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Euler and the Multiplication Formula for the Γ-Function

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1 Introduction

The interpolation of the factorial by the Γ-function was found nearly simultaneously by Daniel Bernoulli (1700-1782) [$Be29$] and Euler in 1729 (see also his 1738 paper [$E19$]) and is undoubtedly one of the most important functions in mathematics. Most of its basic properties were discovered by Euler, who also gave the definition that is often used today to introduce the function originally given in §7 of [$E675$]. In modern notation, we have

$$\Gamma(x) := \int\limits_0^\infty e^{-t} t^{x-1} dt \quad \text{for} \quad \text{Re}(x) > 0.$$

However, a full understanding of Γ as a meromorphic function of its argument could only be achieved after Gauss's contributions in the 19th century; the now universally-adopted notation stems from Adrien Marie Legrendre (1752-1833) [$Le09$].
One of the fundamental properties of the Γ-function is so-called *multiplication formula* that reads, in modern notation

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{x-\frac{1}{2}}}\cdot\Gamma(x). \tag{1}$$

For $n = 2$ one obtains the *duplication formula* that is usually ascribed to Legendre [$Le26$].

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The multiplication formula was first proven rigourously by Gauss in his influential paper [$Ga28$] on the hypergeometric series, in which he also gave a complete account of the factorial function $\Pi(x) := \Gamma(x+1) = x!$. Gauss cited Euler's results very often, but apparently he was not aware of the lesser-known paper [$E421$] of Euler. In that paper Euler presented a formula that is essentially equivalent to (1), as we will now explain.

1.1 The function $\left(\frac{p}{q}\right)$

In §3 of [$E321$] and §44 of [$E421$], Euler studied properties of the function

$$\left(\frac{p}{q}\right) := \int_0^1 \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}}.$$

In his notation the variable n is left implicit, and Euler showed the elegant symmetry property

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right).$$

Of course, by the substitution $x^n = y$ this function is just the Beta-function in disguise:

$$\left(\frac{p}{q}\right) = \frac{1}{n}\int_0^1 y^{\frac{p}{n}-1}dy(1-y)^{\frac{q}{n}-1} = \frac{1}{n}\cdot B\left(\frac{p}{n},\frac{q}{n}\right), \tag{2}$$

where the Beta-function is defined as

$$B(x,y) = \int_0^1 t^{x-1}dt(1-t)^{y-1} \quad \text{for} \quad \text{Re}(x), \text{Re}(y) > 0.$$

Euler implicitly assumed p and q to be natural numbers, but this restriction is of course not necessary.

One of Euler's early discoveries in [$E19$] was that the Beta-integral reduces to a product of Γ-factors:

$$B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}.$$

This result is also given in the supplement to [$E421$].

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1.2 The reflection formula

Euler's version of the reflection formula for the Γ-function,

$$\frac{\pi}{\sin \pi x} = \Gamma(x)\Gamma(1-x),$$

can be found in §43 of [E421] and reads

$$[\lambda] \cdot [-\lambda] = \frac{\pi \lambda}{\sin \pi \lambda},$$

where $[\lambda]$ stands for $\lambda!$, that is $\Gamma(1+\lambda)$.
If one applies the reflection formula for $x = \frac{i}{n}$, $i = 1, 2, \cdots, n-1$, we obtain

$$\begin{aligned}
\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right) &= \frac{\pi}{\sin \frac{\pi}{n}}, \\
\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right) &= \frac{\pi}{\sin \frac{2\pi}{n}}, \\
\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{n-3}{n}\right) &= \frac{\pi}{\sin \frac{3\pi}{n}}, \\
\ldots &= \ldots \\
\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right) &= \frac{\pi}{\sin \frac{(n-1)\pi}{n}}.
\end{aligned}$$

Multiplying these equations together gives our first auxiliary formula

$$\prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right)^2 = \frac{\pi^{n-1}}{\prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right)}.$$

Our second auxiliary formula is

$$\prod_{i=1}^{n-1} \sin\left(\frac{i\pi}{n}\right) = \frac{n}{2^{n-1}},$$

which is a nice exercise and which was certainly known to Euler. For example, in §7 of [E562] and in §240 of [E101], he stated the more general formula

$$\begin{aligned}
\sin n\varphi = 2^{n-1} \sin \varphi \sin\left(\frac{\pi}{n} - \varphi\right) \sin\left(\frac{\pi}{n} + \varphi\right) \\
\sin\left(\frac{2\pi}{n} - \varphi\right) \sin\left(\frac{2\pi}{n} + \varphi\right) \cdot \text{etc.}
\end{aligned}$$

This product has n factors in total. If we divide by $2^{n-1}\sin\varphi$, use $\sin\left(\frac{\pi(n-i)}{n}\right) = \sin\left(\frac{i\pi}{n}\right)$ and take the limit as $\varphi \to 0$, we obtain the second auxiliary formula.

The first and second auxiliary formula were also given by Gauss in [Ga28] and werw used in his proof of the multiplication formula. Combining them and taking the square root, we obtain the beautiful formula

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \sqrt{\frac{(2\pi)^{n-1}}{n}}. \tag{3}$$

This formula was also found by Euler in §46 of [E816], where he stated it in the form

$$\int_0^1 dx\left(\log\frac{1}{x}\right)^{\frac{1}{n}}\int_0^1 dx\left(\log\frac{1}{x}\right)^{\frac{2}{n}}\cdots\int_0^1 dx\left(\log\frac{1}{x}\right)^{\frac{n-1}{n}} = \frac{1\cdot 2\cdot 3\cdots(n-1)}{n^{n-1}}\sqrt{\frac{2^{n-1}\pi^{n-1}}{n}}.$$

2 Euler's version of the Multiplication Formula

In §53 of [E421] Euler gave the formula

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$

As before, $[\lambda]$ is Euler's notation for the factorial of λ, so that $\left[\frac{m}{n}\right] = \Gamma\left(\frac{m}{n}+1\right)$. Euler assumed m and n to be natural numbers, but it is easily seen that we can interpolate $1\cdot 2\cdot 3\cdots(m-1)$ via $\Gamma(m)$. Therefore, if we assume x to be real and positive and substitute x for m in the above formula and express it in terms of the Beta-function using (2), Euler's formula becomes

$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{n-x}\Gamma(x)\frac{1}{n^{n-1}}B\left(\frac{1}{n},\frac{x}{n}\right)B\left(\frac{2}{n},\frac{x}{n}\right)\cdots B\left(\frac{n-1}{n},\frac{x}{n}\right)}.$$

Expressing the Beta-function in terms of the Γ-function, then some rearrangement under the radical yields

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$$\Gamma\left(\frac{x}{n}\right) = \sqrt[n]{n^{1-x}\Gamma(x)\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+1}{n}\right)}\cdot\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+2}{n}\right)}\cdots\frac{\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{x}{n}\right)}{\Gamma\left(\frac{x+n-1}{n}\right)}}.$$

By bringing all Γ-functions with fractional argument to the left-hand side, the expression simplifies to

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = n^{1-x}\Gamma(x)\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right).$$

The product on the right-hand side, $\Gamma\left(\frac{1}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$, was evaluated in (3) and thus we obtain

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\cdots\Gamma\left(\frac{x+n-1}{n}\right) = n^{1-x}\Gamma(x)\sqrt{\frac{(2\pi)^{n-1}}{n}}.$$

Thus, we have arrived at the multiplication formula (1).

3 Summary and Conclusion

From the above sketch it is apparent that in [E421], Euler had a result that is essentially equivalent to the multiplication formula for the Γ-function. He expressed it in terms of the symbol $\left(\frac{p}{q}\right)$, which is expressed in modern notation via the Beta-function. One may wonder why Euler did not express his result in terms of the Γ-function itself. Reading his paper it becomes clear that his main motivation was to express the factorial of rational numbers in terms of integrals of *algebraic functions*, and the formula given by Euler fulfills this purpose. Probably for the same reason, he did not replace $1\cdot 2\cdot 3\cdots(m-1)$ by $\Gamma(m)$.

Euler also expressed $\Gamma(\frac{p}{q})$ in terms of integrals of algebraic algebraic functions in §23 [E19] and §5 of [E122]. That formula reads

$$\int_0^1 (-\log x)^{\frac{p}{q}}\,dx = \sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots\left(\frac{qp}{q}+1\right)}$$

$$\times\sqrt[q]{\int_0^1 dx(x-xx)^{\frac{p}{q}}\cdot\int_0^1 dx(x^2-x^3)^{\frac{p}{q}}\cdot\int_0^1 dx(x^3-x^4)^{\frac{p}{q}}\cdots\int_0^1 dx(x^{q-1}-x^q)^{\frac{p}{q}}}.$$

Despite the similarity to the first formula of section 2, this formula is not as general as the multiplication formula[1]. It appears that Euler was aware that the proofs he indicated in [$E421$] were not completely convincing. He expressed that with characteristic honesty in a concluding Scholium:

Hence infinitely many relations among the integral formulas of the form

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = \left(\frac{p}{q}\right)$$

follow, which are even more remarkable, because we were led to them by a completely singular method. And if anyone does not believe them to be true, he or she should consult my observations on these integral formulas[2] *and will then hence easily be convinced of their truth for any case. But even if this consideration provides some confirmation, the relations found here are nevertheless of even greater importance, because a certain structure is noticed in them and they are easily generalized to all classes, whatever number was assumed for the exponent n, whereas in the first treatment the calculation for the higher classes becomes continuously more cumbersome and intricate.*

The history of the Γ-function is long and complex, and we have seen here that not all parts of the story have been told. We hope that this note provides some motivation for people to examine carefully other papers by Euler and his contemporaries, not only because they make for good reading, but also to find further results, *maybe stated in unfamiliar form*, that were proven rigorously by their successors. This will certainly be of interest for anyone studying the history of mathematics.

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[1] In the foreword of the Opera Omnia, series 1, volume 19, p. LXI A. Krazer and G. Faber claim that these two formulas are equivalent and both are a special case of the multiplication formula. This is incorrect, as it was shown in the preceding sections. The formula given in section 3 does not lead to the multiplication formula, but only interpolates $\Gamma\left(\frac{p}{q}\right)$ in terms of algebraic integrals.

[2] Here Euler refers to his paper [$E321$].

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