
1771 (publ.)	Lagrange	Translation of Euler's <i>Algebra</i>
1773 (pres.)	Euler	<i>Miscellanea Analytica</i> , prob. IV (E560)
1777 (pres.)	Lagrange	<i>Sur quelques problèmes</i> ...
1778 (pres.)	Euler	<i>De tribus pluribusve</i> ... (E763)
1778 (pres.)	Euler	<i>Solutio problematis Fermatiani</i> ... (E769)
1778 (pres.)	Euler	<i>De insigni promotione</i> ... (E772)

Table 1: Timeline of Solutions & Generalizations of the Problem

numeris, quorum summa sit quadratum, quadratorum vero summa biquadratum, ad mentem illustris La Grange adornata [E769] [8] and *De insigni promotione Analysis Diophantaeae* [E772] [9] were read on June 5 and June 12, respectively. The first two of these three specifically mention Lagrange. In E763, Euler indicated only that the problem was proposed by Fermat and studied by Lagrange. But in E769, Euler acknowledged Lagrange's critique and conceded that his own solutions in the *Algebra* were found by "chance and roving efforts," putting their completeness in question.^g Euler then indicated that the subsequent exposition would provide a satisfactory response. In the paper, he described ways to find new values that would make a certain biquadratic (i.e. fourth power) expression equal to the square of a rational number, starting from existing values that have the same property, but did not actually address the completeness issue directly. In E772, Euler only mentioned the problem of Fermat's, and then proceeded to generalize his process for using existing values that make a given biquadratic expression into a perfect square to find new values that do the same. All three of these works from 1780 were published posthumously, in 1824, 1826, and 1830, respectively. All the works mentioned above are summarized in Table 1.

Lagrange's work was the culmination of his decade-long (1767-1777) study of Diophantine problems. Bénédicte Buraux-Bourgeois, in her article on Lagrange's Diophantine period, characterizes Lagrange as playing a *rôle charnière*, a pivotal role leading to a more attentive and rigorous approach to proving the existence of solutions [2]. Buraux-Bourgeois provides a comprehensive survey of Lagrange's Diophantine works, starting with the existence of solutions for the Pell-Fermat equation^h and its generalizations, which Euler also studied with great success. Other notable works are Lagrange's four-square theorem as well as his examination of the quadratic form $Bt^2 + Ctu + Du^2$. For short synopses and commentary on these works, we refer the reader to [2]. No doubt influenced by an extraordinary contemporary like Euler, Lagrange chose to close his Dio-

^gThis wording comes from the translation of E769 by Jordan Bell, available in the Euler Archive.

^h*Solution d'un problème d'arithmétique*, 1768.

phantine period with this work on a problem posed by another great that is Fermat. Indeed, the abrupt manner in which Lagrange ended his article, “*mais en voilà assez sur ce sujet*” signaled the end of his explorations in Diophantine analysis as well.

Synopsis of Sur quelques problèmes de l’analyse de Diophante

In this paper, Lagrange discusses [1] and describes his admiration for Fermat’s famous method of descent, which Fermat used in Observation XXVI of book VI to show that the difference of two fourth powers is never a square. Lagrange goes on to discuss how Euler took advantage of the method of descent in Chapter XIII of [4] to prove many similar results regarding certain polynomial equations that have no integer solutions.

But the difference of twice a fourth power and a fourth power can be a square, i.e. $2x^4 - y^4 = z^2$ has solutions in the positive integers. Lagrange gives the trivial $x = y = z = 1$ solution as well as $x = 13, y = 1, [z = 239]$ and $x = 2,165,017, y = 2,372,159, [z = 3,503,833,734,241]$. Here, Lagrange points out that Euler’s solution methods were not systematic and thus it was not clear that all solutions would be found, nor that the simplest solutions would be found.

At this point, Lagrange introduces the main problem on which the article will focus: to find a right triangle of which the hypotenuse will be a square and the sum of the two sides around the right angle will be a square also, that is to find two numbers of which the sum is a square and of which the sum of the squares is a fourth power. He cites Fermat’s Observation XXIV of book VI in [1] and then connects the Fermat problem to his. Fermat was looking for right triangle legs p and q so that $p + q = y^2$, and $p^2 + q^2 = x^4$. But then

$$2x^4 - y^4 = 2(p^2 + q^2) - (p + q)^2 = p^2 - 2pq + q^2 = (p - q)^2 = z^2.$$

This means that any (p, q) solving the Fermat problem will lead to (x, y, z) solving Lagrange’s problem, and vice versa. Indeed, since $z = p - q$, we have

$$p = \frac{y^2 + z}{2} \quad \text{and} \quad q = \frac{y^2 - z}{2}.$$

Lagrange then lists what he has found so far:

x	y	z	p	q
1	1	1	1	0
13	1	239	120	-119
2,165,017	2,372,159	3,503,833,734,241	1,061,652,293,520	4,565,486,027,761

Of these solutions, only the last solution is allowable because then the legs p and q are positive integers. Indeed, Lagrange points out that Fermat claims these numbers to be the smallest, but does not provide a demonstration. Hence he concludes that the problem has not been resolved.

Lagrange goes on to explain that perhaps Fermat's method of descent can be used to prove that he has found the smallest positive integer solution.

[I]f we can prove that when there are any integer values of x and y that satisfy the equation $2x^4 - y^4 = \square$, then there are necessarily two other smaller values which also satisfy the equation, and when at the same time we have a general method for deducing the larger values from the smaller ones, it is clear that by starting with the smallest possible values of x and y , we could find, by successively reascending, all the other satisfying values in order of their magnitude.

After detailed algebraic manipulations and explanations, Lagrange concludes that a solution to the original $2x^4 - y^4 = z^2$ leads to a solution of $s^4 + 8t^4 = u^2$ via a reversible process, and with s and t each less than the greater of x and y . He then points out that since the form of this equation is different than the original, we must continue our analysis. Using similar algebraic manipulations, and after much work, he concludes that the solution of the equation

$$s^4 + 8t^4 = u^2$$

can be reduced to the solution of one of the equations

$$2q^4 - r^4 = s^2, \text{ or } q^4 - 2r^4 = s^2.$$

Here, q and r satisfy $qr = t$. Moreover, the greater of q and r is less than the greater of s and t . See §7 for more details.

Notice the first of these new equations is exactly like the one we started with ($2x^4 - y^4 = \square$). So Lagrange analyzes the second equation above, and ultimately finds that it leads to an equation of the form $n^4 + 8p^4 = \square$ (using a different n and p than before), and that n and p are each less than q and r . Thus he has finalized the descent pattern by closing the loop. He also has the ability to work from the smaller solutions back up the chain, which he does. Starting from small positive solutions, Lagrange ultimately returns to the Fermat problem and lists the solutions in order.

$$\begin{array}{llll} x = 1, & 13, & 1\,525, & 2\,165\,017, \dots, \\ y = 1, & 1, & 1\,343, & 2\,372\,159, \dots, \\ z = 1, & 239, & 2\,750\,257, & 3\,503\,833\,734\,241, \dots, \end{array}$$

and thus the corresponding legs of the desired right triangle would be

$$\begin{array}{ll} p = 1, & 120, \quad 2\,276\,953, \quad 1\,061\,652\,293\,520, \dots, \\ q = 0, & -119, \quad -473\,304, \quad 4\,565\,486\,027\,761, \dots \end{array}$$

Lagrange has thus shown that these are the smallest positive integer solutions to the problem, therefore verifying the assertion of Fermat.

Lagrange then considers equations of the form $x^4 + ay^4 = \square$ for general a . If a solution exists, then he can always reduce this equation to one in a similar form with

smaller values. Lagrange rejects this as a general approach however, because certain choices he made were a possibility but not a necessity, and so he may not obtain all solutions this way. He then refers the reader to the last chapter of his *Additions to Euler's Algebra* [5] for a simpler and more general approach.

The paper concludes with remarks about solutions of a two-variable polynomial equation of degree higher than two. He builds new solutions from a known solution. His method is reminiscent of perturbations, and he applies it to cubic polynomials in two variables as well as a fourth degree polynomial provided that y^3 and y^4 do not appear and there are no terms xy^2 or x^2y^2 . He ends abruptly.

Translation of “Sur quelques problèmes de l’analyse de Diophante”

On some problems in the analysis of Diophantus ⁱ

1. Among the great number of beautiful Theorems of Arithmetic that Fermat left us in his *Observations on Diophantus*, one of the more remarkable is one which is articulated in the Observation on Question XXVI of Book VI,^j because it is the only one for which Fermat has given a proof.

The Theorem is that *the difference of two biquadratic numbers cannot be a square*; and Fermat's proof consists in showing that, if there were two biquadratic integers whose difference was a square, we could always find two lesser integers that have the same property, and so on, in such a way that one arrives necessarily at small biquadratic numbers whose difference would be a square. But this is impossible, as we can assure ourselves by examining the first few natural numbers. As the Theorem is thus proven for whole numbers, it is clear that this is so for the rational numbers as well, since if the difference of the biquadrates of two rational numbers is a square, and we put the two numbers to a common denominator, it follows that the difference of the biquadrates of the numerators will likewise be a square.

The principle of Fermat's proof is one of the most fruitful in all of the Theory of numbers, and especially for whole numbers; Mr. Euler has further developed this principle, and has applied it to prove other analogous Theorems, to wit: that *the sum of two biquadrates cannot be a square*; that *neither the sum nor the difference of a biquadrate and the quadruple of another biquadrate can be squares*; that *the double of the sum or the difference of two biquadrates can never be a square*; and finally that

ⁱTranslated from the French by Jesse Herche and Christopher Goff, Department of Mathematics, University of the Pacific, Stockton, CA, and Michael P. Saclolo, St. Edwards University, Austin, TX. We have followed the formatting of the version appearing in the Complete Works of Lagrange [11] rather than the original version [10] to enhance readability.

^jSee [1].

the sum of a biquadrate and the double of another biquadrate also cannot be a square. (See Chapter XIII of the second part of his *Elements of Algebra*.^k)

2. But if the sum of a biquadrate and the double of another biquadrate cannot be a square, the same cannot be said for their difference; for it is apparent that the equality

$$2x^4 - y^4 = \square$$

is satisfied by taking $x = 1, y = 1$; and as for the equality

$$x^4 - 2y^4 = \square,$$

one need only take $x = 3, y = 2$.

We have found also for the first equation the other values

$$x = 13, y = 1, \text{ and } x = 2165017, y = 2372159,$$

and for the second these here^l:

$$x = 113, y = 84, \text{ and } x = 57123, y = 2614.$$

We could again find many others by the known method for these sorts of equations, according to which we can deduce new solutions from those that we already have, each solution providing yet another different one, if the Problem admits several (see the Treatise entitled *Doctrinae analyticae inventum novum* in the edition of Diophantus of 1670^m, and Chapters VIII, IX, and X of the second Part of the *Algebra* of Mr. Eulerⁿ). But this method, the only one available to us for equations above the second degree, is merely specific and could never give all possible solutions. We even noticed that, often, the solutions it produces are quite evidently not the simplest. Thus, if it were a matter of solving the two equations above completely, or at least finding all the possible values of x and y that do not exceed given bounds, the method in question would be of almost no use, since we would always be uncertain if the values found through this method are the only ones that answer the question, and we could only remove this doubt by trying successively all integers for x and y .

3. The equation

$$2x^4 - y^4 = \square$$

is especially remarkable, because it contains the solution to a Problem proposed by Fermat as very difficult, in the second Observation on the Question XXIV of Book VI

^kSee [4].

^lThe second value of y is incorrect. It should be 6214, as found in §9.

^mSee [3] in [1].

ⁿSee [4].

of Diophantus.^o The Problem entails finding a right triangle for which the hypotenuse is a square and for which the sum of the two sides around the right angle is also square, in other words to find two numbers whose sum is a square and whose sum of squares is a biquadrate.

Let p and q be the two desired numbers, so that

$$p + q = y^2, \quad p^2 + q^2 = x^4;$$

removing from the double of the latter equations the square of the former, we have

$$p^2 - 2pq + q^2 = 2x^4 - y^4;$$

thus setting

$$p - q = z,$$

we obtain the equation

$$2x^4 - y^4 = z^2,$$

on whose solution, therefore, the solution to the proposed Problem depends: for having found the values of x, y, z , we immediately obtain

$$p = \frac{y^2 + z}{2}, \quad q = \frac{y^2 - z}{2}.$$

If we take for x and y the values given above, we shall obtain^P

- 1^o $x = 1, y = 1$, whence $z = 1$, thus $p = 1, q = 0$;
- 2^o $x = 13, y = 1$, whence $z = 239$, thus $p = 120, q = -119$;
- 3^o $x = 2\,165\,017, y = 2\,372\,159$, whence $z = 1\,560\,590\,745\,759$,
thus $p = 1\,061\,652\,293\,520, q = 4\,565\,486\,027\,761$.

4. Of these three solutions, we see that only the last is admissible when we require that the desired numbers p and q be positive integers. But at the same time we see that the values of p and q are extremely large, and it is natural to believe that we could satisfy the equation with smaller numbers had Fermat not positively assured to the contrary in the place cited above. However, as this assertion has not been proved there, and it does not even seem to me provable by the method Fermat indicates, which is none other than that about which we have spoken above, we can view as not resolved the problem of finding the smallest positive integers that satisfy the double condition that their sum is a square, and that the sum of their squares is a biquadrate. But how might we begin to achieve a complete solution for this Problem and for analogous Problems? It seems

^oSee [1].

^PThe value for z in the third solution is incorrect. It has been corrected to 3 503 833 734 241 in the synopsis. Also, the values of p and q in the third solution have been interchanged. Since p and q are essentially interchangeable in the problem, these were not corrected in the synopsis.

to me that we could only attain this goal by a device similar to that which has served to prove the Theorems that we mentioned at the beginning of this Memoir: for if we can prove that when there are any integer values of x and y that satisfy the equation

$$2x^4 - y^4 = \square,$$

then there are necessarily two other smaller values which also satisfy the equation, and when at the same time we have a general method for deducing the larger values from the smaller ones, it is clear that by starting with the smallest possible values of x and y , which are $x = 1$ and $y = 1$, we could find, by successively reascending, all the other satisfying values in order of their magnitude. All the difficulty therefore consists in reducing the solution of the equation

$$2x^4 - y^4 = \square$$

to that of another similar equation, but in which the numbers x and y are necessarily smaller than in the first. This is the purpose of the following analysis, which seems to me the simplest and most direct that we can use in this investigation.

5. Therefore let us consider the indeterminate equation

$$2x^4 - y^4 = z^2,$$

and let us suppose that we know some integer values of x, y, z that satisfy it. I first note that we can assume x and y are relatively prime; for if they had a common factor, z would be divisible by the square of this common factor, and, once the division is done, the quotients would likewise satisfy the equation.

Moreover I remark that the numbers x, y, z ought to be all odd; because if y were even, then z^2 would be divisible by 2, and so z would also be even also. Thus as y^4 and z^2 would be both divisible by 4, then $2x^4$ would be as well, so that x^4 would be divisible by 2. Thus x would be even and would not be relatively prime to y , contrary to the hypothesis. Now as y is odd, z will also necessarily be odd. Finally, as we know that the square of every odd number is necessarily of the form $8m + 1$, it follows that $z^2 + y^4$ will be of the form $8n + 2$. Thus $2x^4$ would be of the same form and consequently x^4 would be of the form $4n + 1$, so that x will also be odd.

That said, the equation

$$2x^4 - y^4 = z^2$$

gives

$$4x^4 = 2(z^2 + y^4) = (z + y^2)^2 + (z - y^2)^2,$$

whence

$$(z + y^2)^2 = (2x^2)^2 - (z - y^2)^2 = (2x^2 + z - y^2)(2x^2 - z + y^2).$$

These two factors are both even, since y and z are odd; thus let their common factor $= 2m$, so that

$$2x^2 + z - y^2 = 2mp, \quad 2x^2 - z + y^2 = 2mq,$$

with p and q relatively prime. Then

$$(z + y^2)^2 = 4m^2pq;$$

thus the number pq is necessarily a square, and as p and q are relatively prime, both must be squares. Thus putting p^2 and q^2 in place of p and q , we have the equations

$$2x^2 + z - y^2 = 2mp^2, \quad 2x^2 - z + y^2 = 2mq^2, \quad (z + y^2)^2 = 4m^2p^2q^2,$$

whence

$$z + y^2 = 2mpq.$$

Equivalently,

$$z = 2mpq - y^2,$$

and substituting this value of z into the other two equations, we obtain the following

$$x^2 - y^2 = mp(p - q), \quad x^2 + y^2 = mq(p + q).$$

From here we see that as m divides the sum and difference of x^2 and y^2 , it must also divide both $2x^2$ and $2y^2$; but as x and y are relatively prime (hypothesis), m can only be 1 or 2. If $m = 1$, we have

$$x^2 - y^2 = p(p - q), \quad x^2 + y^2 = q(p + q);$$

if $m = 2$, we have

$$x^2 - y^2 = 2p(p - q), \quad x^2 + y^2 = 2q(p + q),$$

and if, for the latter case, we set

$$p + q = q', \quad q - p = p',$$

then we have

$$x^2 - y^2 = p'(p' - q'), \quad x^2 + y^2 = q'(p' + q').$$

Thus, whether m is 1 or 2, we shall have two equations of the form

$$x^2 - y^2 = p(p - q), \quad x^2 + y^2 = q(p + q).$$

First, I consider the former of these equations, and put it in the form

$$\frac{x + y}{p} = \frac{p - q}{x - y};$$

I note now that $x + y$ is an even number, since x and y are both odd, and that p is necessarily odd; for if it were even, then by the second equation q would also be even, so that $q(q+p)$ becomes an even number; but then this number would be evenly even^q, and consequently could not be equal to the sum of two odd squares. Therefore, if we reduce the fraction $\frac{x+y}{p}$ to its lowest terms, it will be of the form $\frac{2m}{n}$, n being odd and relatively prime to m . Hence we have

$$x + y = 2ms, p = ns \text{ and } p - q = 2mt, x - y = nt,$$

for some integers s and t ; and as $x - y$ is necessarily even, and n odd, t must even, so that by putting $2t$ in place of t , we have

$$p - q = 4mt, x - y = 2nt,$$

t being any number that is relatively prime to s , for otherwise x and y would not be relatively prime. From this we obtain^r

$$x = ms + nt, y = ms - nt, p = ns, q = ns - 4mt,$$

values that satisfy the first equation. But they must also satisfy the second equation

$$x^2 + y^2 = q(p + q);$$

thus by substituting them there, we have

$$m^2s^2 + n^2t^2 = (ns - 4mt)(ns - 2mt),$$

and expanding,

$$m^2(s^2 - 8t^2) + 6mnst + n^2(t^2 - s^2) = 0,$$

an equation, which multiplied by $s^2 - 8t^2$ can be put in this form

$$[m(s^2 - 8t^2) + 3nst]^2 = n^2[9s^2t^2 - (t^2 - s^2)(s^2 - 8t^2)],$$

which, dividing by n^2 and manipulating the terms, becomes

$$s^4 + 8t^4 = \left[3st + \frac{m(s^2 - 8t^2)}{n} \right]^2.$$

Therefore, if we set

$$3st + \frac{m(s^2 - 8t^2)}{n} = u,$$

^q "Evenly even" means divisible by 4. "Oddly even" means congruent to 2 mod 4.

^r It is worth noting here, as Lagrange does in §8, that one can replace x with $-x$ or y with $-y$ if needed in order to obtain positive solutions.

which gives

$$\frac{m}{n} = \frac{u - 3st}{s^2 - 8t^2},$$

we obtain the equation

$$s^4 + 8t^4 = u^2.$$

Thus the solution to the proposed equation

$$2x^4 - y^4 = z^2$$

is reduced to that of the equation

$$s^4 + 8t^4 = u^2;$$

and we see by the preceding analysis that, if there are integers that satisfy the first equation, then there will also necessarily be integers that satisfy the second; and vice versa, if we know one integer solution of the latter, we could deduce a solution of the former by means of the formulas

$$\frac{m}{n} = \frac{u - 3st}{s^2 - 8t^2}, \quad x = ms + nt, \quad y = ms - nt.$$

As $\frac{m}{n}$ is assumed to be a fraction reduced to its lowest terms, if $u - 3st$ and $s^2 - 8t^2$ are relatively prime, we have

$$m = u - 3st, \quad n = s^2 - 8t^2;$$

but if the numbers have a common factor, we shall take

$$m = \frac{u - 3st}{l}, \quad n = \frac{s^2 - 8t^2}{l}.$$

And since we can take the numbers s, t, u indiscriminately, together with positive and negative x, y, z , it is easy to see that each solution to the equation

$$s^4 + 8t^4 = u^2$$

will always give two solutions for the equation

$$2x^4 - y^4 = z^2,$$

by taking in the expression for m the number u to be positive or negative. I now note that n can never be zero⁵, and that m can only be zero when $u = 3st$, which gives

$$s^4 + 8t^4 = 9s^2t^2,$$

⁵We know n cannot be zero because $s^2 - 8t^2 = 0$ has no nonzero integer solutions.

whence

$$\frac{s}{t} = \sqrt{\frac{9}{2} \pm \sqrt{\frac{81}{4} - 8}} = \sqrt{\frac{9}{2} \pm \frac{7}{2}} = 1 \text{ or } = \sqrt{8};$$

as the value $\sqrt{8}$ is not admissible due to of its irrationality, leaving $\frac{s}{t} = 1$, and consequently

$$s = 1, t = 1.$$

These values indeed satisfy the equation

$$s^4 + 8t^4 = u^2;$$

but then we will have

$$x = n, y = -n,$$

and as x and y must be relatively prime (hypothesis), we will have $x = 1$ and $y = 1$. Whence we see that when x and y are relatively prime and different from unity, then s and t will also be relatively prime and different from unity, and m will never be zero; so that the greater of the numbers x and y will necessarily be greater than both the two numbers s and t . Consequently, if the equation

$$2x^4 - y^4 = \square$$

is solvable by some integers different from unity, the equation

$$s^4 + 8t^4 = \square$$

will necessarily be solvable in smaller numbers, also different from unity, and vice versa.

6. If the equation that we obtained,

$$s^4 + 8t^4 = u^2,$$

were of the same form as the proposed equation, the Problem would be solved; but since it is not, we must thus continue our analysis by operating henceforth on this latter equation, where s and t are assumed to be relatively prime.

First, I am going to prove that s and u must be odd; because if s were even, s^4 would be divisible by 16; thus u^2 would be by 8; thus u would be by 4; thus u^2 would also be divisible by 16, and $8t^4$ would also be; thus t^4 would be divisible by 2; thus t would be even, and by consequence would not be relatively prime to s , contrary to hypothesis. As s is odd, it is apparent that u must be odd as well. Now I put the equation under examination in the form

$$8t^4 = u^2 - s^4 = (u + s^2)(u - s^2).$$

Since u and s are both odd, the two factors

$$u + s^2 \text{ and } u - s^2$$

are even; thus their common factor will be 2μ , so that

$$u + s^2 = 2\mu\varpi, \quad u - s^2 = 2\mu\rho,$$

with ϖ and ρ relatively prime. Thus

$$8t^4 = 4\mu^2\varpi\rho \text{ and } 2t^4 = \mu^2\varpi\rho,$$

so that μ must divide t^2 . But eliminating u from the two preceding equations leads to

$$s^2 = \mu(\varpi - \rho),$$

whence one sees that μ also divides s^2 . Therefore since t and s are relatively prime (hypothesis), it must be that $\mu = 1$. Thus we shall have

$$2t^4 = \varpi\rho;$$

and as ϖ and ρ are relatively prime, it must be the case that either

$$\varpi = 2q^4, \quad \rho = r^4,$$

or

$$\varpi = q^4, \quad \rho = 2r^4;$$

whence

$$t = qr.$$

In the first case, we have

$$u = 2q^4 + r^4, \quad s^2 = 2q^4 - r^4,$$

and in the second we have

$$u = q^4 + 2r^4, \quad s^2 = q^4 - 2r^4.$$

Whence we see that the solution of the equation

$$s^4 + 8t^4 = u^2$$

is reduced to that of one or the other of the equations

$$2q^4 - r^4 = s^2, \text{ or } q^4 - 2r^4 = s^2.$$

Indeed, if we know some integer values of q, r, s that satisfy one or the other of these equations, then we only need to take set $t = qr$ to obtain the values of s and t that solve the equation

$$s^4 + 8t^4 = \square.$$

I note:

1° That if s and t are relatively prime, s will also be relatively prime to q and to r ; thus q and r will be relatively prime by virtue of the equations

$$2q^4 - r^4 = s^2 \text{ or } q^4 - 2r^4 = s^2;$$

2° That if q and r are different from unity, t will be bigger than q and r ; if q is equal to unity, then $t = r$; but in this case

$$2 - r^4 = s^2 \text{ or } 1 - 2r^4 = s^2.$$

The second of these equations cannot be satisfied by integers, and the first only by

$$r = 1 \text{ and } s = 1;$$

and so we have

$$s = 1, t = 1.$$

If r is equal to unity, then

$$t = q, \text{ and } s^2 = 2q^4 - 1 \text{ or } = q^4 - 2;$$

whence one sees that s will be greater than q . I conclude from this that while s and t , in the equation

$$s^4 + 8t^4 = \square,$$

are relatively prime and different from unity, q and r will also be relatively prime and different from unity, and further that the greater of the numbers s and t will necessarily exceed the greater of the two between q and r .

7. The equation

$$2q^4 - r^4 = s^2$$

is, as we see, similar to the first

$$2x^4 - y^4 = z^2;$$

thus the Problem will be solved, if we had found solely this equation; but, as we have also arrived at the equation

$$q^4 - 2r^4 = s^2,$$

which is different from the two that we just considered, we must again follow the calculation relative to the latter.

We have already seen that q and r must be relatively prime. Now q must be odd, otherwise s would be even, consequently $2r^4$ would be divisible by 4, and r^4 by 2. Thus r and q would be both even and consequently would not be relatively prime. Thus as q is odd, it is apparent that s will be also. Therefore, if we put the equation in this form

$$2r^4 = q^4 - s^2 = (q^2 + s)(q^2 - s),$$

the two factors $q^2 + s$ and $q^2 - s$ will both be even, and consequently of the form $2m\lambda, 2m\mu$, with $2m$ their greatest common factor, and λ, μ two relatively prime numbers. Hence we shall have

$$q^2 + s = 2m\lambda, \quad q^2 - s = 2m\mu, \quad 2r^4 = 4m^2\lambda\mu,$$

or rather

$$r^4 = 2m^2\lambda\mu,$$

so that m divides r^2 , but it also divides q^2 , because

$$q^2 = m(\lambda + \mu);$$

therefore since q and r are relatively prime, it must be that $m = 1$. Therefore

$$r^4 = 2\lambda\mu;$$

and so, r will be even; thus making

$$r = 2h,$$

results in

$$8h^4 = \lambda\mu.$$

Therefore as λ and μ are relatively prime, we shall have necessarily either

$$\lambda = 8n^4, \quad \mu = p^4, \quad \text{or} \quad \lambda = n^4, \quad \mu = 8p^4;$$

whence

$$h = pn, \quad r = 2pn.$$

Hence

$$s = \lambda - \mu = 8n^4 - p^4 \quad \text{or} \quad = n^4 - 8p^4,$$

and

$$q^2 = \lambda + \mu = 8n^4 + p^4 \quad \text{or} \quad = n^4 + 8p^4,$$

where we see that the two values of s and q^2 return to the same form by replacing n by p and s by $-s$. Therefore the solution of the equation

$$q^4 - 2r^4 = s^2$$

reduces itself to that of the equation

$$q^2 = n^4 + 8p^4,$$

by taking

$$r = 2pn, \text{ and } s = n^4 - 8p^4.$$

And we note that n and p must be relatively prime, otherwise r and q would not be, contrary to hypothesis. Moreover, it is apparent that r will always be bigger than p and than n , and as q is necessarily greater than r , it follows that, in the equation

$$n^4 + 8p^4 = \square,$$

the numbers n, p will necessarily be smaller than the numbers q, r in the equation

$$q^4 - 2r^4 = \square.$$

Now, the equation

$$n^4 + 8p^4 = \square$$

is of the same form as that which we have already examined above. Therefore the Problem is solved.

8. Therefore, by the preceding method and formulas, we can solve not only the equations of the form

$$2x^4 - y^4 = \square,$$

but also those of these two other forms

$$x^4 + 8y^4 = \square \text{ and } x^4 - 2y^4 = \square,$$

and with all the generality to which these equations are susceptible; because by beginning with the simplest solutions and passing successively to ones that are more compound, we will be assured to find in order all of the possible integer solutions of these equations, and consequently also all fractional, according the remark at the beginning of this Memoir. Thus, the calculation reduces to the following:

1° With the equation

$$(A) \quad s^4 + 8t^4 = u^2,$$

we shall have the equation

$$(B) \quad x^4 - 2y^4 = z^2,$$

by taking

$$x = u, \quad y = 2st,$$

and the equation

$$(C) \quad 2x^4 - y^4 = z^2,$$

by taking

$$m = \frac{\pm u - 3st}{l}, \quad n = \frac{s^2 - 8t^2}{l}, \quad \pm x = ms + nt, \quad \pm y = ms - nt,$$

l being the greatest common divisor.

2° With either of equations (B), (C), we shall have equation (A) by taking

$$s = z, \quad t = xy.$$

9. Equation (A) gives readily at first

$$s = 1, \quad t = 1, \quad u = 3;$$

thus one has for equation (B)

$$x = 3, \quad y = 2, \quad \text{and from this } z = 7;$$

and for equation (C)

$$m = \frac{\pm 3 - 3}{l}, \quad n = -\frac{7}{l}, \quad \pm x = m + n, \quad \pm y = m - n;$$

thus

$$m = 0, \quad n = -1, \quad l = 7 \quad \text{and } x = 1, \quad y = 1, \quad z = 1,$$

or else

$$m = -6, \quad n = -7, \quad l = 1 \quad \text{and } x = 13, \quad y = 1, \quad z = 239.$$

These values of x, y, z will give others for s, t, u for equation (A). At first

$$x = 1, \quad y = 1, \quad z = 1$$

will give

$$s = 1, \quad t = 1, \quad u = 3;$$

then

$$x = 3, \quad y = 2, \quad z = 7$$

will give

$$s = 7, \quad t = 6, \quad u = 113.$$

Finally

$$x = 13, \quad y = 1, \quad z = 239$$

will give

$$s = 239, t = 13, u = 57\,123.$$

The first of these three solutions to equation (A) is the same that we have adopted at the outset, and we can now disregard them; the two others give therefore new solutions for equations (B) and (C). Therefore first taking

$$s = 7, t = 6, u = 113,$$

we shall have for equation (B)

$$x = 113, y = 84, z = 7\,967;$$

then for equation (C)

$$m = \frac{\pm 113 - 126}{l}, n = -\frac{239}{l}, \pm x = 7m + 6n, \pm y = 7m - 6n.$$

Therefore: either

$$m = -13, n = -239, l = 1, x = 1\,525, y = 1\,343, z = 2\,750\,257;$$

or

$$m = -1, n = -1, l = 239, x = 13, y = 1;$$

this latter solution has already been found above. Next, taking

$$s = 239, t = 13, u = 57\,123,$$

we shall have for equation (B)

$$x = 57\,123, y = 6\,214, z = 3\,262\,580\,153;$$

and for equation (C) we shall have

$$m = \frac{\pm 57\,123 - 9\,321}{l}, n = \frac{55\,769}{l}, \pm x = 239m + 13n, \pm y = 239m - 13n.$$

Therefore, either

$$m = 6, n = 7, l = 7\,967,$$

and from this

$$x = 1\,525, y = 1\,343, z = 2\,750\,257,$$

and this is the solution found above; or

$$m = -9\,492, n = 7\,967, l = 7;$$

thus

$$x = 2\,165\,017, y = 2\,372\,159, \text{ and from this } z = 1\,560\,590\,745\,759.$$

And so on.

10. We see by this calculation, and it would be easy to press further if it is worth the trouble, that the values that satisfy the equation

$$s^4 + 8t^4 = u^2$$

are in order

$$\begin{aligned} s &= 1, & 7, & 239, & \dots, \\ t &= 1, & 6, & 13, & \dots, \\ u &= 3, & 113, & 57\,123, & \dots; \end{aligned}$$

that the values that satisfy the equation

$$x^4 - 2y^4 = z^2$$

are

$$\begin{aligned} x &= 3, & 113, & 57\,123, & \dots, \\ y &= 2, & 84, & 6\,214, & \dots, \\ z &= 7, & 7\,967, & 3\,262\,580\,153, & \dots; \end{aligned}$$

and finally the values that satisfy the equation

$$2x^4 - y^4 = z^2$$

are^t

$$\begin{aligned} x &= 1, & 13, & 1\,525, & 2\,165\,017, & \dots, \\ y &= 1, & 1, & 1\,343, & 2\,372\,159, & \dots, \\ z &= 1, & 239, & 2\,750\,257, & 1\,560\,590\,745\,759, & \dots, \end{aligned}$$

and we can be assured that no numbers smaller than these here can satisfy the proposed formulas.

Now if we deduce from the latter values of x, y, z , those of p and q (N^o. 3), we shall have, in order, all the numbers which can solve the Problem of Fermat, to wit

$$\begin{aligned} p &= 1, & 120, & 2\,276\,953, & 1\,061\,652\,293\,520, & \dots, \\ q &= 0, & -119, & -473\,304, & 4\,565\,486\,027\,761, & \dots \end{aligned}$$

However great these latter numbers may be, they are nevertheless the smallest positive integers that solve the Problem in question, proving Fermat's assertion.

11. In general, we can make the solution of every equation of the form

$$x^4 + ay^4 = z^2$$

^tNote that the incorrect final value of z is included again here. It has been corrected to 3 503 833 734 241 in the synopsis. Similarly, the corresponding values assigned to p and q are still interchanged.

(for a given number a) depend on that of an equation of the same form in which the numbers x, y, z are smaller.

To do so, we need only suppose

$$z = m^2 + an^2,$$

which gives

$$z^2 = (m^2 - an^2)^2 + a(2mn)^2;$$

thus

$$x^2 = m^2 - an^2 \quad \text{and} \quad y^2 = 2mn.$$

Now let

$$x = p^2 - aq^2,$$

whence

$$x^2 = (p^2 + aq^2)^2 - a(2pq)^2;$$

thus

$$m = p^2 + aq^2, n = 2pq;$$

and, substituting into the equation $y = 2mn$, we will have

$$y^2 = 4pq(p^2 + aq^2).$$

To satisfy this equation let us set

$$p = s^2, \quad q = t^2 \quad p^2 + aq^2 = u^2$$

so that

$$y = 2stu,$$

and then comes the equation

$$s^4 + at^4 = u^2,$$

which is similar to the equation in consideration. If this last equation can be solved, we shall obtain

$$x = s^4 - at^4, \quad y = 2stu, \quad z = (s^4 + at^4)^2 + a(2s^2t^2)^2,$$

or

$$z = u^4 + 4as^4t^4;$$

from which we see that y will always be necessarily greater than each of the numbers s, t, u .

Therefore knowing one integer solution of any equation of the form

$$x^4 + ay^4 = z^2,$$

we could derive from these formulas a new solution with larger values, and so on. But we are not assured to find all possible integer solutions through this approach, for the assumptions that we made that take

$$x^4 + ay^4 = z^2$$

to the equation

$$s^4 + at^4 = u^2$$

are merely possible but not absolutely necessary.

Furthermore the simplest and most general method to solve these types of equations is perhaps that of factors, which I have explained in the last Chapter of *Additions to the Algebra* of M. Euler, to which I refer.^u

12. I am going to conclude this Memoir by showing how we can simplify and generalize in some respects the ordinary method for equations beyond the second degree, according to which, from a known solution, we can find several others.

Given the general equation of the third degree in two indeterminates x, y

$$a + bx + cy + dx^2 + exy + fy^2 + gx^3 + hx^2y + kxy^2 + ly^3 = 0,$$

satisfied by the following values

$$x = p, y = q,$$

we have

$$a + bp + cq + dp^2 + epq + fq^2 + gp^3 + hp^2q + kpq^2 + lq^3 = 0.$$

Setting

$$x = p + t, y = q + u$$

and substituting into the given equation, it is transformed into

$$Bt + Cu + Dt^2 + Etu + Fu^2 + Gt^3 + Ht^2u + Ktu^2 + Lu^3 = 0,$$

where the coefficients B, C, \dots are some rational functions of p and q , which we determine easily from the expansion of the terms of the given equation; but we can find them even more easily by using the differential method. For if we suppose

$$a + bp + cq + dp^2 + epq + fq^2 + gp^3 + hp^2q + kpq^2 + lq^3 = A,$$

we will have

$$B = \frac{dA}{dp}, C = \frac{dA}{dq}, D = \frac{1}{2} \frac{d^2A}{dp^2}, E = \frac{d^2A}{dpdq}, F = \frac{1}{2} \frac{d^2A}{dq^2},$$

^uSee [5].

$$G = \frac{1}{2.3} \frac{d^3 A}{dp^3}, \quad H = \frac{1}{2} \frac{d^3 A}{dp^2 dq}, \quad K = \frac{1}{2} \frac{d^3 A}{dp dq^2}, \quad L = \frac{1}{2.3} \frac{d^3 A}{dq^3}.$$

Now, to be able to determine rational values for u and t , I first set equal to zero, in the equation in t and u , the first two terms where t and u are linear; I thus have

$$Bt + Cu = 0, \quad \text{whence } u = -\frac{Bt}{C};$$

so what remains is the equation

$$Dt^2 + Etu + Fu^2 + Gt^3 + Ht^2u + Ktu^2 + Lu^3 = 0;$$

therefore substituting in place of u its value, the whole equation will become divisible by t^2 , and we will have, upon dividing,

$$D - \frac{BE}{C} + \frac{B^2F}{C^2} + \left(G - \frac{BH}{C} + \frac{B^2K}{C^2} - \frac{B^3L}{C^3} \right) t = 0$$

from which we obtain

$$t = \frac{-C^3D + BC^2E - B^2CF}{C^3G - BC^2H + B^2CK - B^3L},$$

thus

$$u = \frac{BC^2D - B^2CE + B^3F}{C^3G - BC^2H + B^2CK - B^3L}.$$

Therefore we will have two new values for x and y , and taking these latter ones in place of p and q , we can deduce new ones, and so on.

13. If the indeterminate equation was of the fourth degree, it would not be generally possible to solve it by the preceding method. But we could arrive at a solution, if it contains only the first two powers of one of the two unknowns, and moreover, by considering this unknown as having two dimensions, there would not be any term of more than four.

Indeed, let the equation

$$0 = a + bx + cy + dx^2 + exy + fy^2 + gx^3 + hx^2y + kx^4$$

meet the required conditions, and let us suppose that the values

$$x = p, \quad y = q$$

satisfy it; substituting $p + t$ in place of x , and $q + u$ in place of y , we will have an equation of the form

$$Bt + Cu + Dt^2 + Etu + Fu^2 + Gt^3 + Ht^2u + Kt^4 = 0.$$

Setting

$$u = tz$$

and dividing the entire equation by t , it becomes

$$B + Cz + Dt + Etz + Ftz^2 + Gt^2 + Ht^2z + Kt^3 = 0,$$

which, as we see, is nothing but an equation of the third degree in z and t . Thus, we could then apply the preceding method to it, provided we know a value for z and t . These values arise quite readily, for we only need to set

$$t = 0, \quad B + Cz = 0, \quad \text{whence } z = -\frac{B}{C}.$$

Therefore, . . . ; but enough on this subject.

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