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# Translation of Euler's Paper E421

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# Expansion of the integral $\int x^{f-1}dx(\log x)^{\frac{m}{n}}$ having extended the integration from the value $x=0$ to $x=1$ *

Leonhard Euler

## Theorem 1

**§1** *If $n$ denotes a positive integer and the integral*

$$\int x^{f-1}dx(1-x^g)^n$$

*is extended from the value $x=0$ to $x=1$, the value of the integral will be*

$$=\frac{g^n}{f}\cdot\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

## Proof

It is known that the integral $\int x^{f-1}dx(1-x^g)^m$ can in general be reduced to this one $\int x^{f-1}dx(1-x^g)^{m-1}$, since it is possible to define constant quantities $A$ and $B$ in such a way that

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$$\int x^{f-1}dx(1-x^g)^m = A\int x^{f-1}dx(1-x^g)^{m-1} + Bx^f(1-x^g)^m;$$

having taken the differentials, this equation results

$$\begin{aligned}&x^{f-1}dx(1-x^g)^m\\ &= Ax^{f-1}dx(1-x^g)^{m-1} + Bfx^{f-1}dx(1-x^g)^m - Bmgx^{f+g-1}dx(1-x^g)^{m-1},\end{aligned}$$

which divided by $x^{f-1}dx(1-x^g)^{m-1}$ gives

$$1-x^g = A + Bf(1-x^g) - Bmgx^g$$

or

$$1-x^g = A - Bmg + B(f+mg)(1-x^g);$$

in order for this equation to hold, it is necessary that

$$1 = B(f+mg) \quad \text{and} \quad A = Bmg,$$

whence we conclude

$$B = \frac{1}{f+mg} \quad \text{and} \quad A = \frac{mg}{f+mg}.$$

Therefore, we will have the following general reduction

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1} + \frac{1}{f+mg}x^f(1-x^g)^m;$$

because it vanishes for $x=0$, if $f>0$, of course, the addition of a constant is not necessary. Hence having extended both integrals to $x=1$, the last absolute part vanishes and for the case $x=1$ it will be

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}.$$

Since for $m=1$

$$\int x^{f-1}dx(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f}$$

having put $x=1$, we obtain the following values for the same case $x=1$

$$\int x^{f-1}dx(1-x^g)^1 = \frac{g}{f} \cdot \frac{1}{f+g},$$
$$\int x^{f-1}dx(1-x^g)^2 = \frac{g^2}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g},$$
$$\int x^{f-1}dx(1-x^g)^3 = \frac{g^3}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g} \cdot \frac{3}{f+3g}$$

and hence we conclude that for any positive integer $n$ it will be

$$\int x^{f-1}dx(1-x^g)^n = \frac{g^n}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g} \cdot \frac{3}{f+3g} \cdots \frac{n}{f+ng}$$

if only the numbers $f$ and $g$ are positive.

## Corollary 1

§2 In turn, the value of a product of this kind, formed from an arbitrary amount of factors, can be expressed by an integral so that

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f}{g^n} \int x^{f-1}dx(1-x^g)^n$$

having extended this integral from the value $x=0$ to $x=1$.

## Corollary 2

§3 Therefore, if one considers a progression of this kind

$$\frac{1}{f+g}; \quad \frac{1 \cdot 2}{(f+g)(f+2g)}; \quad \frac{1 \cdot 2 \cdot 3}{(f+g)(f+2g)(f+3g)}; \quad \frac{1 \cdot 2 \cdot 3 \cdot 4}{(f+g)(f+2g)(f+3g)(f+4g)} \quad \text{etc.},$$

its general term corresponding to the indefinite index $n$ is conveniently represented by this integral $\frac{f}{g^n} \int x^{f-1}dx(1-x^g)^n$; and using this formula, the progression and its terms corresponding to fractional indices can be exhibited.

## Corollary 3

§4 If we write $n-1$ instead of $n$, we will have

$$\frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{(f+g)(f+2g)(f+3g)\cdots(f+(n-1)g)} = \frac{f}{g^{n-1}} \int x^{f-1}dx(1-x^g)^{n-1};$$

multiplication by $\frac{n}{f+ng}$ yields

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f \cdot ng}{g^n(f+ng)} \int x^{f-1}dx(1-x^g)^{n-1}.$$

## Scholium 1

§5 It would have been possible to derive this last formula immediately from the preceding one, since we just proved that

$$\int x^{f-1}dx(1-x^g)^n = \frac{ng}{f+ng} \int x^{f-1}dx(1-x^g)^{n-1},$$

if both integrals are extended from the value $x = 0$ to $x = 1$; this is to be kept in mind for all the integrals everywhere in the following. Furthermore, it is to be noted that the quantities $f$ and $g$ are positive, a condition the proof absolutely requires, of course. Concerning the number $n$, if it denotes the index of a certain term of the progression (§3), that index can also be negative, because all terms, also those corresponding to negative indices, of the progression are considered to be exhibited by the given integral formula. Nevertheless, it is to be noted that this reduction

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg} \int x^{f-1}dx(1-x^g)^{m-1}$$

is only true, if $m > 0$, because otherwise the algebraic part $\frac{1}{f+mg}x^f(1-x^g)^m$ would not vanish for $x = 1$.

## Scholium 2

§6 I already studied series of this kind, which can be called transcendental, because the terms corresponding to fractional indices are transcendental quantities, in Comment. acad. sc. Petrop., book 5 in more detail[1]; therefore, I will not investigate those progressions here again but focus on the remarkable comparisons of the integral formulas that are derived from it. After I had shown that the value of the indefinite product $1 \cdot 2 \cdot 3 \cdots n$ is expressed by the integral formula $\int dx \left(\log \frac{1}{x}\right)^n$ extended from $x = 0$ to $x = 1$, which, if $n$ is a positive integer, is manifest by direct integration, I subjected to examination the cases in which fractional numbers are taken for $n$; in these cases it is indeed not obvious at all, to which kind of

[1] Euler refers to his paper "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt." This is paper E19 in the Eneström-Index


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transcendental quantities these terms are to be referred. But, using a singular trick, I reduced the same terms to more familiar quadratures; therefore, this seems to be most worthy to consider it with all eagerness.

## Problem 1

§7 *Since it was demonstrated that*

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n,$$

*having extended the integral from $x=0$ to $x=1$, to assign the value of the same product in the case $g=0$ by means of an integral.*

### Solution

Having put $g=0$ in the integral formula, the term $(1-x^g)^n$ vanishes, but at the same time also the denominator $g^n$ vanishes, whence the question reduces to the task to define the value of the fraction $\frac{(1-x^g)^n}{g^n}$ in the case $g=0$, in which both the numerator and the denominator vanish. Therefore, let us consider $g$ as an infinitely small quantity, and because $x^g = e^{g\log x}$, it will be $x^g = 1 + g\log x$ and hence $(1-x^g)^n = g^n(-\log x)^n = g^n\left(\log\frac{1}{x}\right)^n$; hence our integral formula becomes $f\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n$ for this case so that one now has this expression

$$\frac{1\cdot 2\cdot 3\cdots n}{f^n} = f\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n$$

or

$$1\cdot 2\cdot 3\cdots n = f^{n+1}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n.$$

### Corollary 1

§8 If $n$ is a positive integer, the integration of the integral $\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n$ succeeds and, having extended it from $x=0$ to $x=1$, indeed the product we found to be equal to it results. But if fractional numbers are taken for $n$, the same formula can be applied to interpolate this hypergeometric progression

$$1, \quad 1\cdot 2, \quad 1\cdot 2\cdot 3, \quad 1\cdot 2\cdot 3\cdot 4, \quad 1\cdot 2\cdot 3\cdot 4\cdot 5 \quad \text{etc.}$$

or

$$1, \quad 2, \quad 6, \quad 24, \quad 120, \quad 720, \quad 5040 \quad \text{etc.}$$

## Corollary 2

**§9** If the expression just found is divided by the principal one, a product whose factors proceed in an arithmetic progression will emerge, namely

$$(f+g)(f+2g)(f+3g)\cdots(f+ng) = f^n g^n \frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n}{\int x^{f-1}dx\,(1-x^g)^n},$$

whose values can also be assigned, using the integral, if $n$ is a fractional number.

## Corollary 3

**§10** Since

$$\int x^{f-1}dx(1-x^g)^n = \frac{ng}{f+ng}\int x^{f-1}dx(1-x^g)^{n-1},$$

in like manner, for the case $g=0$ it will be

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n = \frac{n}{f}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}$$

and hence by those other integrals

$$1\cdot 2\cdot 3\cdots n = nf^n\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}$$

and

$$(f+g)(f+2g)\cdots(f+ng) = f^{n-1}g^{n-1}(f+ng)\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx(1-x^g)^{n-1}}.$$

## Scholium

**§11** Because we found that

$$1\cdot 2\cdot 3\cdots n = f^{n+1}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n,$$

it is plain that this integral does not depend on the value of the quantity $f$, which is also easily seen by putting $x^f = y$, whence first we find

$$f x^{f-1} dx = dy \quad \text{and} \quad \log\frac{1}{x} = -\log x = -\frac{1}{f}\log y = \frac{1}{f}\log\frac{1}{y}$$

and therefore

$$f^n \left(\log\frac{1}{x}\right)^n = \left(\log\frac{1}{y}\right)^n$$

such that

$$1\cdot 2\cdot 3\cdots n = \int dy \left(\log\frac{1}{y}\right)^n,$$

which expression results from the first by putting $f = 1$. Therefore, for an interpolation of the formulas of this kind the whole task is reduced to the definition of the values of the integral $\int dx \left(\log\frac{1}{x}\right)^n$ for the cases, in which the exponent $n$ is a fractional number. For example, if $n = \frac{1}{2}$, one has to assign the value of the formula $\int dx\sqrt{\log\frac{1}{x}}$, which value I already once showed to be $= \frac{1}{2}\sqrt{\pi}$, while $\pi$ denotes the circumference of the circle whose diameter is $= 1$; but for other fractional numbers I taught how to reduce its value to quadratures of algebraic curves of higher order. Because this reduction is by no means obvious and is only valid if the integration of the formula $\int dx \left(\log\frac{1}{x}\right)^n$ is extended from the value $x = 0$ to $x = 1$, it seems to be worth of one's unique attention. But even though I already treated this subject once[2], nevertheless, because I was led to the results in a rather non straight-forward way, I decided take on this subject here again and explain everything in more detail.

## Theorem 2

**§12** *If the integrals are extended from the value $x = 0$ to $x = 1$ and $n$ denotes a positive integer, it will be*

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)(n+3)\cdots 2n} = \frac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}},$$

*whatever positive numbers are taken for $f$ and $g$.*

[2] Euler considered this expression also in E19 "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt" mentioned already in the footnote above.

## Proof

Because above (§4) we showed that

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f \cdot ng}{g^n(f+ng)} \int x^{f-1}dx(1-x^g)^{n-1},$$

if we write $2n$ instead of $n$, we will have

$$\frac{1 \cdot 2 \cdot 3 \cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)} = \frac{f \cdot 2ng}{g^{2n}(f+2ng)} \int x^{f-1}dx(1-x^g)^{2n-1}.$$

Now divide the first equation by the second one and this third one will result

$$\frac{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)}{(n+1)(n+2)\cdots 2n} = \frac{g^n(f+2ng)}{2(f+ng)} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

But if one writes $f+ng$ instead of $f$ in the first equation, this fourth equation will result

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)} = \frac{(f+ng)ng}{g^n(f+2ng)} \int x^{f+ng-1}dx(1-x^g)^{n-1}.$$

Multiply this fourth equation by the third and one will find the equation to be demonstrated, namely

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)(n+3)\cdots 2n} = \frac{1}{2}ng \int x^{f+ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

## Corollary 1

**§13** If one sets $f=n$ and $g=1$ in the first equation, the same product will result, namely

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}n \int x^{n-1}dx(1-x)^{n-1};$$

having compared this equation to the one mentioned above we obtain

$$\frac{\int x^{n-1}dx(1-x)^{n-1}}{g\int x^{f+ng-1}dx(1-x^g)^{n-1}} = \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$


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## Corollary 2

§14 If we write $x^g$ instead of $x$ in that equation, it will be

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots 2n}=\frac{1}{2}ng\int x^{ng-1}dx(1-x^g)^{n-1}$$

such that we find this comparison of the following integral formulas

$$\int x^{ng-1}dx(1-x^g)^{n-1}=\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

## Corollary 3

§15 If we set $g=0$ in the equation of the theorem, because of $(1-x^g)^m=g^m\left(\log\frac{1}{x}\right)^m$, the powers of $g$ will cancel each other and this equation will result

$$\frac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots 2n}=\frac{1}{2}n\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}},$$

whence we conclude

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}=g\int x^{ng-1}dx(1-x^g)^{n-1}$$

or, because of

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}=\frac{f}{n}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n},$$

this equality

$$\frac{2f}{n}\cdot\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n}}=g\int x^{ng-1}dx(1-x^g)^{n-1}.$$

## Corollary 4

**§16** Let us set $f = 1$, $g = 2$ and $n = \frac{m}{2}$ here so that $m$ is a positive integer, and, because of

$$\int dx \left(\log \frac{1}{x}\right)^m = 1 \cdot 2 \cdot 3 \cdots m,$$

it will be

$$\frac{4}{m} \cdot \frac{\left(\int dx \left(\log \frac{1}{x}\right)^{\frac{m}{2}}\right)^2}{1 \cdot 2 \cdot 3 \cdots m} = 2 \int x^{m-1} dx (1-x^2)^{\frac{m}{2}-1}$$

and hence

$$\int dx \left(\log \frac{1}{x}\right)^{\frac{m}{2}} = \sqrt{1 \cdot 2 \cdot 3 \cdots m \cdot \frac{m}{2} \int x^{m-1} dx (1-x^2)^{\frac{m}{2}-1}}$$

and by taking $m = 1$, because of

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2},$$

one will have

$$\int dx \sqrt{\log \frac{1}{x}} = \sqrt{\frac{1}{2} \int \frac{dx}{\sqrt{1-xx}}} = \frac{1}{2}\sqrt{\pi}.$$

## Scholium

**§17** So lo and behold this succinct proof of the theorem I proved some time ago[3], which says that $\int dx \sqrt{\log \frac{1}{x}} = \frac{1}{2}\sqrt{\pi}$, and which is free from an argument of interpolation, which I had used back then. Here, it was deduced from this theorem I found, which states that

$$\frac{\left(\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{2n-1}} = g \int x^{ng-1} dx (1-x^g)^{n-1}.$$

But the principal theorem, from which this one is deduced, reads as follows

[3]Euler proved this theorem in E19: "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt."

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}=\int x^{n-1}dx(1-x)^{n-1};$$

for, each side of the equation is transformed through integration from 0 to 1 into this numerical product

$$\frac{1\cdot 2\cdot 3\cdots(n-1)}{(n+1)(n+2)\cdots(2n-1)}.$$

but if we want to assign an extended class [of integrals] to one side, the theorem can thus be displayed so that it becomes

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1};$$

and, if one takes $g=0$ here,

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1}.$$

Therefore, first it has to be noted that this equality holds, whatever numbers are taken for $f$ and $g$; in the case $f=g$, this is indeed clear, since

$$\int x^{g-1}dx(1-x^g)^{n-1}=\frac{1-(1-x^g)^n}{ng}=\frac{1}{ng};$$

for, it will be

$$2g\int x^{ng+g-1}dx(1-x^g)^{n-1}=k\int x^{nk-1}dx(1-x^k)^{n-1},$$

and because

$$\int x^{ng+g-1}dx(1-x^g)^{n-1}=\frac{1}{2}\int x^{ng-1}dx(1-x^g)^{n-1},$$

the equality is obvious, because $k$ can be taken arbitrarily. But in the same way I arrived at this theorem, it is possible to extend to other similar ones.

# Theorem 3

§18 *If the following integrals are extended from the value $x = 0$ to $x = 1$ and $n$ denotes any positive integer, it will be*

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(2n+1)(2n+2)\cdots 3n} = \frac{2}{3} ng \int x^{f+2ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{2n-1}}{\int x^{f-1} dx (1-x^g)^{3n-1}},$$

*whatever positive numbers are taken for $f$ and $g$.*

## Proof

In the preceding theorem, we already saw that

$$\frac{1 \cdot 2 \cdot 3 \cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)} = \frac{f \cdot 2ng}{g^{2n}(f+2ng)} \int x^{f-1} dx (1-x^g)^{2n-1};$$

if, in like manner, we write $3n$ instead of $n$ in the principal formula, we will have

$$\frac{1 \cdot 2 \cdot 3 \cdots 3n}{(f+g)(f+2g)\cdots(f+3ng)} = \frac{f \cdot 3ng}{g^{3n}(f+3ng)} \int x^{f-1} dx (1-x^g)^{3n-1};$$

from which that equation [i.e. the first] divided by this one [i.e., the second] produces

$$\frac{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)}{(2n+1)(2n+2)\cdots 3n} = \frac{2g^n(f+3ng)}{3(f+2ng)} \cdot \frac{\int x^{f-1} dx (1-x^g)^{2n-1}}{\int x^{f-1} dx (1-x^g)^{3n-1}}.$$

But if we write $f + 2gn$ instead of $f$ in the principal equation (§4), we obtain this equation

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)} = \frac{(f+2ng)ng}{g^n(f+3ng)} \int x^{f+2ng-1} dx (1-x^g)^{n-1}.$$

Now this equation is multiplied by the preceding and the equation to be proved will result

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(2n+1)(2n+2)\cdots 3n} = \frac{2}{3} ng \int x^{f+2ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{2n-1}}{\int x^{f-1} dx (1-x^g)^{3n-1}}.$$


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### Corollary 1

§19 We obtain the same value from the principal equation by putting $f = 2n$ and $g = 1$ so that

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(2n+1)(2n+2)\cdots 3n} = \frac{2}{3}n \int x^{2n-1}dx(1-x)^{n-1},$$

which integral formula, by writing $x^k$ instead of $x$, is transformed into this one

$$\frac{2}{3}nk \int x^{2nk-1}dx(1-x^k)^{n-1}$$

such that

$$g \int x^{f+2ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx\,(1-x^g)^{2n-1}}{\int x^{f-1}dx\,(1-x^g)^{3n-1}} = k \int x^{2nk-1}dx(1-x^k)^{n-1}.$$

### Corollary 2

§20 If we set $g = 0$ here, because of $1 - x^g = g \log \frac{1}{x}$, we will have this equation

$$\int x^{f-1}dx \left(\log \frac{1}{x}\right)^{n-1} \cdot \frac{\int x^{f-1}dx\,\left(\log \frac{1}{x}\right)^{2n-1}}{\int x^{f-1}dx\,\left(\log \frac{1}{x}\right)^{3n-1}} = k \int x^{2nk-1}dx(1-x^k)^{n-1};$$

because we had found before that

$$\frac{\left(\int x^{f-1}dx\,\left(\log \frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\,\left(\log \frac{1}{x}\right)^{2n-1}} = k \int x^{nk-1}dx(1-x^k)^{n-1},$$

by multiplying both expressions by each other we will have this equation

$$\frac{\left(\int x^{f-1}dx\,\left(\log \frac{1}{x}\right)^{n-1}\right)^3}{\int x^{f-1}dx\,\left(\log \frac{1}{x}\right)^{3n-1}} = k^2 \int x^{nk-1}dx(1-x^k)^{n-1} \cdot \int x^{2nk-1}dx(1-x^k)^{n-1}.$$

## Corollary 3

§21 Without any restriction one can put $f = 1$ here; because, then for $n = \frac{1}{3}$ and $k = 3$ it will be

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{-\frac{2}{3}}\right)^3}{\int dx \left(\log \frac{1}{x}\right)^0} = 9 \int dx(1-x^3)^{-\frac{2}{3}} \cdot \int xdx(1-x^3)^{-\frac{2}{3}}$$

and, because of

$$\int dx \left(\log \frac{1}{x}\right)^{-\frac{2}{3}} = 3 \int dx \left(\log \frac{1}{x}\right)^{\frac{1}{3}} \quad \text{and} \quad \int dx \left(\log \frac{1}{x}\right)^0 = 1,$$

$$\left(\int dx \left(\log \frac{1}{x}\right)^{\frac{1}{3}}\right)^3 = \frac{1}{3} \int dx(1-x^3)^{-\frac{2}{3}} \cdot \int xdx(1-x^3)^{-\frac{2}{3}};$$

but then for $n = \frac{2}{3}$ and $k = 3$ it will be

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{-\frac{1}{3}}\right)^3}{\int dx \log \frac{1}{x}} = 9 \int xdx(1-x^3)^{-\frac{1}{3}} \cdot \int x^3dx(1-x^3)^{-\frac{1}{3}}$$

or

$$\left(\int dx \left(\log \frac{1}{x}\right)^{\frac{2}{3}}\right)^3 = \frac{4}{3} \int xdx(1-x^3)^{-\frac{1}{3}} \cdot \int x^3dx(1-x^3)^{-\frac{1}{3}}.$$

## General Theorem

§22 *If the following integrals are extended from the value $x = 0$ to $x = 1$ and $n$ denotes any positive integer, it will be*

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(\lambda n+1)(\lambda n+2) \cdots (\lambda+1)n} = \frac{\lambda}{\lambda+1} ng \int x^{f+\lambda ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}},$$

*whatever positive numbers are taken for the letters $f$ and $g$.*

## Proof

Since, as we showed above,

$$\frac{1\cdot 2\cdots n}{(f+g)(f+2g)\cdots(f+ng)}=\frac{f\cdot ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1},$$

if we write $\lambda n$ instead of $n$ here at first, but then $(\lambda+1)n$ instead of $n$, we will obtain these two equations

$$\frac{1\cdot 2\cdots \lambda n}{(f+g)(f+2g)\cdots(f+\lambda ng)}=\frac{f\cdot \lambda ng}{g^{\lambda n}(f+\lambda ng)}\int x^{f-1}dx(1-x^g)^{\lambda n-1},$$

$$\frac{1\cdot 2\cdots (\lambda+1)n}{(f+g)(f+2g)\cdots(f+(\lambda+1)ng)}=\frac{f\cdot (\lambda+1)ng}{g^{(\lambda+1)n}(f+(\lambda+1)ng)}\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1};$$

the first equation, divided by this one, gives

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)}=g^n\frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

But if we write $f+\lambda ng$ instead of $f$ in the first equation, we will obtain

$$\frac{1\cdot 2\cdots n}{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)}=\frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)}\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1},$$

which two equations multiplied by each other produce the equation to be demonstrated

$$\frac{1\cdot 2\cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)}=\frac{\lambda ng}{\lambda+1}\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

## Corollary 1

**§23** If we set $f=\lambda n$ and $g=1$ in the principal equation, we will also find

$$\frac{1\cdot 2\cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)}=\frac{\lambda n}{\lambda+1}\int x^{\lambda n-1}dx(1-x)^{n-1},$$

which form writing $x^k$ instead of $x$ changes into this one

$$\frac{\lambda nk}{\lambda+1}\int x^{\lambda nk-1}dx(1-x^k)^{n-1}$$

such that we have this very far-extending theorem

$$g\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{\lambda n+n-1}}=k\int x^{\lambda nk-1}dx(1-x^k)^{n-1}.$$

## Corollary 2

§24 This theorem now holds, even if $n$ is not an integer; because the number $\lambda$ can be taken arbitrarily, let us even write $m$ instead of $\lambda n$ and we will find this theorem

$$\frac{\int x^{f-1}dx(1-x^g)^{m-1}}{\int x^{f-1}dx(1-x^g)^{m+n-1}}=\frac{k\int x^{mk-1}dx(1-x^k)^{n-1}}{g\int x^{f+mg-1}dx(1-x^g)^{n-1}}.$$

## Corollary 3

§25 If we set $g=0$, because of $1-x^g=g\log\frac{1}{x}$, that theorem will take this form

$$\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{m-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{m+n-1}}=\frac{k\int x^{mk-1}dx(1-x^k)^{n-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}},$$

which is more conveniently represented as follows

$$\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{m-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{m+n-1}}=k\int x^{mk-1}dx(1-x^k)^{n-1};$$

here, it is evident that the numbers $m$ and $n$ can be permuted between themselves.

## Scholium

§26 Thus, we found two sources, from which many comparisons of integrals formulas can be derived; the one way, laid in § 24, contains integrals of this kind

$$\int x^{p-1}dx(1-x^g)^{q-1},$$

which I already treated some time ago in my observations on the integrals of the formulas[4]

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1},$$

extended from the value $x=0$ to $x=1$; there I showed at first that the letters $p$ and $q$ can be interchanged such that

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1} = \int x^{q-1}dx(1-x^n)^{\frac{p}{n}-1},$$

but then that

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{p}{n}}} = \frac{\pi}{n\sin\frac{p\pi}{n}};$$

But especially, I demonstrated that

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int \frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}} = \int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int \frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}};$$

the comparison found in § 24 is already contained in this equation such that nothing new, which I have not already explained, can be deduced from this. Therefore, here I mainly attempt to follow the other way explained in § 25; since without any restriction one can take $f=1$, our primary equation will be

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{m-1}}{\int dx\left(\log\frac{1}{x}\right)^{m+n-1}} = k\int x^{mk-1}dx(1-x^k)^{n-1},$$

by means of which the values of the integral formulas $\int dx\left(\log\frac{1}{x}\right)^{\lambda}$, if $\lambda$ is not an integer, can be reduced to quadratures of algebraic curves; since, if $\lambda$ is an integer, the integration is obtained in absolute terms, because

$$\int dx\left(\log\frac{1}{x}\right)^{\lambda} = 1\cdot 2\cdot 3\cdots\lambda.$$

But the question of greatest importance concerns the cases, in which $\lambda$ is a rational number. Therefore, I will define these here successively for some small denominators.

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[4]Euler again refers to his paper "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x=1$". This is paper E321 in the Eneström-Index.

## Problem 2

§27 *While $i$ denotes a positive integer, to define the value of the integral $\int dx\left(\log\frac{1}{x}\right)^{\frac{i}{2}}$, having extended the integration from $x=0$ to $x=1$.*

### Solution

Let us put $m=n$ in our general equation and it will be

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int dx\left(\log\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1}.$$

Now let $n-1=\frac{i}{2}$ and, because of $2n-1=i+1$, it will be

$$\int dx\left(\log\frac{1}{x}\right)^{2n-1}=1\cdot 2\cdot 3\cdots(i+1);$$

now further take $k=2$ so that $nk-1=i+1$, and it will be

$$\frac{\left(\int dx\sqrt{\left(\log\frac{1}{x}\right)^i}\right)^2}{1\cdot 2\cdot 3\cdots(i+1)}=2\int x^{i+1}dx(1-x^2)^{\frac{i}{2}}$$

and hence

$$\frac{\int dx\sqrt{\left(\log\frac{1}{x}\right)^i}}{\sqrt{1\cdot 2\cdot 3\cdots(i+1)}}=\sqrt{2\int x^{i+1}dx(1-x^2)^{\frac{i}{2}}},$$

where it is evidently sufficient to take only odd numbers for $i$, because for the even ones the expansion is immediately obvious.

### Corollary 1

§28 But all cases are easily reduced to $i=1$ or even to $i=-1$; for, if $i+1$ is not a negative number, the reduction we found holds. For this case it will therefore be

$$\int\frac{dx}{\sqrt{\log\frac{1}{x}}}=\sqrt{2\int\frac{dx}{\sqrt{1-xx}}}=\sqrt{\pi},$$

because of $\int\frac{dx}{\sqrt{1-xx}}=\frac{\pi}{2}$.

### Corollary 2

§29 But having covered this principal case, because of

$$\int dx\left(\log\frac{1}{x}\right)^{n}=n\int dx\left(\log\frac{1}{x}\right)^{n-1},$$

we will have

$$\int dx\sqrt{\log\frac{1}{x}}=\frac{1}{2}\sqrt{\pi},\quad \int dx\left(\log\frac{1}{x}\right)^{\frac{3}{2}}=\frac{1\cdot 3}{2\cdot 2}\sqrt{\pi},$$

and in general

$$\int dx\left(\log\frac{1}{x}\right)^{\frac{2n+1}{2}}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\frac{2n+1}{2}\sqrt{\pi}.$$

## Problem 3

§30 *While i denotes a positive integer, to define the value of the integral* $\int dx\left(\log\frac{1}{x}\right)^{\frac{i}{3}-1}$, *having extended the integration from* $x=0$ *to* $x=1$.

### Solution

Let us start from the equation of the preceding problem

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{n-1}\right)^{2}}{\int dx\left(\log\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}dx(1-x^{k})^{n-1}$$

and let us set $m=2n$ in the general formula such that one has

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{2n-1}}{\int dx\left(\log\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}dx(1-x^{k})^{n-1},$$

and by multiplying these two equations we obtain

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{n-1}\right)^{3}}{\int dx\left(\log\frac{1}{x}\right)^{3n-1}}=kk\int x^{nk-1}dx(1-x^{k})^{n-1}\cdot\int x^{2nk-1}dx(1-x^{k})^{n-1}.$$

Now just set $n = \frac{i}{3}$ here such that

$$\int dx \left(\log \frac{1}{x}\right)^{i-1} = 1 \cdot 2 \cdot 3 \cdots (i-1),$$

and take $k = 3$ and it will result

$$\frac{\left(\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{i-3}}\right)^3}{1 \cdot 2 \cdot 3 \cdots (i-1)} = 9 \int x^{i-1} dx \sqrt[3]{(1-x^3)^{i-3}} \cdot \int x^{2i-1} dx \sqrt[3]{(1-x^3)^{i-3}},$$

whence we conclude

$$\frac{\int dx \sqrt[3]{\left(\log \frac{1}{x}\right)^{i-3}}}{\sqrt[3]{1 \cdot 2 \cdot 3 \cdots (i-1)}} = \sqrt[3]{9 \int \frac{x^{i-1} dx}{\sqrt[3]{(1-x^3)^{3-i}}} \cdot \int \frac{x^{2i-1} dx}{\sqrt[3]{(1-x^3)^{3-i}}}}.$$

### Corollary 1

**§31** Here, two principal cases occur, on which all remaining ones depend, namely by putting either $i = 1$ or $i = 2$, which are;

$$\text{I.} \quad \int \frac{dx}{\sqrt[3]{\left(\log \frac{1}{x}\right)^2}} = \sqrt[3]{9 \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x dx}{\sqrt[3]{(1-x^3)^2}}},$$

$$\text{II.} \quad \int \frac{dx}{\sqrt[3]{\log \frac{1}{x}}} = \sqrt[3]{9 \int \frac{dx}{\sqrt[3]{1-x^3}} \cdot \int \frac{x^3 dx}{\sqrt[3]{1-x^3}}};$$

which last formula, because of

$$\int \frac{x^3 dx}{\sqrt[3]{1-x^3}} = \frac{1}{3} \int \frac{dx}{\sqrt[3]{1-x^3}},$$

is transformed into this one

$$\int \frac{dx}{\sqrt[3]{\log \frac{1}{x}}} = \sqrt[3]{\int \frac{dx}{\sqrt[3]{1-x^3}} \cdot \int \frac{x dx}{\sqrt[3]{1-x^3}}}$$


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### Corollary 2

§32 If, for the sake of brevity, as in my observations mentioned before[5] we set

$$\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$$

and, as we did it there, for this class also set

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\sin\frac{\pi}{3}} = \alpha,$$

but then put

$$\left(\frac{1}{1}\right) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A,$$

it will be

$$\text{I.} \quad \int \frac{dx}{\sqrt[3]{\left(\log\frac{1}{x}\right)^2}} = \sqrt[3]{9\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)} = \sqrt[3]{9\alpha A},$$

$$\text{II.} \quad \int \frac{dx}{\sqrt[3]{\left(\log\frac{1}{x}\right)^1}} = \sqrt[3]{3\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)} = \sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

### Corollary 3

§33 Therefore, for the first case we will have

$$\int dx\sqrt[3]{\left(\log\frac{1}{x}\right)^{-2}} = \sqrt[3]{9\alpha A}, \quad \int dx\sqrt[3]{\log\frac{1}{x}} = \frac{1}{3}\sqrt[3]{9\alpha A}$$

and

$$\int dx\sqrt[3]{\left(\log\frac{1}{x}\right)^{3n+1}} = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\frac{3n+1}{3}\sqrt[3]{9\alpha A},$$

but for the other case

[5]Euler refers to his paper E321 again.

$$\int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{-1}} = \sqrt[3]{\frac{3\alpha\alpha}{A}}, \quad \int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{2}} = \frac{2}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}$$

and

$$\int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{3n-1}} = \frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdots\frac{3n-1}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

## Problem 4

§34 *While $i$ denotes a positive integer, to define the value of the integral $\int dx \left(\log\frac{1}{x}\right)^{\frac{i}{4}-1}$, having extended the integration from $x=0$ to $x=1$.*

### Solution

In the solution of the preceding problem, we were led to this equation

$$\frac{\left(\int dx \left(\log\frac{1}{x}\right)^{n-1}\right)^3}{\int dx \left(\log\frac{1}{x}\right)^{3n-1}} = kk\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}};$$

but the general formula, setting $m=3n$ in it, yields

$$\frac{\int dx \left(\log\frac{1}{x}\right)^{n-1}\cdot\int dx \left(\log\frac{1}{x}\right)^{3n-1}}{\int dx \left(\log\frac{1}{x}\right)^{4n-1}} = k\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}};$$

combining these formulas we obtain

$$\frac{\left(\int dx \left(\log\frac{1}{x}\right)^{n-1}\right)^4}{\int dx \left(\log\frac{1}{x}\right)^{4n-1}} = k^3\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}.$$

Let $n=\frac{i}{4}$ and take $k=4$ and it will be

$$\frac{\int dx \left(\log\frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1\cdot 2\cdot 3\cdots(i-1)}} = \sqrt[4]{4^3\int\frac{x^{i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}}.$$

### Corollary 1

**§35** So if $i=1$, we will have

$$\int dx \sqrt[4]{\left(\log\frac{1}{x}\right)^{-3}} = \sqrt[4]{4^3 \int \frac{dx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{xdx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{x^2dx}{\sqrt[4]{(1-x^4)^3}}};$$

if this expression is denoted by the letter $P$, it will be in general

$$\int dx \sqrt[4]{\left(\log\frac{1}{x}\right)^{4n-3}} = \frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdots\frac{4n-3}{4}P.$$

### Corollary 2

**§36** For the other principal case, let us take $i=3$ and it will be

$$\int dx \sqrt[4]{\left(\log\frac{1}{x}\right)^{-1}} = \sqrt[4]{2\cdot 4^3 \int \frac{x^2dx}{\sqrt[4]{1-x^4}} \cdot \int \frac{x^5dx}{\sqrt[4]{1-x^4}} \cdot \int \frac{x^8dx}{\sqrt[4]{1-x^4}}}$$

or, after a reduction to simpler forms is done,

$$\int dx \sqrt[4]{\left(\log\frac{1}{x}\right)^{-1}} = \sqrt[4]{8 \int \frac{xxdx}{\sqrt[4]{1-x^4}} \cdot \int \frac{xdx}{\sqrt[4]{1-x^4}} \cdot \int \frac{dx}{\sqrt[4]{1-x^4}}};$$

if this expression is denoted by the letter $Q$, in general, it will be

$$\int dx \sqrt[4]{\left(\log\frac{1}{x}\right)^{4n-1}} = \frac{3}{4}\cdot\frac{7}{4}\cdot\frac{11}{4}\cdots\frac{4n-1}{4}Q.$$

### Scholium

**§37** If we indicate the integral formula $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}}$ by the sign $\left(\frac{p}{q}\right)$, in general, the solution will be as follows

$$\int dx \sqrt[4]{\log\left(\frac{1}{x}\right)^{i-4}} = \sqrt[4]{1\cdot 2\cdot 3\cdots(i-1)4^3\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)}$$

and for the two expanded cases it becomes

$$P = \sqrt[4]{4^3 \left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)} \quad \text{and} \quad Q = \sqrt[4]{8 \left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}.$$

Now for the formulas depending on the circle, let us set

$$\left(\frac{3}{1}\right) = \frac{\pi}{4\sin\frac{\pi}{4}} = \alpha \quad \text{and} \quad \left(\frac{2}{2}\right) = \frac{\pi}{4\sin\frac{2\pi}{4}} = \beta,$$

but for the transcendental ones of higher order let

$$\left(\frac{2}{1}\right) = \int \frac{xdx}{\sqrt[4]{(1-x^4)^3}} = \int \frac{dx}{\sqrt[2]{1-x^4}} = A,$$

on which certainly all the remaining ones depend; hence we will find

$$P = \sqrt[4]{4^3 \frac{\alpha\alpha}{\beta} AA} \quad \text{and} \quad Q = \sqrt[4]{4\alpha\alpha\beta \frac{1}{AA}},$$

whence it is clear that

$$PQ = 4\alpha = \frac{\pi}{\sin\frac{\pi}{4}}.$$

But because $\alpha = \frac{\pi}{2\sqrt{2}}$ and $\beta = \frac{\pi}{4}$, it will be

$$P = \sqrt[4]{32\pi AA} \quad \text{and} \quad Q = \sqrt[4]{\frac{\pi^3}{8AA}} \quad \text{and} \quad \frac{P}{Q} = \frac{4A}{\sqrt{\pi}}.$$

## Problem 5

§38 *While i denotes a positive integer, to define the value of the integral* $\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{i-5}}$, *having extended the integration from* $x = 0$ *to* $x = 1$.

### Solution

From the preceding solutions it is already clear that for this case one will arrive at this formula at the end

$$\frac{\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{i-5}}}{\sqrt[5]{1\cdot 2\cdot 3\cdots(i-1)}} = \sqrt[5]{5^4 \int \frac{x^{i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}} \cdot \int \frac{x^{2i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}} \cdot \int \frac{x^{3i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}} \cdot \int \frac{x^{4i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}},$$

which integral formulas belong to the fifth class introduced in my dissertation mentioned above[6]. Hence, if in the same way as it was done there the sign $\left(\frac{p}{q}\right)$ denotes this formula $\int \frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}}$, the value in question can be more conveniently expressed in such a way that

$$\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{i-5}} = \sqrt[5]{1\cdot 2\cdot 3\cdots (i-1)5^4\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\left(\frac{4i}{i}\right)};$$

here it indeed suffices to have assigned values smaller than five to $i$; for, if the numerators exceed five just note that

$$\left(\frac{5+m}{i}\right) = \frac{m}{m+i}\left(\frac{m}{i}\right),$$

but then further

$$\left(\frac{10+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\left(\frac{m}{i}\right),$$

$$\left(\frac{15+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\cdot\frac{m+10}{m+i+10}\left(\frac{m}{i}\right).$$

Furthermore, for this class two formulas indeed involve the quadrature of the circle; these formulas are

$$\left(\frac{4}{1}\right) = \frac{\pi}{5\sin\frac{\pi}{5}} = \alpha \quad \text{and} \quad \left(\frac{3}{2}\right) = \frac{\pi}{5\sin\frac{2\pi}{5}} = \beta,$$

but then two contain higher quadratures, which we want to put as

$$\left(\frac{3}{1}\right) = \int\frac{xxdx}{\sqrt[5]{(1-x^5)^4}} = \int\frac{dx}{\sqrt[5]{(1-x^5)^2}} = A \quad \text{and} \quad \left(\frac{2}{2}\right) = \int\frac{xdx}{\sqrt[5]{(1-x^5)^3}} = B,$$

and using these I assigned the values of all remaining formulas of this class[7], namely

[6] Euler again refers to his paper E321 "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x=1$". This is paper E321 in the Eneström-Index.
[7] Euler took the following list out of E321.

$$\left(\frac{5}{1}\right)=1,\quad \left(\frac{5}{2}\right)=\frac{1}{2},\quad \left(\frac{5}{3}\right)=\frac{1}{3}\quad \left(\frac{5}{4}\right)=\frac{1}{4},\quad \left(\frac{5}{5}\right)=\frac{1}{5};$$
$$\left(\frac{4}{1}\right)=\alpha,\quad \left(\frac{4}{2}\right)=\frac{\beta}{A},\quad \left(\frac{4}{3}\right)=\frac{\beta}{2B}\quad \left(\frac{4}{4}\right)=\frac{\alpha}{3A};$$
$$\left(\frac{3}{1}\right)=A,\quad \left(\frac{3}{2}\right)=\beta,\quad \left(\frac{3}{3}\right)=\frac{\beta\beta}{\alpha B};$$
$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{2}{2}\right)=B;$$
$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

## Corollary 1

**§39** Having taken the exponent $i=1$, it will be

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{-4}}=\sqrt[5]{5^4\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)}=\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}A^2B},$$

whence we conclude in general, while $n$ denotes a positive integer, that

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-4}}=\frac{1}{5}\cdot\frac{6}{5}\cdot\frac{11}{5}\cdots\frac{5n-4}{5}\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}A^2B}.$$

## Corollary 2

**§40** Now let $i=2$, and since then this equation results

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{-3}}=\sqrt[5]{5^4\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)\left(\frac{8}{2}\right)},$$

because of

$$\left(\frac{6}{2}\right)=\frac{1}{3}\left(\frac{1}{2}\right)=\frac{1}{3}\left(\frac{2}{1}\right)\quad\text{and}\quad\left(\frac{8}{2}\right)=\frac{3}{3}\left(\frac{3}{2}\right),$$

it will be


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$$\sqrt[5]{5^3\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)}=\sqrt[5]{5^3\alpha\beta\frac{BB}{A}}$$

and in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-3}}=\frac{2}{5}\cdot\frac{7}{5}\cdot\frac{12}{5}\cdots\frac{5n-3}{5}\sqrt[5]{5^3\alpha\beta\frac{BB}{A}}.$$

## Corollary 3

**§41**[8] Let $i=3$ and the form found

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{-2}}=\sqrt[5]{2\cdot 5^4\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{9}{3}\right)\left(\frac{12}{3}\right)},$$

because of

$$\left(\frac{6}{3}\right)=\frac{1}{4}\left(\frac{3}{1}\right),\quad\left(\frac{9}{3}\right)=\frac{4}{7}\left(\frac{4}{3}\right),\quad\left(\frac{12}{3}\right)=\frac{2}{5}\cdot\frac{7}{10}\left(\frac{3}{2}\right),$$

changes to

$$\sqrt[5]{2\cdot 5^2\left(\frac{3}{3}\right)\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)\left(\frac{3}{2}\right)}=\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}},$$

whence it is concluded that in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-2}}=\frac{3}{5}\cdot\frac{8}{5}\cdot\frac{13}{5}\cdots\frac{5n-2}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}}.$$

## Corollary 4

**§42** Finally, for $i=4$ our equation

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{-1}}=\sqrt[5]{6\cdot 5^4\left(\frac{4}{4}\right)\left(\frac{8}{4}\right)\left(\frac{12}{4}\right)\left(\frac{16}{4}\right)},$$

because of

[8] This is erroneously listed as § 51 in the original.

$$\left(\frac{8}{4}\right)=\frac{3}{7}\left(\frac{4}{3}\right),\quad \left(\frac{12}{4}\right)=\frac{2}{6}\cdot\frac{7}{11}\left(\frac{4}{2}\right),\quad \left(\frac{16}{4}\right)=\frac{1}{5}\cdot\frac{6}{10}\cdot\frac{11}{15}\left(\frac{4}{1}\right),$$

will be transformed into this form

$$\sqrt[5]{6\cdot 5\left(\frac{4}{4}\right)\left(\frac{4}{3}\right)\left(\frac{4}{2}\right)\left(\frac{4}{1}\right)}=\sqrt[5]{5\frac{\alpha\alpha\beta\beta}{AAB}}$$

such that in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-1}}=\frac{4}{5}\cdot\frac{9}{5}\cdot\frac{14}{5}\cdots\frac{5n-1}{5}\sqrt[5]{5\alpha\alpha\beta\beta\frac{1}{AAB}}.$$

## Scholium

§43 If we represent the value of the integral formula $\int dx\left(\log\frac{1}{x}\right)^{\lambda}$ by the sign $[\lambda]$, the cases expanded up to now yield

$$\begin{aligned}
\left[-\frac{4}{5}\right]&=\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}\cdot A^2B}, & \left[+\frac{1}{5}\right]&=\frac{1}{5}\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}\cdot A^2B},\\
\left[-\frac{3}{5}\right]&=\sqrt[5]{5^3\alpha\beta\cdot\frac{BB}{A}}, & \left[+\frac{2}{5}\right]&=\frac{2}{5}\sqrt[5]{5^3\alpha\beta\frac{BB}{A}},\\
\left[-\frac{2}{5}\right]&=\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}}, & \left[+\frac{3}{5}\right]&=\frac{3}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}},\\
\left[-\frac{1}{5}\right]&=\sqrt[5]{5\alpha^2\beta^2\cdot\frac{1}{AAB}}, & \left[+\frac{4}{5}\right]&=\frac{4}{5}\sqrt[5]{5\alpha^2\beta^2\cdot\frac{1}{AAB}},
\end{aligned}$$

whence by combining two, whose indices add up to $0$, we conclude

$$\left[+\frac{1}{5}\right]\cdot\left[-\frac{1}{5}\right]=\alpha\ \ =\frac{\pi}{5\sin\frac{\pi}{5}},$$

$$\left[+\frac{2}{5}\right]\cdot\left[-\frac{2}{5}\right]=2\beta=\frac{2\pi}{5\sin\frac{2\pi}{5}},$$

$$\left[+\frac{3}{5}\right]\cdot\left[-\frac{3}{5}\right]=3\beta=\frac{3\pi}{5\sin\frac{3\pi}{5}},$$

$$\left[+\frac{4}{5}\right]\cdot\left[-\frac{4}{5}\right]=4\alpha=\frac{4\pi}{5\sin\frac{4\pi}{5}}.$$

But from the preceding problem, in like manner, we deduce:

$$\left[-\frac{3}{4}\right]=P=\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA},\quad \left[+\frac{1}{4}\right]=\frac{1}{4}\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA},$$
$$\left[-\frac{1}{4}\right]=Q=\sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}},\quad \left[+\frac{3}{4}\right]=\frac{3}{4}\sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}}$$

and hence

$$\left[+\frac{1}{4}\right]\cdot\left[-\frac{1}{4}\right]=\alpha\ \ =\frac{\pi}{4\sin\frac{\pi}{4}},$$
$$\left[+\frac{3}{4}\right]\cdot\left[-\frac{3}{4}\right]=3\alpha=\frac{3\pi}{4\sin\frac{3\pi}{4}},$$

whence, in general, we obtain this theorem that

$$[\lambda]\cdot[-\lambda]=\frac{\lambda\pi}{\sin\lambda\pi};$$

the reason for this can be given from the interpolation method explained some time ago[9] as follows. Since

$$[\lambda]=\frac{1^{1-\lambda}\cdot 2^{\lambda}}{1+\lambda}\cdot\frac{2^{1-\lambda}\cdot 3^{\lambda}}{2+\lambda}\cdot\frac{3^{1-\lambda}\cdot 4^{\lambda}}{3+\lambda}\cdot\text{etc.},$$

[9] Euler explains the interpolation method he talks about here also in E19: "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt."

it will be

$$[-\lambda] = \frac{1^{1+\lambda}\cdot 2^{-\lambda}}{1-\lambda}\cdot\frac{2^{1+\lambda}\cdot 3^{-\lambda}}{2-\lambda}\cdot\frac{3^{1+\lambda}\cdot 4^{-\lambda}}{3-\lambda}\cdot\text{etc.}$$

and hence

$$[\lambda]\cdot[-\lambda] = \frac{1\cdot 1}{1-\lambda\lambda}\cdot\frac{2\cdot 2}{4-\lambda\lambda}\cdot\frac{3\cdot 3}{9-\lambda\lambda}\cdot\text{etc.} = \frac{\lambda\pi}{\sin\lambda\pi},$$

as I demonstrated elsewhere[10].

## Problem 6 - General Problem

§44 *If the letters $i$ and $n$ denote positive integers, to define the value of the integral*

$$\int dx\left(\log\frac{1}{x}\right)^{\frac{i-n}{n}} \quad \textit{or} \quad \int dx\sqrt[n]{\left(\log\frac{1}{x}\right)^{i-n}}$$

*having extended the integration from $x=0$ to $x=1$.*

### Solution

The method explained up to this point will exhibit the value in question expressed via quadratures of algebraic curves in the following way

$$\frac{\int dx\sqrt[n]{\left(\log\frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1\cdot 2\cdot 3\cdots(i-1)}} = \sqrt[n]{n^{n-1}\int\frac{x^{i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}\cdots\int\frac{x^{(n-1)i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}}.$$

Hence, if, for the sake of brevity, we denote the integral formula $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ by this character $\left(\frac{p}{q}\right)$, but on the other hand the formula $\int dx\sqrt[n]{\left(\log\frac{1}{x}\right)^m}$ by this character $\left[\frac{m}{n}\right]$ such that $\left[\frac{m}{n}\right]$ denotes the value of this indefinite product $1\cdot 2\cdot 3\cdots z$, while $z=\frac{m}{n}$, the value in question will be expressed more succinctly in the following way

[10] Euler proved this relation in his paper "Methodus facilis computandi angulorum sinus ac tangentes tam naturales quam artificiales". This is paper E128 in the Eneström-Index.

$$\left[\frac{i-n}{n}\right] = \sqrt[n]{1\cdot 2\cdot 3\cdots (i-1)n^{n-1}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right)},$$

whence it is also concluded that

$$\left[\frac{i}{n}\right] = \frac{i}{n}\sqrt[n]{1\cdot 2\cdot 3\cdots (i-1)n^{n-1}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right)}.$$

Here, it will always suffice to take the number $i$ smaller than $n$, because it is known for larger numbers that

$$\left[\frac{i+n}{n}\right] = \frac{i+n}{n}\left[\frac{i}{n}\right], \quad \text{in the same way} \quad \left[\frac{i+2n}{n}\right] = \frac{i+n}{n}\cdot\frac{i+2n}{n}\left[\frac{i}{n}\right] \quad \text{etc.},$$

and and in this way the whole investigation is hence reduced to those cases in which the numerator $i$ of the fraction $\frac{i}{n}$ is smaller than the denominator $n$. In addition, it will be helpful to have noted the following properties of the integral formulas

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right):$$

I. That letters $p$ and $q$ are interchangeable so that

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right).$$

II. If one of the two numbers $p$ or $q$ is equal to the exponent $n$, that value of the integral formula will be algebraic, namely

$$\left(\frac{n}{p}\right) = \left(\frac{p}{n}\right) = \frac{1}{p} \quad \text{or} \quad \left(\frac{n}{q}\right) = \left(\frac{q}{n}\right) = \frac{1}{q}.$$

III. If the sum of the numbers $p+q$ is equal to the exponent $n$, that value of the integral formula $\left(\frac{p}{q}\right)$ can be exhibited by means of the the circle, because

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \frac{\pi}{n\sin\frac{p\pi}{n}} \quad \text{and} \quad \left(\frac{q}{n-q}\right) = \left(\frac{n-q}{q}\right) = \frac{\pi}{n\sin\frac{q\pi}{n}}.$$

IV. If one of the numbers $p$ or $q$ is greater than the exponent $n$, that integral formula $\left(\frac{p}{q}\right)$ can be reduced to another one whose terms are smaller than $n$; this is achieved using this reduction

$$\left(\frac{p+n}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. That there is a relation among many of these integral formulas of such a kind that

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right) = \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right) = \left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right);$$

by means of which all reductions I gave in my observations on these formulas[11] are found.

## Corollary 1

**§45** If we apply the formula found to individual cases in this way, by means of reduction IV, we will be able to exhibit them in the most simple way in the following calculation. And for the case $n = 2$, in which no further reduction is necessary, we will have

$$\left[\frac{1}{2}\right] = \frac{1}{2}\sqrt[2]{2\left(\frac{1}{1}\right)} = \frac{1}{2}\sqrt[2]{\frac{\pi}{\sin\frac{\pi}{2}}} = \frac{1}{2}\sqrt{\pi}.$$

## Corollary 2

**§46** For the case $n = 3$ we will have these reductions

$$\left[\frac{1}{3}\right] = \frac{1}{3}\sqrt[3]{3^2\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)}$$
$$\left[\frac{2}{3}\right] = \frac{2}{3}\sqrt[3]{3\cdot 1\left(\frac{2}{2}\right)\left(\frac{1}{2}\right)}.$$

[11] Euler is again referring to E321.

### Corollary 3

**§47** For the case $n = 4$ are obtained these three reductions

$$\left[\frac{1}{4}\right] = \frac{1}{4}\sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)},$$

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)} = \frac{1}{2}\sqrt[2]{4\left(\frac{2}{2}\right)}$$

because of $\left(\frac{4}{2}\right) = \frac{1}{2}$,

$$\left[\frac{3}{4}\right] = \frac{3}{4}\sqrt{4\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)};$$

because in the middle equation $\left(\frac{2}{2}\right) = \left(\frac{4-2}{2}\right) = \frac{\pi}{4}$, it will, of course as before, be

$$\left[\frac{2}{4}\right] = \left[\frac{1}{2}\right] = \frac{1}{2}\sqrt{\pi}.$$

### Corollary 4

**§48** Now let $n = 5$ and these four reductions result

$$\left[\frac{1}{5}\right] = \frac{1}{5}\sqrt[5]{5^4\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)},$$

$$\left[\frac{2}{5}\right] = \frac{2}{5}\sqrt[5]{5^3\cdot 1\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)},$$

$$\left[\frac{3}{5}\right] = \frac{3}{5}\sqrt[5]{5^2\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)},$$

$$\left[\frac{4}{5}\right] = \frac{4}{5}\sqrt[5]{5\cdot 1\cdot 2\cdot 3\left(\frac{4}{4}\right)\left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)}.$$

### Corollary 5

**§49** Let $n = 6$ and we will have these reductions

$$\left[\frac{1}{6}\right] = \frac{1}{6}\sqrt[6]{6^5\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)},$$

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2\left(\frac{6}{2}\right)} = \frac{1}{3}\sqrt[3]{6^2\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)},$$

$$\left[\frac{3}{6}\right] = \frac{3}{6}\sqrt[6]{6^3\cdot 3\cdot 3\left(\frac{3}{3}\right)^3\left(\frac{6}{3}\right)^2} = \frac{1}{2}\sqrt[2]{6\left(\frac{3}{3}\right)},$$

$$\left[\frac{4}{6}\right] = \frac{4}{6}\sqrt[8]{6^2\cdot 2\cdot 4\cdot 2\left(\frac{4}{4}\right)^2\left(\frac{2}{4}\right)^2\left(\frac{6}{4}\right)} = \frac{2}{3}\sqrt[3]{6\cdot 2\left(\frac{4}{4}\right)\left(\frac{2}{4}\right)},$$

$$\left[\frac{5}{6}\right] = \frac{5}{6}\sqrt[6]{6\cdot 1\cdot 2\cdot 3\cdot 4\left(\frac{5}{5}\right)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{5}\right)}.$$

## Corollary 6

**§50** Having put $n = 7$ the following six equations result

$$\left[\frac{1}{7}\right] = \frac{1}{7}\sqrt[7]{7^6\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)\left(\frac{6}{1}\right)},$$

$$\left[\frac{2}{7}\right] = \frac{2}{7}\sqrt[7]{7^5\cdot 1\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)},$$

$$\left[\frac{3}{7}\right] = \frac{3}{7}\sqrt[7]{7^4\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)},$$

$$\left[\frac{4}{7}\right] = \frac{4}{7}\sqrt[7]{7^3\cdot 1\cdot 2\cdot 3\left(\frac{4}{4}\right)\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)\left(\frac{2}{4}\right)\left(\frac{6}{4}\right)\left(\frac{3}{4}\right)},$$

$$\left[\frac{5}{7}\right] = \frac{5}{7}\sqrt[7]{7^2\cdot 1\cdot 2\cdot 3\cdot 4\left(\frac{5}{5}\right)\left(\frac{3}{5}\right)\left(\frac{1}{5}\right)\left(\frac{6}{5}\right)\left(\frac{4}{5}\right)\left(\frac{2}{5}\right)},$$

$$\left[\frac{6}{7}\right] = \frac{6}{7}\sqrt[7]{7\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\left(\frac{6}{6}\right)\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\left(\frac{3}{6}\right)\left(\frac{2}{6}\right)\left(\frac{1}{6}\right)}.$$

## Corollary 7

**§51** Now let $n = 8$ and these seven reductions will be obtained

$$\left[\frac{1}{8}\right]=\frac{1}{8}\sqrt[8]{8^7\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)\left(\frac{6}{1}\right)\left(\frac{7}{1}\right)},$$

$$\left[\frac{2}{8}\right]=\frac{2}{8}\sqrt[8]{8^6\cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2\left(\frac{6}{2}\right)^2\left(\frac{8}{2}\right)}=\frac{1}{4}\sqrt[4]{8^3\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)},$$

$$\left[\frac{3}{8}\right]=\frac{3}{8}\sqrt[8]{8^5\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)},$$

$$\left[\frac{4}{8}\right]=\frac{4}{8}\sqrt[8]{8^4\cdot 4\cdot 4\cdot 4\left(\frac{4}{4}\right)^4\left(\frac{8}{4}\right)^3}=\frac{1}{2}\sqrt[2]{8\left(\frac{4}{4}\right)},$$

$$\left[\frac{5}{8}\right]=\frac{5}{8}\sqrt[8]{8^3\cdot 1\cdot 2\cdot 3\cdot 4\left(\frac{5}{5}\right)\left(\frac{2}{5}\right)\left(\frac{7}{5}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)\left(\frac{6}{5}\right)\left(\frac{3}{5}\right)},$$

$$\left[\frac{6}{8}\right]=\frac{6}{8}\sqrt[8]{8^2\cdot 4\cdot 2\cdot 6\cdot 4\cdot 2\left(\frac{6}{6}\right)^2\left(\frac{4}{6}\right)^2\left(\frac{2}{6}\right)^2\left(\frac{8}{6}\right)}=\frac{3}{4}\sqrt[4]{8\cdot 2\cdot 4\left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right)},$$

$$\left[\frac{7}{8}\right]=\frac{7}{8}\sqrt[8]{8\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\left(\frac{7}{7}\right)\left(\frac{6}{7}\right)\left(\frac{5}{7}\right)\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)\left(\frac{2}{7}\right)\left(\frac{1}{7}\right)}.$$

### Scholium

§52 It would be superfluous to expand these cases any further, because the structure of these formulas is already seen very clearly from the ones given. If the numbers $m$ and $n$ are coprime in the proposed formula $\left[\frac{m}{n}\right]$, the rule is obvious, because

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)};$$

but if these numbers $m$ and $n$ have a common divisor, it will indeed be useful to reduce this fraction $\frac{m}{n}$ to the smallest form and extract the value in question from the preceding cases; however, the operation can also be done as follows. Because the expression in question certainly has this form

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}PQ},$$

where $Q$ is the product of the $n-1$ integral formulas, $P$ on the other hand the product of some absolute numbers, in order to find that product $Q$ let us just continue this series of formulas $\left(\frac{m}{m}\right)\left(\frac{2m}{m}\right)\left(\frac{3m}{m}\right)$etc. until the numerator exceeds the exponent $n$, and instead of this numerator let us write its excess over $n$; if this excess is set $=\alpha$ such that our formula is $\left(\frac{\alpha}{m}\right)$, this numerator $\alpha$ will give a factor of a product $P$; then let us continue this series of formulas $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$etc. until one again gets to a numerator greater than the exponent $n$, and the formula $\left(\frac{n+\beta}{m}\right)$ emerges; instead of this formula one then has to write $\left(\frac{\beta}{m}\right)$, and hence the factor $\beta$ is introduced into the product $P$ and one has to continue like this until $n-1$ formulas for $Q$ will have emerged.

To understand these operations more easily, let us expand the case of the formula

$$\left[\frac{9}{12}\right]=\frac{9}{12}\sqrt[12]{12^3PQ}$$

in this way; an investigation of the letters $Q$ and $P$ being instituted in the following way:

$$\begin{array}{llllllllllll} \text{for } Q\ldots & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right) & \left(\frac{12}{9}\right) & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right) & \left(\frac{12}{9}\right) & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right), \\ \text{for } P\ldots & & 6 \cdot & 3 & & 9 \cdot & 6 \cdot & 3 & & 9 \cdot & 6 \cdot & 3 \end{array}$$

and so one finds

$$Q=\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3\left(\frac{12}{9}\right)^2 \quad \text{and} \quad P=6^3\cdot 3^3\cdot 9^2.$$

Because $\left(\frac{12}{9}\right)=\frac{1}{9}$, $PQ=6^3\cdot 3^3\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3$ and hence

$$\left[\frac{9}{12}\right]=\frac{3}{4}\sqrt[4]{12\cdot 6\cdot 3\left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)}.$$

## Theorem

§53 *Whatever positive numbers are indicated by the letters $m$ and $n$, in the notation introduced and explained before it will always be*

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$


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### Proof

For the cases in which $m$ and $n$ are coprime numbers, the validity of this theorem was shown in the preceding theorems; but the fact that it also holds, if those numbers $m$ and $n$ enjoy a common divisor, is not evident from that theorem; but since the formula was already proved to be true in the cases in which $m$ and $n$ are mutually prime, it is safe to conclude that this theorem is true in general. I am completely aware that this kind to deduce something is completely unusual and must seem suspect to most people. In order to clear those doubts, because for the cases, in which the numbers $m$ and $n$ composite between themselves, we obtained two expressions, it will be useful to have shown the agreement for the cases explained before. And the case $m = n$ is already a huge confirmation, in which case our formula obviously becomes the unity.

### Corollary 1

**§54** The first case requiring a demonstration of the agreement is that one, in which $m = 2$ and $n = 4$, for which we found above (§47)

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\left(\frac{2}{2}\right)^2};$$

but now by the power of the theorem

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2 \cdot 1\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)},$$

whence, after the comparison is done, becomes

$$\left(\frac{2}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right),$$

whose validity was confirmed in my observations mentioned[12] above.

### Corollary 2

**§55** If $m = 2$ and $n = 6$, from the results above (§49) we have

[12]Euler again refers to his paper "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x = 1$". This is paper E321 in the Eneström-Index.

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2};$$

now on the other hand by means of the theorem

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\cdot 1\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)\left(\frac{5}{2}\right)}$$

and therefore it has to be

$$\left(\frac{2}{2}\right)\left(\frac{4}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right),$$

whose validity is clear from the same source.

### Corollary 3

**§56** If $m = 3$ and $n = 6$, one arrives at this equation

$$\left(\frac{3}{3}\right)^2 = 1\cdot 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{5}{3}\right);$$

but if $m = 4$ and $n = 6$, in like manner,

$$2^2\left(\frac{4}{4}\right)\left(\frac{2}{4}\right) = 1\cdot 2\cdot 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)$$

or

$$\left(\frac{4}{4}\right)\left(\frac{2}{4}\right) = \frac{3}{2}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right),$$

which is also found to be true.

### Corollary 4

**§57** The case $m = 2$ and $n = 8$ yields this equality

$$\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right),$$

but the case $m = 4$ and $n = 8$ this one

$$\left(\frac{4}{4}\right)^3 = 1\cdot 2\cdot 3\left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{6}{4}\right)\left(\frac{7}{4}\right)$$

and finally the case $m = 6$ and $n = 8$ gives

$$2\cdot 4\left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right) = 1\cdot 3\cdot 5\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right),$$

which is also in agreement with the truth.

## Scholium

**§58** But if in general the numbers $m$ and $n$ have the common factor 2 and the proposed formula is $\left[\frac{2m}{2n}\right] = \left[\frac{m}{n}\right]$, because

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)},$$

after having reduced the same for the exponent $2n$ it will be

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}\cdot 2^2\cdot 4^2\cdot 6^2\cdots(2m-2)^2\left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2\cdots\left(\frac{2n-2}{2m}\right)^2}.$$

By the theorem, the same expression on the other hand becomes

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}\cdot 1\cdot 2\cdot 3\cdots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{2}{2m}\right)\left(\frac{3}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right)},$$

whence for the exponent $2n$ it will be

$$\begin{aligned}2\cdot 4\cdot 6\cdots(2m-2)&\left(\frac{2}{2m}\right)\left(\frac{4}{2m}\right)\left(\frac{6}{2m}\right)\cdots\left(\frac{2n-2}{2m}\right)\\ = 1\cdot 3\cdot 5\cdots(2m-1)&\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right).\end{aligned}$$

If in the same way the common divisor is 3, for the exponent $3n$ one will find

$$3^2\cdot 6^2\cdot 9^2\cdots(3m-3)^2\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\left(\frac{9}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2$$

$$= 1 \cdot 2 \cdot 4 \cdot 5 \cdots (3m-2)(3m-1)\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\cdots\left(\frac{3n-1}{3m}\right),$$

which equation can be more conveniently exhibited as follows

$$\frac{1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot 10 \cdots (3m-2)(3m-1)}{3^2 \cdot 6^2 \cdot 9^2 \cdots (3m-3)^2} = \frac{\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\left(\frac{7}{3m}\right)\cdots\left(\frac{3n-2}{3m}\right)\left(\frac{3n-1}{3m}\right)}.$$

But if in general the common divisor is $d$ and the exponent $dn$, one will have

$$\left(d \cdot 2d \cdot 3d \cdots (dm-d)\left(\frac{d}{dm}\right)\left(\frac{2d}{dm}\right)\left(\frac{3d}{dm}\right)\cdots\left(\frac{dn-d}{dm}\right)\right)^d$$
$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (dm-1)\left(\frac{1}{dm}\right)\left(\frac{2}{dm}\right)\left(\frac{3}{dm}\right)\cdots\left(\frac{dn-1}{dm}\right),$$

which equation can easily be accommodated to any cases, whence the following theorem deserves to be noted.

## Theorem

§59 *If $\alpha$ is a common divisor of the numbers $m$ and $n$ and the formula $\left(\frac{p}{q}\right)$ denotes the value of the integral $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ extended from $x=0$ to $x=1$, it will be*

$$\left(\alpha \cdot 2\alpha \cdot 3\alpha \cdots (m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n-\alpha}{m}\right)\right)^\alpha$$
$$= 1 \cdot 2 \cdot 3 \cdots (m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right).$$

## Proof

The validity of this theorem is seen from the preceding Scholium; while the common divisor was $= d$ and the two propounded numbers were $dm$ and $dn$ there, here I just wrote $m$ and $n$ instead of them, but instead of their divisor $d$ I wrote the letter $\alpha$, which nature of the divisor the stated equality contains in such a way that one assumes the numbers $m$ and $n$ and hence also $m-\alpha$ and $n-\alpha$ to occur in the continued arithmetic progression $\alpha$, $2\alpha$, $3\alpha$ etc. In addition, I am forced to confess that this proof is, of course, mainly based on induction and cannot be considered to be rigorous by any means; but because we are nevertheless convinced of its truth, this theorem seems to be worth of one's greater attention; nevertheless, there is no doubt that a further expansion of integral formulas of this kind will finally lead to a complete proof; because it was possible for us to see its truth before we had the complete proof, from which the notable example of analytical investigation shines forth.

### Corollary 1

**§60**[13] Thus, if we substitute the integrals for the signs we introduced, our theorem will be as follows

$$\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\int\frac{x^{\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x^{2\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\int\frac{x^{n-\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}$$

$$=\sqrt[\alpha]{1\cdot 2\cdot 3\cdots(m-1)\int\frac{dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{xdx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\int\frac{x^{n-2}dx}{\sqrt[n]{(1-x^n)^{n-m}}}}.$$

### Corollary 2

**§61** Or if, for the sake of brevity, we set $\sqrt[n]{(1-x^n)^{n-m}}=X$, it will be

$$\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\int\frac{x^{\alpha-1}dx}{X}\cdot\int\frac{x^{2\alpha-1}dx}{X}\cdots\int\frac{x^{n-\alpha-1}dx}{X}$$

$$=\sqrt[\alpha]{1\cdot 2\cdot 3\cdots(m-1)\int\frac{dx}{X}\cdot\int\frac{xdx}{X}\cdot\int\frac{x^2dx}{X}\cdots\int\frac{x^{n-2}dx}{X}}.$$

## General Theorem

**§62** *If the common divisors of the two numbers $m$ and $n$ are $\alpha$, $\beta$, $\gamma$ etc. and the formula $\left(\frac{p}{q}\right)$ denotes the value of the integral $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ extended from $x=0$ to $x=1$, the following expressions consisting of integral formulas of this kind will be separately equal to each other*

[13]The original erroneously states this paragraph as §69.

$$\begin{aligned}
&\left[\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n-\alpha}{m}\right)\right]^{\alpha}\\
=&\left[\beta\cdot 2\beta\cdot 3\beta\cdots(m-\beta)\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n-\beta}{m}\right)\right]^{\beta}\\
=&\left[\gamma\cdot 2\gamma\cdot 3\gamma\cdots(m-\gamma)\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n-\gamma}{m}\right)\right]^{\gamma}\\
&\text{etc.}
\end{aligned}$$

## Proof

The validity of this theorem obviously follows from the preceding theorem, because each of these expressions is equal to this one

$$1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right),$$

which corresponds to the unity as smallest common divisor of the numbers $m$ and $n$. Therefore, as many expressions of this kind can be shown equal to each other as there are common divisors of the two numbers $m$ and $n$.

## Corollary 1

**§63** Because this formula $\left(\frac{n}{m}\right)$ is $=\frac{1}{m}$ and hence $m\left(\frac{n}{m}\right)=1$, our equal expressions can be represented more succinctly as follows

$$\begin{aligned}
&\left[\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right]^{\alpha}\\
=&\left[\beta\cdot 2\beta\cdot 3\beta\cdots m\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n}{m}\right)\right]^{\beta}\\
=&\left[\gamma\cdot 2\gamma\cdot 3\gamma\cdots m\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n}{m}\right)\right]^{\gamma}.
\end{aligned}$$

For, even if the number of factors was increased here, the structure of the arrangement falls more easily on the eyes.

### Corollary 2

**§64** Thus, if $m=6$ and $n=12$, because of the common divisors of these numbers, 6, 3, 2, 1, one will have the following four forms that are all equal to each other

$$=\left[6\left(\frac{6}{6}\right)\left(\frac{12}{6}\right)\right]^6=\left[3\cdot 6\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\right]^3$$
$$=\left[2\cdot 4\cdot 6\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)\right]^2$$
$$=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{3}{6}\right)\cdots\left(\frac{12}{6}\right).$$

### Corollary 3

**§65** If the last formula is combined with the penultimate, this equation will result

$$\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}=\frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},$$

but the last compared to the second yields

$$\frac{1\cdot 2\cdot 4\cdot 5}{3\cdot 3\cdot 6\cdot 6}=\frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.$$

### Scholium

**§66** Hence infinitely many relations among the integral formulas of the form

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{p}{q}\right)$$

follow, which are even more remarkable, because we were led to them by a completely singular method. And if anyone does not believe them to be true, he or she should consult my observations on these integral formulas[14] and will then easily be convinced of their truth for any case. But even if this treatment provides some confirmation, the relations found here are nevertheless of even greater importance, because a certain structure is noticed in them and they are easily generalised to all classes,

[14] Euler again refers to his paper Öbservationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x=1$". This is paper E321 in the Eneström-Index.

whatever number was assumed for the exponent $n$, whereas in the first treatment the calculation for the higher classes becomes continuously more cumbersome and intricate.

## Supplement containing the Proof of the theorem proposed in §53

It is convenient to derive this proof from the results mentioned above; let us take the equation given in §25, which for $f = 1$, having changed the letters, reads as follows

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\nu-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{\mu-1}}{\int dx\left(\log\frac{1}{x}\right)^{\nu+\mu-1}} = \varkappa\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}},$$

and, using known reductions, represent it in this form

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\nu}\cdot\int dx\left(\log\frac{1}{x}\right)^{\mu}}{\int dx\left(\log\frac{1}{x}\right)^{\nu+\mu}} = \frac{\varkappa\mu\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Now let $\nu = \frac{m}{n}$ and $\mu = \frac{\lambda}{n}$, but then $\varkappa = n$ so that we have

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\frac{m}{n}}\cdot\int dx\left(\log\frac{1}{x}\right)^{\frac{\lambda}{n}}}{\int dx\left(\log\frac{1}{x}\right)^{\frac{\lambda+m}{n}}} = \frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}dx}{\sqrt[\varkappa]{(1-x^{n})^{n-m}}},$$

which, for the sake of brevity having used notation introduced above, is more conveniently expressed as follows

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]} = \frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right).$$

Now let us successively write the numbers 1, 2, 3, $4\ldots n$ instead of $\lambda$ and multiply all these equations in turn, whose number is $= n$, and the resulting equation will be

$$\begin{aligned}&\left[\frac{m}{n}\right]^{n}\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+1}{n}\right]\left[\frac{m+2}{n}\right]\left[\frac{m+3}{n}\right]\cdots\left[\frac{m+n}{n}\right]}\\
&= m^{n}\frac{1}{m+1}\cdot\frac{2}{m+2}\cdot\frac{3}{m+3}\cdots\frac{n}{m+n}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)\\
&= m^{n}\frac{1\cdot2\cdot3\cdots m}{(n+1)(n+2)(n+3)\cdots(m+n)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).\end{aligned}$$

But in like manner, let us just transform the left-hand side such that

$$\left[\frac{m}{n}\right]^{n}\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

whose agreement with the preceding form reveals itself freely by multiplying through the cross, as they say. But because from the nature of these formulas

$$\left[\frac{n+1}{n}\right] = \frac{n+1}{n}\left[\frac{1}{n}\right], \quad \left[\frac{n+2}{n}\right] = \frac{n+2}{n}\left[\frac{2}{n}\right], \quad \left[\frac{n+3}{n}\right] = \frac{n+3}{n}\left[\frac{3}{n}\right] \quad \text{etc.},$$

and since the number of these formulas $= m$, this left-hand side will become

$$\left[\frac{m}{n}\right]^n \frac{n^m}{(n+1)(n+2)(n+3)\cdots(n+m)};$$

because this one is equal to the other part exhibited before

$$m^n \frac{1\cdot 2\cdot 3\cdots m}{(n+1)(n+2)(n+3)\cdots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

we obtain this equation

$$\left[\frac{m}{n}\right]^n = \frac{m^n}{n^m} 1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)$$

such that

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1\cdot 2\cdot 3\cdots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)};$$

because this equation $\left(\frac{n}{m}\right) = \frac{1}{m}$ totally agrees with the one proposed in §53, its truth is now indeed proved from most certain principles.

## Proof of the theorem proposed in §59

Also this theorem needs a more rigorous proof which I give using the equation established before, i.e.

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]} = \frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right),$$

as follows. While $\alpha$ is a common divisor of the numbers $m$ and $n$, let us successively write the numbers $\alpha$, $2\alpha$, $3\alpha$ etc. up to $n$ instead of $\lambda$, whose total amount is $= \frac{n}{\alpha}$, and now let us multiply all equations resulting in this way in turn such that this equation emerges

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\left[\frac{m+n}{n}\right]}$$

$$= m^{\frac{n}{\alpha}} \frac{\alpha}{m+\alpha}\cdot\frac{2\alpha}{m+2\alpha}\cdot\frac{3\alpha}{m+3\alpha}\cdots\frac{n}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Now let us transform the left-hand side into this one equal to it

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+\alpha}{n}\right]\left[\frac{n+2\alpha}{n}\right]\left[\frac{n+3\alpha}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

which, because of $\left[\frac{n+\alpha}{n}\right]=\frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$ and similarly for the remaining ones, is reduced to this one

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{n}{n+\alpha}\cdot\frac{n}{n+2\alpha}\cdot\frac{n}{n+3\alpha}\cdots\frac{n}{n+m},$$

In like manner, the right-hand side of the equation is transformed into this one

$$m^{\frac{n}{\alpha}} \frac{\alpha}{n+\alpha}\cdot\frac{2\alpha}{n+2\alpha}\cdot\frac{3\alpha}{n+3\alpha}\cdots\frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right),$$

whence this equation results

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} n^{\frac{m}{\alpha}} = n^{\frac{n}{\alpha}}\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)$$

and hence

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1}{n^m}\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha}}$$

which expression[15] compared to the preceding yields this equation

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha} = 1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

which is to be understood for all common divisors of the two numbers $m$ and $n$.

[15]The Latin original says $m^n$ under the radical sign, which was corrected in this translation.